Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 16, Number 2 (2020), pp. 325-339

http://www.ripublication.com/gjpam.htm

A Powerful Method for Obtaining Exact Solutions of Volterra Integral Equation's Types

Mohamed A. Ramadan[1], Galal M. Moatimid, Mahmoud H.Taha[2]

[1]*Department of Mathematics, Faculty of Science, Menoufia University, Menoufia, Egypt.*

[2]*Department of Mathematics, Faculty of Education, Ain Shams University, Cairo, Egypt.*

Corresponding author: mahmoud.h.taha@edu.asu.edu.eg

Abstract

The purpose of this paper is to obtain an exact solution of linear as well as nonlinear Volterra integral (VIE) and Volterra integro-differential (VIDE) equations. These equations exist in many applications, namely here, the motion of a charged particle in an oscillating magnetic field. Also, the probability density that is associated with a continuous-time random walk in three dimensions. For some nonlinear problems, the classical homotopy perturbation method (HPM) is divergent. Therefore, a modified algorithm is made on the (HPM). The method will be named as a power series homotopy perturbation method (PSHPM). This method is convergent when solving nonlinear VIE or VIDE. In all studied cases, it converges to a closed form which resulted in an exact solution. Through this method, an algorithm is successfully established to solve linear and nonlinear VIE and VIDE. Finally, several examples are presented to validate the applicability of the deducted technique.

Keywords: Homotopy perturbation method; Volterra integro-differential equation; Heat conduction; Crystal growth; Radiation of heat from a semi-infinite solid; Charged particle in oscillating magnetic fields.

Mathematics Subject Classification (2010): 44A05, 65L05.

1. INTRODUCTION

Integral and integro-differential equations arise in many branches in physics and in engineering. For instance, in the potential theory, acoustics, elasticity, fluid mechanics, irradiative transfer, and theory of population. The target of this paper is to obtain exact solutions of some problems in the electromagnetic theory and probability density that is associated with a continuous-time random walk in three dimensions as those given in References [1- 3]. In many areas, the integral equation originates from the conversion of a boundary-value problem or an initial-value problem that are associated with a partial or an ordinary differential equation. Many problems lead directly to integral or integro-differential equations. Along these topics, they cannot be modulated through differential equations. Integral equations have several types; here, we considered Volterra integro-differential equation of second kind which is a general case of the Volterra integral equation if the differentiation occurs on the unknown function. The Volterra integro-differential equation may take the following form:

$$u^{(k)}(x) = f(x) + \int_0^x k(x,t,u(t))dt, \tag{1.1}$$

with the initial conditions $u(0) = \alpha,\ u'(0) = \beta,\ \ldots..,\ u^{(k-1)} = \delta,$ (1.2)

where $u^{(k)}(x)$ is the k^{th} derivative of the function $u(x)$. Also, $k(x,t,u(t))$ is called the kernel of the integral equation.

The functions $f(x)$ and $k(x,t,u(t))$ are analytic functions on the closed subset of $\mathbb{R}$ where $\mathbb{R}$ is the set of all real numbers. The existence and uniqueness of Eq. (1.1) are depicted by Pachpatta [3].

During the last decade, many attempts, to solve the linear or nonlinear versions of Eq. (1.1), were carried out by several researches by using numerical or perturbed methods. The time collocation method and the projection method did a great job in solving it [3]. Adomian decomposition method is used by Wazwaz [4] with a lot of modifications on the method. He managed to solve and get approximate solution for most cases of that equation. Others applied an iterative method to solve this equation. He [5] tried to solve linear differential and integral equations using a new method which is called (HPM) in 1999. Also, he managed to solve some nonlinear porblems [6]. After that he developed his technique to work on more complicated problems and introduced a solution to a lot of applications [7].

Recently, El-Dib and Moatimid [8] introduced a coupling between a power series formula and the (HPM) to obtain an exact solution of linear as well as nonlinear differential equations. They obtained exact solution of a singular Lane-Emden equation. Also, for a singular one-dimension pseudohyperbolic equation. Finally, for linear singular one-dimensional thermo-elasticity coupled system. HPM has been rapidly growing in the recent years. A new technique with a modification is done on (HPM). In the current method, the solution of the functional equations is considered as a summation of an infinite series. Usually, this series converges to the solution and, therefore, the modification is done. The first order term of the series is cut off equated to zero; consequently, all higher order terms, which are dependent on the first order term, will vanish. In other words; the whole series will fall directly into exactly one term which will be the exact solution itself. All of that is done with the aid of the initial-guess function which has been formulated as a power series which involves the undetermined coefficients. Later, these coefficients will lead to the exact solution.

