Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 16, Number 2 (2020), pp. 271--284

http://www.ripublication.com/gjpam.htm

Not Necessarily Semi Normalized Greedy Bases

Nashat Faried [1] and Fatma Mahmoud Helmi[2]

[1]*Professor of pure mathematics in Mathematics Department,
Faculty of Science, Ain Shams University, Egypt.*

[2]*Demonstrator in Basic engineering science Department
Benha Faculty of Engineering, Benha University, Egypt.*

Abstract

In 1999 Konyagin and Temlyakov introduced the greedy algorithm and applied their method to seminormalized basis in Banach spaces. We make new approach for studying greedy approximations by getting rid of seminormalized condition. For this goal we do some modifications of basic definitions such as democracy, greediness and partial greediness. This method allows us to study unbounded basis as well. We reprove some known theorems using the new definitions. We proved that a basis is greedy if and only if it is unconditional and democratic, almost greedy basis is equivalent to being quasi greedy and democratic and partially greedy is being quasi greedy and conservative. We give an example to show the differences between the two methods.

1. **INTRODUCTION AND BACKGROUND**:

For any semi normalized basis $(e_n)_{n=1}^{\infty}$ of a Banach space X (i.e., $1/c \leq \|e_n\| \leq c$ for some c)with their biorthogonal functionals $(e_n^*)_{n=1}^{\infty}$. And for each $= 0, 1, 2, \ldots$, we let Σ_m represent the collection of elements of X which can be described as a linear combination of m elements of $(e_n)_{n=1}^{\infty}$:

$$\Sigma_m = \left\{ y = \sum_{j\in B} \alpha_j\, e_j \,:\, B \subset \mathbb{N}, |B| = m, \alpha_j \in \mathbb{R} \right\}.$$

For $x \in X$, its best m −term approximation error (in respect of the given basis) as

$$\sigma_m(x) = \inf_{y\in\Sigma m} \|x - y\|\,.$$

The greedy algorithm, $(G_m)_{m=1}^{\infty}$, a sequence of maps from X to X whenever, for each x, $G_m(x)$ is obtained by picking up the largest m cofficients of x.

To be explicit, for $x \in X$ express

$$G_m(x) = \sum_{j\in B} e_j^* (x)ej,$$

Where the set $B \subset \mathbb{N}$ is chosen in a way that $|B| = m$ and $|e_j^* (x)| \geq |e_k^*(x)|$ whenever $j \in B$ and $k \notin B$.

Note that $S_m$is the partial sum operators

$$S_m = \sum_{j=1}^{m} e_j^*(x)e_j.$$

Notice that R_m is called the remainder operators $R_m = I - S_m$. Where for each $x \in X$ the map $\rho: \mathbb{N} \to \mathbb{N}$ is the greedy ordering such that $\rho(\mathbb{N}) \subset \{j: e_j^*(x) \neq 0\}$ and if $j < k$, then $|e_j^*(x)| > |e_k^*(x)|$ or $|e_j^*(x)| = |e_k^*(x)|$ and $\rho(j) < \rho(k)$.

Definition1.1. [1]: A basis $(e_n)$is greedy if there is a constant C where $C \geq 1$ such that for any $x \in X$ and $m \in \mathbb{N}$ we have

$$\|x - G_m(x)\| \leq C\sigma_m(x).$$

The smallest constant C will be called the greedy constant of (e_n). Where mth greedy remainder $H_m(x) = x - G_m(x)$.

Definition1.2 [1]: A basis (e_n) is named democratic if there exists a constant $D \geq 1$ such that for any two finite subsets A, B of $\mathbb{N}$ with $|A| = |B|$ we possess

$$\left\|\sum_{n\in A} e_n\right\| \leq D \left\|\sum_{n\in B} e_n\right\|.$$

They also exhibited in [1] that a basis is greedy if and only if being unconditional and democratic. Remember that a basis (e_n) is unconditional if for any $x \in X$ the series $\sum_{n=1}^{\infty} e_n^*(x)\, e_n$converges in norm to x irrespective of its terms ordering. The property of being unconditional is easily recognized to be equivalent to that being suppression-unconditional, in the sense that the natural projections onto any subsequence of the basis

$$P_A = \sum_{n\in A} e_n^*(x)e_n, \qquad A \subset \mathbb{N},$$

Are uniformly bounded, i.e. there exists a constant K in a way that for all $x = \sum_{n=1}^{\infty} e_n^*(x)\, e_n$ and all $A \subset \mathbb{N}$,

$$\left\|\sum_{n\in A} e_n^*(x)\, e_n\right\| \leq K \left\|\sum_{n=1}^{\infty} e_n^*(x)\, e_n\right\|.$$

The smallest K is the suppression unconditional constant of the basis.

An unconditional basis of c_0 and ℓ_p for $1 \leq p < \infty$ is the standard unit vector basis .

It is recognized that for every infinite dimensional Banach space with bases has a conditional basis (i.e., not unconditional). This is a result of Pelczynski and Singer [6]. A proof of this can be found in ([4] page 235).

An example of a basis which is conditional is the summing basis of c_0, $(f_n)_{n=1}^{\infty}$ defined as

$$f_n = e_1 + \cdots + e_n, \ n \in \mathbb{N}.$$

In ℓ_1, the sequence (x_n) defined by $x_1 = e_1$ and $x_n = e_{n-1} - e_n$ for $n > 1$ forms a conditional basis.

For an example of conditional bases for the other ℓ_p spaces, see Lindenstrauss and Tzafriri [5], also contains examples (in particular, they show ℓ_2 has a conditional basis in Proposition 2.b.11).

Konyagin and Temlyakov initiated in [1] the quasi greedy basis definition. A basis is known as quasi-greedy if for $x \in X$, $\lim_{n\to\infty} G_m(x) = x$ in other words, the series converges to x. Later on, Wojtaszczyk [3] proved precisely that the bases for which the greedy operators $(G_m)_{m=1}^{\infty}$ are uniformly bounded (despite the fact of being nonlinear on x), i.e., there is a constant $C \geq 1$ so that for all $x \in X$ and $m \in \mathbb{N}$,

$$\|G_m(x)\| \leq C\|x\|.$$

Quasi-greedy bases are not unconditional generally; in fact, most classical spaces include conditional quasi-greedy bases. Wojtaszczyk provided in [3] a general construction (enhanced in [8]) of quasi-greedy bases in some Banach spaces. Dilworth and Mitra illustrated in [7] that ℓ_1 has a quasi-greedy basis which is conditional. In any case, quasi-greedy bases retain some vestiges of unconditionality and, in particular, they are unconditional for constant coefficients (see [3]). Conversely, unconditional bases are quasi-greedy all the time. To be precise, if $(e_n)$is K-suppresion unconditional then (e_n) is K-quasi-greedy

The class of almost greedy bases was introduced in [2]. A basis (e_n) is called almost greedy if there is a constant C such that for any $x \in X$, $n \in \mathbb{N}$

$$\|x - G_n(x)\| \leq C \inf \left\{ \left\| x - \sum_{j \in A} e_j^*(x) e_j \right\| : |A| = n \right\}.$$

In this paper, first we redefine G_m's taking into account not only magnitude but also the norms of the elements of the basis. Giving an example using the new definitions, second, we show that we can extend characterization of quasi greedy, almost greedy and partially greedy for non-necessarily seminormalized basis.

2. GREEDY BASES WHICH ARE NOT SEMI NORMALIZED BASES :

We define $\left(\tilde{G}_m\right)_{m=1}^{\infty}$; a sequence of maps from X to X where, for every x, $\tilde{G}_m(x)$ is got by taking the largest m components of x. To be explicit, for $x \in X$ put

$$\tilde{G}_m(x) = \sum_{j\in B} e_j^*(x)ej,$$

Where the set $B \subset \mathbb{N}$ is chosen in a way that $|B| = m$ and $\left\|e_j^*(x)\ e_j\right\| \geq \|e_k^*(x)\ e_k\|$ whenever $j \in B$ and $k \notin B$. Where the mth greedy remainder

$$\widetilde{H}_m(x) = x - \tilde{G}_m(x).$$

For each $x \in X$ we define greedy ordering as the map $\rho: \mathbb{N} \to \mathbb{N}$ such that

$\rho(\mathbb{N}) \subset \left\{j: e_j^*(x) \neq 0\right\}$ and if $j < k$, then $\left\|e_j^*(x)e_j\right\| > \|e_k^*(x)\ e_k\|$ or

$\left\|e_j^*(x)\ e_j\right\| = \|e_k^*(x)\ e_k\|$ and $\rho(j) < \rho(k)$.