The aim of this paper is to obtain exact solutions of Volterra integral equation's types. We shall proceed with similar arguments as given in our previous work [8]. To clarify the paper, the rest of this article will be divided into four sections as follows: Section 2 will go through the basic idea of the method (algorithm of solution). Section 3 is divided into three parts, part one depicts some examples on linear and nonlinear integral equations and their exact solution using our new technique which is known as a power series homotopy perturbation method (PSHPM), part two handles integro-differential equations and their exact solutions, and part three solves system of Volterra integral equations. Finally, Section 4 is devoted to graph the conclusion.

2. BASIC IDEA OF THE METHOD (ALGORITHM OF SOLUTION)

We will start with the basic concepts of the homotopy technique for Eq. (1.1). It may be decomposed into two parts; namly, L and N which are known as a linear and nonlinear parts, respectively, as follows:

$$L(u)+N(u)=0, \tag{2.1}$$

then construct a homotopy equation in the following form:

$$H(u,p)=L(u)-L(U)+\rho\left[L(U)+N(u)\right]=0, \rho\in[0,1], \tag{2.2}$$

where U is an initial guess, or sometimes called a trial function, for the solution of Eq. (1.1) with the use of the artificial homotopy parameter ρ to expand $u(x,\rho)$ as

$$u(x,\rho)=u_0(x)+\rho u_1(x)+\rho^2 u_2(x)+\ldots, \tag{2.3}$$

take the initial guess as a power series in x as

$$U(x)=\sum_{n=0}^{\infty} a_n x^{\frac{n}{m}}, \tag{2.4}$$

where $m\neq 0$ is a real number which is taken according to the nature of the case study. Now, consider

$$L(u)=u^{(k)}(x) \text{ and } N(u)=-f(x)-\int_0^x k(x,t,u(t))dt, \tag{2.5}$$

Combining equations (2.3-5) and Eq. (2.2), one finds

$$u_0^{(k)}(x)+\rho u_1^{(k)}(x)+\ldots-U^{(k)}(x)+\rho\left[U^{(k)}(x)-f(x)-\int_0^x k(x,t,u_0(t)+\rho u_1(t)+\ldots)dt\right]=0, \tag{2.6}$$

Equating the coefficients of like powers of ρ, one gets

$$\rho^0 : u_0^{(k)}(x)=\sum_{n=0}^{\infty} a_n x^{\frac{n}{m}}, \tag{2.7}$$

integrate both sides of Eq. (2.7), $k-$times, one gets

$$u_0(x) = \sum_{n=0}^{\infty} \frac{a_n x^{\frac{n}{m}+k}}{\left(\frac{n}{m}+1\right)\left(\frac{n}{m}+2\right)\cdots\left(\frac{n}{m}+k\right)} + \delta \frac{x^{k-1}}{(k-1)!} + \ldots + \beta x + \alpha, \tag{2.8}$$

$$\rho^1 : u_1^{(k)}(x) = -U^{(k)}(x) + f(x) + \int_0^x k(x,t,u_0(t))dt, \tag{2.9}$$

$$\rho^2 : u_2^{(k)}(x) = \int_0^x k(x,t,u_1(t))dt, \tag{2.10}$$

$$\vdots$$

$$\rho^n : u_n^{(k)}(x) = \int_0^x k(x,t,u_{n-1}(t))dt. \tag{2.11}$$

Set $u_1(x) = 0$ in Eq. (2.9) then comparing the coefficients of like powers of x to get the undetermined coefficients a_n's.

At this stage, substitute from the coefficients of a_n's into Eq. (2.8) to obtain the required exact solution.

It should be mentioned here that:

- The selection of the guessing function in the classical HPM is rather difficult. Now, PSHPM introduces a simple technique of selecting a proper initial approximation. Therefore, this new technique gives a general method. It is a rapid convergence to the exact solution with simpler calculations.
- The exact solutions, needs a convergent series as given in Eq. (2.8). Otherwise, we obtain only an approximate solution of the given Volterra-differential equation.
- The above method still valid in case of the linear as well as nonlinear Volterra-integral equations.