Definition 2.1: A basis $(e_n)$is greedy if there exists a constant $C \geq 1$ such that for such $x \in X$ and $m \in \mathbb{N}$ we possess

$$\left\|x - \tilde{G}_m(x)\right\| \leq C\sigma_m(x).$$

The smallest constant C will be known as the greedy constant of (e_n).

We give new definitions according to the new greedy approximation $\tilde{G}_m(x)$.

Definition 2.2: A basis (e_n) is called democratic if there is a constant $D \geq 1$ such that for any two finite subsets A, B of $\mathbb{N}$ with $|A| = |B|$ we have

$$\left\|\sum_{n\in A} \frac{e_n}{\|e_n\|}\right\| \leq D \left\|\sum_{n\in B} \frac{e_n}{\|e_n\|}\right\|.$$

We suggest the following example of a basis unbounded from above but not semi normalized.

Example 2.3:

Suppose the weighted sequence space $\ell_{(n)}^1$. We choose $x \in \ell_{(n)}^1$ such that

$$x = \sum_{n=1}^{100} \left(100 - \left\lfloor\frac{n-1}{2}\right\rfloor\right) e_n + \sum_{n=101}^{\infty} \frac{50}{n^3}\, e_n$$

Where $\|e_1\| = 1, \|e_2\| = 2, \ldots, \|e_i\| = i$ for $i = 1,2,3, \ldots$

$$x = 100e_1, 100e_2, 99\, e_3, 99e_4, \ldots .48e_{98}, 49e_{99}, 49e_{100}, \frac{50}{101}\, e_{101}, \ldots \ldots$$

Hence, $G_3(x) = 100\, e_1 + 100e_2 + 99e_3$ using semi normalized basis, but $\tilde{G}_3(x) = 48\, e_{98} + 49\, e_{99} + 49\, e_{100}$ by using greater component of x.

Theorem 2.4: A basis (e_n) that is not semi normalized is being greedy if and only if it is being unconditional and democratic.

Proof: Let us assume, first, that (e_n) is greedy with greedy constant C.

For any finite set $S \subset \mathbb{N}$ we denote P_s the projection

$$P_s(x) = \sum_{n \in s} e_n^*(x)\, e_n.$$

We will prove the unconditionality of $(e_n)$by showing that for every $x \in X$ and any finite set $S \subset \mathbb{N}$ we get

$$\|P_s(x)\| \leq (C+1)\|x\|. \quad (1)$$

We will fix a finite set $S \subset \mathbb{N}$ of cardinality $m,\ x \in X$ and a number

$$\alpha > \sup_{n \in S} \|e_n^*(x)\, e_n\|\ .$$

Consider the vector

$$y = x - P_s(x) + \alpha \sum_{n \in s} \frac{e_n}{\|e_n\|}.$$

Obviously $\sigma_m(y) \leq \|x\|$ and $\tilde{G}_m(y) = \alpha \sum_{n \in S} \frac{e_n}{\|e_n\|}$. Thus, by assuming that (e_n) is greedy, we get

$$\|x - P_S(x)\| = \|y - \tilde{G}_m(y)\| \leq C\sigma_m(y) \leq C\|x\|.$$

That implies (1).