3. ILLUSTRATED EXAMPLES

One type of the stochastic reaction–diffusion model that is used to modulate the cellular processes such as gene expression [9], involves continuous-time random walk of molecules on a lattice. One model is studied from these models [10]. They involve a molecule that can undergo a continuous-time random walk with binding possible at one specific binding site. This model could be formulated as a linear, convolution Volterra equation with a kernel, $K(t,s) = K(t-s)$, that is smooth but contains sharp gradients.

Through these examples, the first one is considered as the problem of calculating the probability density that is associated with a continuous-time random walk in three dimensions. It may be killed at a fixed lattice site [11]. This problem is modulated through a linear Volterra integral equation. So, one sets $k = 0$ in Eq. (1.1).

Example 3.1

Consider the following linear Volterra integral equation:

$$y(t) = \sin(t) - \int_0^t \cos(t-s)y(s)ds. \tag{3.1}$$

For convenience, one may choose the linear and nonlinear parts as

$$L\{y(t)\} = y(t), \text{ and } N\{y(t)\} = -\sin(t) + \int_0^t \cos(t-s)y(s)ds, \tag{3.2}$$

In this example, the initial guess, as given in Eq. (2.4) is taken with $m=1,$ therefore,

$$Y(t) = \sum_{n=0}^{\infty} a_n t^n. \tag{3.3}$$

It follows that the homotopy equation, as given in Eq. (2.2), is written as

$$H(y,p) = y(t) - \sum_{n=0}^{\infty} a_n t^n + \rho\left[\sum_{n=0}^{\infty} a_n t^n - \sin(t) + \int_0^t \cos(t-s)y(s)ds\right] = 0. \tag{3.4}$$

Set $y(t,\rho) = y_0(t) + \rho y_1(t) + \dots$, then comaring the coefficients of like powers of ρ, one gets

$$\rho^0 : y_0(x) = \sum_{n=0}^{\infty} a_n t^n \tag{3.5}$$

and

$$\rho^1 : y_1(x) = -\sum_{n=0}^{\infty} a_n t^n + \sin(t) - \int_0^t \cos(t-s)\sum_{n=0}^{\infty} a_n s^n ds \tag{3.6}$$

The undetermined coefficients that appearing in Eq. (3.5) are found from the cancellation of $y_1(t)$. Therefore, the coefficients a_n 's are directly

$$a_0 = 0, a_1 = 1, a_2 = \frac{-1}{2}, a_3 = 0, a_4 = \frac{1}{4!}, \dots. \tag{3.7}$$

Subsitute from Eq. (3.7) into Eq. (3.3), it follows that the exact solution becomes

$$y(t) = \frac{2\sqrt{3}}{3}\sin\left(\frac{\sqrt{3}t}{2}\right)e^{-t/2}. \tag{3.8}$$

It is remarkable that Isaacson and Kirby [11] used the standard collocation method to solve the same problem. However, they got only a numerical approximate solution with relatively large error. Meanwhile, here, after using the PSHPM, the exact solution is obtained easily and without any computation difficulties.

Abel in 1826 [12] investigated the motion of a particle that slides down along a smooth unknown curve, in a vertical plane, under the influence of the gravity. The particle takes the time $f(x)$ to move from the highest point of vertical height x to the lowest point 0 on the curve. The Abel's problem is derived to find the equation of that curve. Abel derived the equation of motion of the sliding particle along a smooth curve by the singular integral equation

$$u(x)=\int_0^x \frac{u(t)}{\sqrt{x-t}}dt. \tag{3.9}$$

The general form of Abel's equation is a generalized weakly singular Volterra equation (WSVIE) in the form [13]

$$u(x)=f(x)+\int_0^x \frac{u(t)}{\sqrt{x-t}}dt. \tag{3.10}$$

These equations arise in many mathematical physics and chemistry applications such as stereology, heat conduction, crystal growth and the radiation of heat from a semi-infinite solid [13]. It is also assumed that the function $u(x)$ is sufficiently smooth, so that it guarantees a unique solution. The weakly-singular and the generalized weakly-singular equations fall under the category of singular equations with singular kernels.