To show that (e_n) is democratic, let us pick two finite sets P, Q of the same cardinality m.Take a third subset S such that $|S| = m$ and$P \cap S = \emptyset = Q \cap S$. Fix any $\epsilon > 0$ and consider

$$x = (1+\epsilon) \sum_{n \in p} \frac{e_n}{\|e_n\|} + \sum_{n \in s} \frac{e_n}{\|e_n\|}\ .$$

We have

$$\sigma_m(x) \leq (1+\epsilon) \left\| \sum_{n \in P} \frac{e_n}{\|e_n\|} \right\|.$$

and

$$\left\| \sum_{n \in S} \frac{e_n}{\|e_n\|} \right\| = \|x - \tilde{G}_m(x)\| \leq C\sigma_m(x)$$

$$\leq C(1+\epsilon) \left\| \sum_{n \in P} \frac{e_n}{\|e_n\|} \right\|. \quad (2)$$

Analogously we have

$$\left\|\sum_{n\in Q}\frac{e_n}{\|e_n\|}\right\| \leq C(1+\epsilon)\left\|\sum_{n\in S}\frac{e_n}{\|e_n\|}\right\|. \qquad (3)$$

Linking (2), (3) and considering that ϵ is arbitrarily small, we obtain

$$\left\|\sum_{n\in Q}\frac{e_n}{\|e_n\|}\right\| \leq C^2\left\|\sum_{n\in P}\frac{e_n}{\|e_n\|}\right\|.$$

Presently we will prove the opposite section of the theorem. Assume that $(e_n)$is K −unconditional and D −democratic. Fix $x \in X$ and $m = 1, 2, \dots$.

Given any $\epsilon > 0$ we choose

$$p_m = \sum_{n\in B}\alpha_n e_n \in \Sigma_m .$$

such that $\|x - p_m\| \leq \sigma_m(x) + \epsilon$.

Clearly, we can write

$$\tilde{G}_m(x) = \sum_{n\in S} e_n^*(x)e_n = P_S(x),$$

for some $S \subset N$ with $|S| = m$. Then,

$$\|x - G_m(x)\| = \|x - P_Sx + P_Bx - P_Bx\| = \|x - P_Bx + P_{B\setminus S}x - P_{S\setminus B}x\|. \qquad (4)$$

The assumption that $(e_n)$is K −unconditional implies that

$$\begin{aligned}\|x - P_Bx - P_{S\setminus B}x\| &= \|x - P_{B\cup S}x\|\\ &= \|P_{N\setminus(B\cup S)}(x - p_m)\|\\ &\leq K\|x - p_m\| \qquad (5)\\ &\leq K(\sigma_m(x) + \epsilon),\end{aligned}$$

and that

$$\|P_{S\setminus B}x\| \leq K\|x - p_m\| \leq K(\sigma_m(x) + \epsilon). \qquad (6)$$

From the definition of $\tilde{G}_m$ it is immediate to see that

$$\gamma := \min_{j\in S\setminus B}\|e_j^*(x)\, e_j\| \geq \max_{j\in B\setminus S}\|e_j^*(x)\, e_j\| := \beta,$$

So, from the unconditionality of (e_n), we obtain

$$\gamma\left\|\sum_{j\in S\setminus B}\frac{e_j}{\|e_j\|}\right\| \leq K\|P_{S\setminus B}x\| \qquad (7)$$

And

$$\|P_{B\setminus S}x\| \le K\beta \left\| \sum_{j\in B\setminus S} \frac{e_j}{\|e_j\|} \right\|. \qquad (8)$$

Since $|B \setminus S| = |S \setminus B|$, using the inequality of D −democracy of the basis

$$\left\| \sum_{j\in S\setminus B} \frac{e_j}{\|e_j\|} \right\| \le D \left\| \sum_{j\in B\setminus S} \frac{e_j}{\|e_j\|} \right\|,$$

(7) and (8) we get

$$\|P_{B\setminus S}x\| \le K^2 D \, \|P_{S\setminus B}x\|. \qquad (9)$$

From (4), (6), (9) and considering that ϵ was arbitrarily small,

$$\begin{aligned}\|x - \tilde{G}_m(x)\| &\le \|x - P_B x - P_{S\setminus B}x\| + \|P_{B\setminus S}\, x\| \\ &\le K(\sigma_m(x) + \epsilon) + K^2 D \, \|P_{S\setminus B}x\| \\ &\le K(\sigma_m(x) + \epsilon) + K^2 D \, K(\sigma_m(x) + \epsilon). \\ &\le (K + K^3 D)(\sigma_m(x) + \epsilon).\end{aligned}$$

3. NEW QUASI-GREEDY AND NEW ALMOST-GREEDY CHARACTERIZATION.

We give characterizations of quasi greedy, almost and partially greedy bases with the new definitions and approaches.