In the following examples the PSHPM is applied on weakly singular Volterra integral equation (WSVIE) as in [14] setting $k=0$ in Eq. (2.5) to get Volterra integral equation.

Example 3.2

Consider the following linear WSVIE equation:

$$u(x)=x(1-x)+\frac{16}{105}x^{7/2}(7-6x)-\int_0^x \frac{xt}{\sqrt{x-t}}u(t)dt. \tag{3.11}$$

For convenience, one may choose the linear and nonlinear parts as

$$L\{u(x)\}=u(x),\ \text{and}\ N(u(x))=-x(1-x)-\frac{16}{105}x^{7/2}(7-6x)+\int_0^x \frac{xt}{\sqrt{x-t}}u(t)dt. \tag{3.12}$$

In this example, for the initial guess as given in Eq. (2.4), it is appropriate to take $m=2$. Therefore, one gets

$$U(x)=\sum_{n=0}^{\infty} a_n x^{n/2}, \tag{3.13}$$

It follows that the homotopy equation, as given in Eq. (2.2), may be written as

$$H(u,p) = u(x) - \sum_{n=0}^{\infty} a_n x^{n/2} + \rho\left[\sum_{n=0}^{\infty} a_n x^{n/2} - x(1-x) - \frac{16}{105}x^{7/2}(7-6x) + \int_0^x \frac{xt}{\sqrt{x-t}}u(t)dt\right] = 0 \quad (3.14)$$

As given before, one gets

$$\rho^0 : u_0(x) = \sum_{n=0}^{\infty} a_n x^{n/2}, \quad (3.15)$$

and

$$\rho^1 : u_1(x) = -\sum_{n=0}^{\infty} a_n x^{n/2} + x(1-x) + \frac{16}{105}x^{7/2}(7-6x) - \int_0^x \frac{xt}{\sqrt{x-t}}u_0(t)dt. \quad (3.16)$$

The cancellation of $u_1(x)$ results

$$-\sum_{n=0}^{\infty} a_n x^{n/2} + x(1-x) + \frac{16}{105}x^{7/2}(7-6x) - \int_0^x \frac{xt}{\sqrt{x-t}}\sum_{n=0}^{\infty} a_n t^{n/2} dt = 0. \quad (3.17)$$

As shown before, one finds

$$a_2 = 1, \quad a_4 = -1, \text{ and all other } a_n\text{'s} = 0. \quad (3.18)$$

Therefore, the exact solution of Eq. (3.11) may be written as

$$u(x) = x - x^2. \quad (3.19)$$

Example 3.3

Consider the nonlinear WSVIE equation

$$u(x) = x^{1/2} + \frac{3\pi}{8}x^2 - \int_0^x \frac{u^3(t)}{\sqrt{x-t}}dt. \quad (3.20)$$

For convenience, one may choose the linear and nonlinear parts as

$$L\{u(x)\} = u(x), \text{ and } N\{u(x)\} = -x^{1/2} - \frac{3\pi}{8}x^2 + \int_0^x \frac{u^3(t)}{\sqrt{x-t}}dt, \quad (3.21)$$

In this example, for the initial guess as given in Eq. (2.4), it is appropriate to take $m = 2$. Therefore, one gets

$$U(x) = \sum_{n=0}^{\infty} a_n x^{n/2}, \quad (3.22)$$

It follows that the homotopy equation, as given in Eq. (2.2), may be written as

$$H(u,p) = u(x) - \sum_{n=0}^{\infty} a_n x^{n/2} + \rho\left[\sum_{n=0}^{\infty} a_n x^{n/2} - x^{1/2} - \frac{3\pi}{8}x^2 + \int_0^x \frac{u^3(t)}{\sqrt{x-t}}dt\right] = 0 \quad (3.23)$$

As given before, one gets

$$\rho^0 : u_0(x) = \sum_{n=0}^{\infty} a_n x^{n/2} \tag{3.24}$$

and

$$\rho^1 : u_1(x) = -\sum_{n=0}^{\infty} a_n x^{n/2} + x^{1/2} + \frac{3\pi}{8} x^2 - \int_0^x \frac{u_0^3(t)}{\sqrt{x-t}} dt \tag{3.25}$$

The cancellation of $u_1(x)$ yields

$$-\sum_{n=0}^{\infty} a_n x^{n/2} + x^{1/2} + \frac{3\pi}{8} x^2 - \int_0^x \frac{u_0^3(t)}{\sqrt{x-t}} dt = 0 \tag{3.26}$$

As shown before, one finds

$$a_0 = 1, \text{ and all other } a's = 0. \tag{3.27}$$

Therefore, the exact solution of Eq. (3.20) may be written as

$$u(x) = \sqrt{x}. \tag{3.28}$$

In the following, we consider an integro–differential equation which describes a charged particle in a motion of certain configurations of oscillating magnetic fields. The problems are successfully solved on using PSHPM to get their exact solution. These examples were solved by Dehghan and Shakeri [15] by using the traditional HPM. Unfortunatly, they got only approximate solutions .

Now, the integro-differential equation as given by Machado, and Hida [16] is given as:

$$\frac{d^2u(t)}{dx^2} = -a(t)u(t) + b(t)\int_0^x \cos(w_p s)u(s)ds + g(t) \tag{3.29}$$

where $a(t), b(t)$ and $g(t)$ are periodic functions of time.

Eq. (3.29) may be easily found in the motion of a charged particle for some field configurations. The initial-value conditions may be considered as

$$u(0) = \alpha, \text{ and } u'(0) = \beta. \tag{3.30}$$

Example 3.4

Through the following example, we domenstrate the description of the above problem to show the efficiency of the mentioned method for solving Eq. (3.29). Their characteristics and initial conditions may be listed in the following equations:

$$w_p = 2, a(t) = \cos t, b(t) = \sin(t/2),$$

$$g(t) = \cos t - t\sin t + (t\sin t + \cos t)\cos t - \sin(t/2)\left(\frac{2}{9}\sin(3t) - \frac{t}{6}\cos(3t) + \frac{t}{2}\cos t\right), \tag{3.31}$$

and

$$\alpha = 1, \text{ and } \beta = 0.$$

For convenience, one may choose the linear and nonlinear parts as

$$L\{u(t)\} = u''(t), \tag{3.32}$$

and

$$\begin{aligned} N\{u(t)\} = {} & \cos t\, u(t) - \sin(t/2)\int_0^x \cos(2s)u(s)ds - \cos t + t\sin t - (t\sin t + \cos t) - \\ & \sin(t/2)\left(\frac{2}{9}\sin(3t) - \frac{t}{6}\cos(3t) + \frac{t}{2}\cos t\right)\cos t, \end{aligned} \tag{3.33}$$

In this example, for the initial guess as given in Eq. (2.4), it is appropriate to take $m = 1$. Therefore, one gets

$$U(t) = \sum_{n=0}^{\infty} a_n t^n, \tag{3.34}$$

It follows that the homotopy equation, as given in Eq. (2.2), is written as

$$H(u, p) = u''(t) - \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + \rho\left[\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + N\{u(t)\}\right] = 0 \tag{3.35}$$

As given before, one gets

$$\rho^0 : u_0''(t) = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} \tag{3.36}$$

and

$$\rho^1 : u_1''(t) = -\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} - N(u(t)), \tag{3.37}$$

In accordance with the intial conditions, the special solution of Eq. (3.36) may be written as

$$u_0(t) = \sum_{n=2}^{\infty} a_n t^n + 1, \tag{3.38}$$

Cancelling $u_1(x)$, and then use similar arguments as given before, one finds to the closed form of the solution in the form.

$$u(t) = t\sin t + \cos t. \tag{3.39}$$

Example 3.5

Another case study due to the previous phenomenon , in which their functions are formulated as follows:

$$w_p = 1, a(t) = -\sin t, b(t) = \sin t,$$

$$g(t) = \frac{1}{9}e^{-t/3} - (e^{-t/3} + t)\sin t - \left(\frac{-3}{10}e^{-t/3}\cot + \frac{9}{10}e^{-t/3}\sin t + \cos t + t\sin t - \frac{7}{10}\right)\sin t, \tag{3.40}$$

and

$$\alpha = 1, \text{and } \beta = \frac{2}{3}.$$

For convenience, one may choose the linear and nonlinear parts as

$$L\{u(t)\} = u''(t), \tag{3.41}$$

and

$$\begin{aligned} N\{u(t)\} = u(t)\sin t - \sin t\int_0^x \cos(s)u(s)ds + \frac{1}{9}e^{-t/3} - (e^{-t/3} + t)\sin t - \\ \left(\frac{-3}{10}e^{-t/3}\cos t + \frac{9}{10}e^{-t/3}\sin t + \cos t + t\sin t - \frac{7}{10}\right)\sin t \end{aligned} \tag{3.42}$$