Definition3.1: quasi greedy

A basis is called quasi-greedy if there exists a constant C where

$\|\tilde{G}_m(x)\| \le C\|x\|$, for all $x \in X$ and $n \in \mathbb{N}$ (10)

Definition3.2: almost greedy

A basis is said to be almost greedy if there is a constant C so that for all $x \in X$,$n \in \mathbb{N}$

$$\|x - \tilde{G}_m(x)\| \le C\tilde{\sigma}_m(x).$$

Where

$$\tilde{\sigma}_m(x) = \inf \left\{ \left\| x - \sum_{j\in A} e_j^*(x) e_j \right\| : |A| = m \right\}.$$

Notice that

$\sigma_m(x) \le \tilde{\sigma}_m(x) \le \|R_m(x)\| \to 0 \;\; as \;\; m \to \infty.$

It was shown in [2] that (e_n) is almost greedy if and only if (e_n) is quasi-greedy and democratic.

Definition3.3: A system $(x_n, x_n^*)_{n\in\mathbb{N}}$ is named unconditional for constant coefficients if there are constants C and $c > 0$ such that for any finite subset $A \subset \mathbb{N}$ and each sequence of signs $(\varepsilon_n)_{n\in A} = \pm 1$ we have

$$c\left\|\sum_{n\in A}\frac{x_n}{\|x_n\|}\right\| \leq \left\|\sum_{n\in A}\varepsilon_n\frac{x_n}{\|x_n\|}\right\| \leq C\left\|\sum_{n\in A}\frac{x_n}{\|x_n\|}\right\|.$$

The latter lemma states that every quasi-greedy basis is unconditional for constant coefficient.

Lemma 3.3: Suppose $(e_n)_{n\in\mathbb{N}}$ has a quasi – greedy constant k. Assume A is a finite subset of $\mathbb{N}$. Then, for every selection of signs $\varepsilon_j = \pm 1$, we have

$$\frac{1}{2K}\left\|\sum_{j\in A}\frac{e_j}{\|e_j\|}\right\| \leq \left\|\sum_{j\in A}\varepsilon_j\frac{e_j}{\|e_j\|}\right\| \leq 2K\left\|\sum_{j\in A}\frac{e_j}{\|e_j\|}\right\|. \quad (11)$$

Therefore, for every real numbers $(a_j)_{j\in A}$,

$$\left\|\sum_{j\in A}a_j\frac{e_j}{\|e_j\|}\right\| \leq 2K\max|a_j|\left\|\sum_{j\in A}\frac{e_j}{\|e_j\|}\right\|. \quad (12)$$

Proof: First for any subset B such that $B \subset A$, $|B| = m$ and $\varepsilon > 0$, suppose the vector x described as

$$x = \sum_{j\in B}(1+\varepsilon)\frac{e_j}{\|e_j\|} + \sum_{j\in A\setminus B}\frac{e_j}{\|e_j\|}.$$

$$\tilde{G}_m(x) = \sum_{j\in B}(1+\varepsilon)\frac{e_j}{\|e_j\|}$$

Then, from (10) we have

$$\left\|\sum_{j\in B}(1+\varepsilon)\frac{e_j}{\|e_j\|}\right\| \leq K\left\|\sum_{j\in B}(1+\varepsilon)\frac{e_j}{\|e_j\|} + \sum_{j\in A\setminus B}\frac{e_j}{\|e_j\|}\right\| =$$

$$\leq K\left(\left\|\sum_{j\in A}\frac{e_j}{\|e_j\|}\right\| + \varepsilon\left\|\sum_{j\in B}\frac{e_j}{\|e_j\|}\right\|\right).$$

Taking $\varepsilon \to 0$, we have $\left\|\sum_{j\in B}\frac{e_j}{\|e_j\|}\right\| \leq K\left\|\sum_{j\in A}\frac{e_j}{\|e_j\|}\right\|$.