In this example, for the initial guess as given in Eq. (2.4), it is appropriate to take $m = 1$. Therefore, one gets

$$U(t) = \sum_{n=0}^{\infty} a_n t^n, \tag{3.43}$$

It follows that the homotopy equation, as given in Eq. (2.2), is written as

$$H(u, p) = u''(t) - \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + \rho\left[\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + N(u(t))\right] = 0 \tag{3.44}$$

As before, one gets

$$\rho^0 : u_0''(t) = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} \tag{3.45}$$

and

$$\rho^1 : u_1''(t) = -\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} - N(u(t)), \tag{3.46}$$

In accordance with the intial conditions, the special solution of Eq. (3.40) may be written as

$$u_0(t) = \sum_{n=2}^{\infty} a_n t^n + \frac{1}{3}t^2 + 1, \tag{3.47}$$

Integrate Eq. (3.45) twice with respect to t. Then cancelling $u_1(x)$, and use similar arguments as given before, one finds the closed form of the solution is in the form.

$$u(t) = e^{-t/3} + t. \tag{3.48}$$

As mentioned before, Dehghan and Shakeri [15] managed to solve the above example using HPM. They only reached for a numerical approximate solution.

In the next examples, a simple system of linear and nonlinear integral equations are considered to be solved by PSHPM is order to prove the accuracy and efficiency of the method in handling systems of Volterra integral equations.

Example 3.6

Consider the following system of linear integral equations as given by Biazara et al. [17]

$$f(x) = -x^2 - \frac{2}{3}x^4 + xg(x) + 2\int_0^x (xf(t) + g(t))dt, \tag{3.49}$$

and

$$g(x) = x - \frac{1}{4}x^2 + \frac{2}{3}x^3 + \frac{1}{2}x^4 - \frac{1}{2}(x + x^2)f(x) - \frac{1}{2}\int_0^x (f(t) - g(t))dt. \tag{3.50}$$

The procedure of finding the exact solution by PSHPM for a system of integral equations consists of a scheme similar to that we use in single integral equation but with slight modifications.

Let us begin by letting

$$L\{f(x)\} = f(x), \text{ and } N\{f(x)\} = x^2 + \frac{2}{3}x^4 - xg(x) - 2\int_0^x (xf(t) + g(t))dt, \tag{3.51}$$

and

$L\{g(x)\} = g(x)$, and

$$N\{g(x)\} = -x + \frac{1}{4}x^2 - \frac{2}{3}x^3 - \frac{1}{2}x^4 + \frac{1}{2}(x + x^2)f(x) + \frac{1}{2}\int_0^x (f(t) - g(t))dt, \tag{3.52}$$

In this example, for the initial guess as given in Eq. (2.4), it is appropriate to take $m=1$. Therefore, one gets

$$F(x)=\sum_{n=0}^{\infty}a_n x^n, \text{ and } G(x)=\sum_{n=0}^{\infty}b_n x^n \tag{3.53}$$

It follows that the homotopy equation, as given in Eq. (2.2), is written as

$$H(f,p)=f(x)-\sum_{n=0}^{\infty}a_n x^n+\rho\left[\sum_{n=0}^{\infty}a_n x^n+x^2+\frac{2}{3}x^4-xg(x)-2\int_0^x(xf(t)+g(t))dt\right]=0 \tag{3.54}$$

$$H(g,p)=g(x)-\sum_{n=0}^{\infty}b_n x^n$$

$$+\rho\left[\sum_{n=0}^{\infty}b_n x^n+x-\frac{1}{4}x^2+\frac{2}{3}x^3+\frac{1}{2}x^4-\frac{1}{2}(x+x^2)f(x)-\frac{1}{2}\int_0^x(f(t)-g(t))dt\right] \tag{3.55}$$

as $f(x,\rho)=f_0(x)+\rho f_1(x)+\ldots$ and $g(x,\rho)=g_0(x)+\rho g_1(x)+\ldots$. Using similar arguments as before, one finds

$$\rho^0: f_0(x)=\sum_{n=0}^{\infty}a_n x^n \quad \text{and } g_0(x)=\sum_{n=0}^{\infty}a_n x^n \ , \tag{3.56}$$

and

$$\rho^1: f_1(x)=\sum_{n=0}^{\infty}a_n x^n-N\{f(x)\}, \text{ and } g_1(x)=\sum_{n=0}^{\infty}b_n x^n-N\{g(x)\}. \tag{3.57}$$

Equating $f_1(x)=0$ and $g_1(x)=0$, after comparing coefficients of like powers of ρ, one gets simultaneous equations of the undetermined coefficients a_n 's and b_n 's.

Inserting these coefficents on Eq. (3.56), one tends to the closed form of the solutions which are given by

$$f(x)=x^2 \text{ and } g(x)=x. \tag{3.58}$$

Biazara et al. [17] solved the above system on using the Adomian decomposition method (ADM). They only reached for a numerical approximate solution.

Example 3.7

Consider the following system of nonlinear integral equations as given by Biazara et al. [17]

$$f(x)=-\frac{1}{2}x^5-\frac{2}{3}x^4+x^3+x^2-g^3(x)+2x\int_0^x(f(t)+g^3(t))dt, \tag{3.59}$$

$$g(x) = x + \frac{1}{10}x^2 - x^6 - \frac{1}{35}x^7 + f^3(x) + \frac{1}{5}\int_0^x (f^3(t) - g(t))dt\,. \tag{3.60}$$

The procedure of finding the exact solution by PSHPM for a system of integral equations consists of a scheme similar to that we use in single integral equation but with slight modifications, begin by letting

$$L\{f(x)\} = f(x), \text{ and } N\{f(x)\} = \frac{1}{2}x^5 + \frac{2}{3}x^4 - x^3 - x^2 + g^3(x) - 2x\int_0^x (f(t) + g^3(t))dt, \tag{3.61}$$

$$L\{g(x)\} = g(x), \text{ and } N\{g(x)\} = -x - \frac{1}{10}x^2 + x^6 + \frac{1}{35}x^7 - f^3(x) - \frac{1}{5}\int_0^x (f^3(t) - g(t))dt, \tag{3.62}$$

in this example take two initial guesses in Eq. (2.4) with $m = 1$ in the form

$$F(x) = \sum_{n=0}^{\infty} a_n x^n, \text{ and } G(x) = \sum_{n=0}^{\infty} b_n x^n \tag{3.63}$$

construct the homotopy equation as in Eq. (2.2)

$$H(f, p) =$$

$$f(x) - \sum_{n=0}^{\infty} a_n x^n + \rho\left[\sum_{n=0}^{\infty} a_n x^n + \frac{1}{2}x^5 + \frac{2}{3}x^4 - x^3 - x^2 + g^3(x) - 2x\int_0^x (f(t) + g^3(t))dt\right] = 0, \tag{3.64}$$

and

$$H(g, p) = g(x) - \sum_{n=0}^{\infty} b_n x^n + \rho\left[\sum_{n=0}^{\infty} b_n x^n - x - \frac{1}{10}x^2 + x^6 + \frac{1}{35}x^7 - f^3(x) - \frac{1}{5}\int_0^x (f^3(t) - g(t))dt\right]. \tag{3.65}$$

AS shown in the previous example, one gets

$$\rho^0 : f_0(x) = \sum_{n=0}^{\infty} a_n x^n \;\; and \;\; g_0(x) = \sum_{n=0}^{\infty} a_n x^n \;, \tag{3.66}$$

and

$$\rho^1 : f_1(x) = \sum_{n=0}^{\infty} a_n x^n - N\{f(x)\}, and \; g_1(x) = \sum_{n=0}^{\infty} b_n x^n - N\{g(x)\}.` \tag{3.67}$$

Equating $f_1(x) = 0$ and $g_1(x) = 0$ then after comparing coefficients of like powers of x to get the undetermined coefficients a_n's and b_n's. Substitute into Eq. (3.62) to get the closed form of the exact solutions as given by

$$f(x) = x^2 \text{ and } g(x) = x. \tag{3.68}$$

As given by Biazara et al. [17], they solved the same system using the Adomian decomposition method (ADM). They only reached for a numerical approximate solution, meanwhile, we get here the exact solution which proves that PSHPM is a powerful tool to solve system of Volterra integral equations.