Hence, for any choice of signs $\varepsilon_j = \pm 1$, $A^+, A^- \subset A$ we have

$$\left\|\sum_{j\in A} \varepsilon_j \frac{e_j}{\|e_j\|}\right\| = \left\|\sum_{j\in A^+} \frac{e_j}{\|e_j\|}\right\| + \left\|\sum_{j\in A^-} \frac{e_j}{\|e_j\|}\right\|$$

$$\left\|\sum_{j\in A} \varepsilon_j \frac{e_j}{\|e_j\|}\right\| \leq 2K \left\|\sum_{j\in A} \frac{e_j}{\|e_j\|}\right\|.$$

This supplies the right hand side of the inequality (11) and its left-hand side is similar.

Lemma 3.4: Suppose $(e_n)_{n\in\mathbb{N}}$ has quasi –greedy constant K. Suppose $x \in X$ has greedy ordering ρ of x and every $m \in \mathbb{N}$. Then

$$\left|e^*_{\rho(m)}(x)\right| \left\|\sum_{j=1}^{m} e_{\rho(j)}\right\| \leq 4K^2\|x\| \qquad (13)$$

And hence, if A is any subset of N and $\left(a_j\right)_{j\in A}$ are any real numbers,

$$\min\left|a_j\right|_{j\in A} \left\|\sum_{j\in A} e_j\right\| \leq 4K^2 \left\|\sum_{j\in A} a_j e_j\right\| \qquad (14)$$

Proof: we prove (13), and then (14) is immediate. Let $a_j = e^*_{\rho(j)}(x)$ and $b_j = \frac{1}{\left|e^*_{\rho(j)}(x)\right|}$. let $\varepsilon_j = sgn\, a_j$ such that $b_j\left(G_j(x) - G_{j-1}(x)\right) = \varepsilon_j\, e_{\rho(j)}$ and put $b_0 = b_{m+1} = 0$. Then by (11)

$$\left\|\sum_{j=1}^{m} e_{\rho(j)}\right\| \leq 2K \left\|\sum_{j=1}^{m} \varepsilon_j\, e_{\rho(j)}\right\|$$

$$\leq 2K \left\|\sum_{j=1}^{m} b_j\left(G_j(x) - G_{j-1}(x)\right)\right\|$$

$$= 2K \left\|\sum_{j=0}^{m} \left(b_j - b_j + 1\right)\left(G_j(x)\right)\right\|$$

$$\leq 2K \sum_{j=0}^{m} \left|b_j - b_j + 1\right| \left\|G_j(x)\right\|$$

$$\leq 2K^2 \sum_{j=0}^{m} \left|b_j - b_j + 1\right| \|x\| = 4K^2 b_m \|x\|.$$

We redefine the fundamental function $\varphi(n)$ of a basis (e_n) which is not semi normalized by

$$\varphi(n) = \sup_{|A|\le n}\left\|\sum_{K\in A}\frac{e_K}{\|e_K\|}\right\|$$

It is obvious that (e_n) is democratic with constant Δ if and only if

$$\Delta^{-1}\varphi(|A|) \le \sum_{k\in A}\frac{e_K}{\|e_K\|} \le \varphi(|A|),\ \ |A| < \infty \quad (15)$$

Lemma3.5: Let (e_n) be a quasi-greedy basis that is democratic. With the quasi-greedy constant K and the democratic constant Δ. Then, for $x \in X$, if ρ is the quasi greedy ordering

$$\left|e^*_{\rho(m)}(x)\right| \le \frac{4K^2\Delta}{\varphi(m)}\ \|x\| \qquad (16)$$

$$\sup|e^*_K(H_m x)|_{K\in\mathbb{N}} \le \frac{4K^2\Delta}{\varphi(m+1)}\ \|x\| \qquad (17)$$

Proof: it follows from (15)

Theorem 3.6: Let (e_n) be a not semi normalized basis of a Banach space. The following characterizations with respect to our new definitions are equivalent:

1. (e_n) is almost greedy.
2. (e_n) is quasi-greedy and democratic.
3. $\lambda > 1$ there is a constant $C = C_\lambda$ such that

$$\left\|H_{[\lambda m]}(x)\right\| \le C_\lambda\, \sigma_m(x)$$

Proof: We start by showing (1) implies (2). Suppose (e_n) be almost greedy. It is immediate that (e_n) is quasi-greedy. We prove that it is democratic. Now suppose $|A| \le |B|$. Let $\delta > 0$ and define

$$x = \sum_{j\in A}\frac{e_j}{\|e_j\|} + \sum_{j\in B\backslash A}(1+\delta)\frac{e_j}{\|e_j\|}.$$