4. CONCLUSION

The aim of this article is to provide a basic concept on obtaining exact solution on the class of all types of Volterra integral equations. Therefore, we have presented a modification to the homotopy perturbation method by combining it with a power series formula which has firstly formulated by Taylor. The basic idea in this approach is to choose a suitable trial function in a form of a power series as that earlier obtained by Taylor. The results reveal that He's HPM is very effective and convenient. The exact solution arises if the power series has summed and resulted in a closed form. The procedure is dependent mainly, on the cancellation of the first approximate solution. Consequently, all the higher solution-orders will be excluded. Therefore, the first-order solution will be supported and becomes an exact solution, provided that the given infinte series is convergent to a compact form. The newly technique is named as (PSHPM). It is used to solve all types of Volterra integral (VIE) as well as integro-differential (VIDE) equations. The technique is tested over several application problems and managed to get the exact solution for every single problem. This technique proved to be reliable in solving VIE and VIDE with proved accuracy and efficiency for a lot of daily life applications that connected to integral equations. All problems, in this paper, are solved with the aid of Mathematica software.

REFERENCES

[1] O. Diekmann, Thresholds and traveling for the geographical spread of infection, Journal of Mathematical Biology, **6**,109–130 (1978).

[2] J.P. Kauthen, Continuous time collocation method for Volterra–Fredholm integral equations, Numerical Mathematics, **56**, 409–424 (1989).

[3] B.G. Pachpatta, On mixed Volterra–Fredholm type integral equations, Indian Journal of Pure and Applied Mathematics, **17**, 488–496 (1986).

[4] A.M. Wazwaz., Linear and Non-linear Integral equations (methods and applications), Saint Xavier University, Chicago (2011).

[5] JH. He, Homotopy perturbation technique. Computer Methods in Applied Mechanics and Engineering, **62**, 178-257 (1999).

[6] JH. He, A coupling method of homotopy technique and perturbation technique for nonlinear problems. International Journal of Non-Linear Mechanics, **35**(1), 37–43 (2000).

[7] JH. He, Application of homotopy perturbation method to nonlinear wave equations. Chaos, Solitons and Fractals, **26**, 695–700 (2005).

[8] Y. O. El-Dib, and G. M. Moatimid, On the coupling of the homotopy perturbation and Frobenius method for exact solutions of singular nonlinear differential equations, Accepted in Nonlinear Science Letters A (2018).

[9] S.A. Isaacson, and C.S. Peskin, Incorporating diffusion in complex geometries into stochastic chemical kinetics simulations, SIAM Journal on Scientific

Computing, **27** (1), 47–74 (2006).

[10] S.A. Isaacson, The reaction–diffusion master equation as an asymptotic approximation of diffusion to a small target, SIAM Journal on Applied Mathematics, **70**(1), 77–111 (2009).

[11] A. S. Isaacson, and R. M. Kirby, Numerical solution of linear Volterra integral equations of the second kind with sharp gradients. Journal of Computational and Applied Mathematics, **235**, 4283–4301 (2011).

[12] N. H. Abel, Auflosung einer mechanischen aufgabe, Journal für Reine Angewandte Mathematik, **1**, 153–157, (1826).

[13] R. Estrada and R. Kanwal, Singular Integral Equations, Birkhauser, Berlin, (2000).

[14] M.A. AL-Jawary, and A. M. Shehan, Exact solutions for weakly singular Volterra integral equations by using an efficient iterative method, International Journal of Science and Research, **12**, 67-76 (2015).

[15] M. Dehghan and F. Shakeri, Solution of an integro-differential equation arising in oscillating magnetic fields using He's homotopy perturbation method, Progress In Electromagnetics Research, **78**, 361-376 (2008).

[16] J.M. Machado, and M.T. Suc Hida, Solutions for a class of integro–differential equations with time periodic coefficients, Applied Mathematics E-Notes, **2**, 66–71 (2002).

[17] J. Biazara, E. Babolianb, and R.Islamc, Solution of a system of Volterra integral equations of the first kind by Adomian method, Applied Mathematics and Computation, **139**, 249-258 (2003).