Then, if $|B\backslash A| = r$ we have

$$\widetilde{H}_r(x) = \sum_{j\in A}\frac{e_j}{\|e_j\|}$$

However,

$$\tilde{\sigma}_r(x) \le \left\|\sum_{j\in B} e^*_j(x)e_j\right\| \le \left\|\sum_{j\in B}\frac{e_j}{\|e_j\|}\right\| + \delta\left\|\sum_{j\in B\backslash A}\frac{e_j}{\|e_j\|}\right\|.$$

Letting $\delta \to 0$, from almost greedy definition $\|\widetilde{H}_r(x)\| \le C\, \tilde{\sigma}_r(x)$ then (e_n) is democratic.

Next we show that (2) implies (1). Let (e_n) is quasi and democratic then (e_n) is almost greedy suppose $x \in X$ and $m \in \mathbb{N}$. Let

$$\tilde{G}_m(x) = \sum_{j \in A} e_j^*(x) e_j$$

Where $|A| = m$. Suppose $|B| = r \leq m$. Then

$$x - \tilde{G}_m(x) = x - \sum_{j \in A} e_j^*(x) e_j + \sum_{j \in B} e_j^*(x) e_j - \sum_{j \in B} e_j^*(x) e_j$$

$$\widetilde{H}_m(x) = \left(x - \sum_{j \in B} e_j^*(x) e_j \right) + \sum_{j \in B \setminus A} e_j^*(x) e_j - \sum_{j \in A \setminus B} e_j^*(x) e_j$$

Then $|B \setminus A| \leq s := |A \setminus B|$. Thus

$\left\| \sum_{j \in B \setminus A} e_j^*(x) e_j \right\| \leq 2K \left(\max \left| e_j^*(x) \right| \right)_{j \in B \setminus A} \varphi(s)$ by (12)

$\leq 2K \left(\min \left| e_j^*(x) \right| \right)_{j \in A \setminus B} \varphi(s)$

From lemma 3.5 eq. (16) we get

$$\begin{aligned}
&\leq 8K^3 \Delta \left\| \sum_{j \in A \setminus B} e_j^*(x) e_j \right\| \\
&= 8K^3 \Delta \left\| \tilde{G}_s(x) \right\| \\
&= 8K^3 \Delta \left\| \tilde{G}_s \left(x - \sum_{j \in B} e_j^*(x) e_j \right) \right\|
\end{aligned}$$

From quasi greedy definition (10)

$$\leq 8K^4 \Delta \left\| x - \sum_{j \in B} e_j^*(x) e_j \right\|$$

We also have

$$\left\| \sum_{j \in A \setminus B} e_j^*(x) e_j \right\| = \left\| \tilde{G}_s(x - \sum_{j \in B} e_j^*(x) e_j) \right\|$$

Thus it follows that

$$\left\| \widetilde{H}_m(x) \right\| \leq (8K^4 \Delta + K + 1) \left\| x - \sum_{j \in B} e_j^*(x) e_j \right\|$$

And so, optimizing over B with $|B| \leq m$,

$\|\widetilde{H}_m(x)\| \leq (8\, K^4\, \Delta + K + 1)\tilde{\sigma}_m(x)$.

For the proof of (3) (see [3] thoerem3.3).

If A,B are subsets of $\mathbb{N}$ we use the notation $A < B$ to mean that $m \in A$, $n \in B$ imples $m < n$.

The basis (e_n) is conservative if there is a constant Γ such that

$$\left\|\sum_{k\in A} \frac{e_k}{\|e_k\|}\right\| \leq \left\|\sum_{k\in B} \frac{e_k}{\|e_k\|}\right\|$$

If $|A| \leq |B|$ and $A < B$

Definition3.7:

A basis (e_n) is partially greedy if there is a constant C such that for any $x \in X$,$m \in \mathbb{N}$

$\|\widetilde{H}_m(x)\| \leq C\|R_m x\|$ (18)

Theorem3.8:

A basis (e_n) is partially greedy if and only if it is quasi- greedy and conservative.

Proof: clearly partially greedy basis is also quasi-greedy.

First, we prove if the basis is partially greedy then it is conservative.

Suppose $(e_n)$is partially greedy with constant C and $A < B$ with $|A| = |B| = m$. Let $r = \max A$. Let $D = [1, r]\backslash A$ and then, for $\delta > 0$, let

$$x = \sum_{k\in A} \frac{e_k}{\|e_k\|} + (1+\delta) \sum_{k\in B\cup D} \frac{e_k}{\|e_k\|}.$$

Then

$$\|\widetilde{H}_r(x)\| = \left\|\sum_{k\in A} \frac{e_k}{\|e_k\|}\right\|.$$

And

$$\|R_r(x)\| = \left\|(1+\delta) \sum_{k\in B} \frac{e_k}{\|e_k\|}\right\|,$$

From (18), so that taking $\delta \to 0$ gives conservative with $\Gamma = C$.

Conversely, suppose$(e_n)$is quasi greedy with constant $K$and conservative with constant Γ. Suppose $x \in X$ and $m \in \mathbb{N}$. Let ρ be the greedy ordering for x. Then, let $D = \{\rho(j): j \leq m,\ \rho(j) \leq m\}$ and $B = \{\rho(j): j \leq m,\ \rho(j) > m\}$. Let $A = [1, r]\backslash D$.Then $|A| = |B| = r$, say, and $A < B$. Now

$$\left\|\sum_{k\in B} e_k^*(x)e_k\right\| = \left\|\tilde{G}_r(R_m x)\right\| \leq K\, \|R_m x\|. \quad (19)$$

By (12)

$$\left\|\sum_{k\in A} e_k^*(x)e_k\right\| \leq 2K\, \left(\max_{k\in A}|e_k^*(x)|\right)\left\|\sum_{k\in A} e_k\right\|$$

$$\leq 2K\, \left(\min_{k\in B}|e_k^*(x)|\right)\left\|\sum_{k\in B} e_k\right\|$$

$$\leq 8K^3\Gamma\left\|\sum_{k\in B} e_k^*(x)e_k\right\|$$

Also by (19)

$$\leq 8\, K^4\Gamma\|R_m x\|.$$

Combining gives

$$\left\|\widetilde{H}_m x\right\| \leq \|R_m x\| + \left\|\sum_{k\in A} e_k^*(x)e_k\right\| + \left\|\sum_{k\in B} e_k^*(x)e_k\right\|$$

$\leq (8\, K^4\Gamma + K + 1)\|R_m x\|$,

This completes the proof.

REFERENCES

[1] S. V. Konyagin and V. N. Temlyakov, (1999), "A remark on greedy approximation in Banach spaces," East J. Approx. 5, no. 3, 365–379.

[2] S. J. Dilworth, N. J. Kalton, D. Kutzarova, and V. N. Temlyakov, (2003), "The thresholding greedy algorithm, greedy bases, and duality," Constr. Approx. 19, no. 4, 575–597.

[3] P. Wojtaszczyk, (2000), "Greedy algorithm for general biorthogonal systems," J. Approx. Theory 107, no. 2, 293–314.

[4] F. Albiac and N. J. Kalton, (2006), "Topics in Banach space theory," Graduate Texts in Mathematics, vol. 233, Springer, New York.

[5] Lindenstrauss, J. Tzafriri, L., (1977), "Classical Banach Spaces I sequence spaces," springer.

[6] A. Pelczynski and I. Singer and I. Singer, (1964/1965), "On non-equivalent bases and conditional bases in Banach spaces," Studia Math. 25, 5–25.

[7] S. J. Dilworth and D. Mitra, (2001), "A conditional quasi-greedy basis of l1," Studia Math. 144, no. 1, 95–100.

[8] G. Garrigós and P. Wojtaszczyk, (2014), "Conditional quasi-greedy bases in Hilbert and Banach spaces", Indiana Univ. Math. J. 63 no. 4, 1017–1036.