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The Quenching Behavior of a Semilinear Parabolic Equation with a Singular Boundary Outflux *

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Abstract

In this paper, we consider the quenching behavior of the solution of a semilinear parabolic equation with a singular boundary outflux. First we prove a finite-time quenching for the solution. then, we assert that quenching occurs on the boundary under certain conditions .Futhermore,we show that the time derivative blows up at a quenching point. Finally, we get a quenching rate and a lower bound for the quenching time.
Keywords: semilinear parabolic equation, singular boundary outflux, quenching, maximum principles

1. INTRODUCTION

In this paper, we consider the quenching behavior of solutions of the following semilinear parabolic equation with a singular boundary outflux:

$$\begin{cases} u_t = u_{xx} + f(x)(1-u)^{-p}, & 0 < x < 1, 0 < t < T, \\ u_x(0,t) = 0, u_x(1,t) = -u^{-q}(1,t), & 0 < t < T, \\ u(x,0) = u_0(x), & 0 \le x \le 1, \end{cases} \tag{1.1}$$

where p, q are positive constants and T≤ ∞.The initial function $u_0 : [0,1] \to (0,1)$

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satisfies the compatibility conditions $u_0'(0) = 0, u_0'(1) = -u_0^{-q}(1).$, And $f(x)$ is a continuous and positive function. Throughout this paper,we assume that the initial function u_0 satisfies inequalities

$$u_0''(x) + f(x)(1 - u_0(x)))^{-p} \geq 0. \tag{1.2}$$

$$u_0'(x) \leq 0. \tag{1.3}$$

$$f'(x) \leq 0. \tag{1.4}$$

Since 1975,quenching problems with various boundary conditions have been studied extensively(cf. the survey by Chan[1, 2] and Kirk and Roberts[14] and[3, 4, 6−9, 12]).In the literature,quenching problems have been less studied with two nonlinear heat sources.we give as examples five of these papers.

Chan and Yuen [5]considered the problem

$$\begin{cases} u_t = u_{xx}, in \ \ \Omega, \\ u_x(0,t) = (1 - u(0,t))^{-p}, u_x(a,t) = (1 - u(a,t))^{-q}, 0 < t < T, \\ u(x,0) = u_0(x), 0 \leq u_0(x) < 1, in \ \ \Omega, \end{cases}$$

where $a, p, q > 0, T \leq \infty, D = (0,a), \Omega = D \times (0,T)$.
They showed that $x = a$ is the unique quenching point in finite time if u_0 is a lower solution ,and u_t blow up at quenching.Finally,they obtained criteria for nonquenching and quenching by using the positive steady states.

Burhan selcuk and Nuri ozalp [15]considered the problem

$$\begin{cases} u_t = u_{xx} + (1 - u)^{-p}, \quad 0 < x < 1, 0 < t < T, \\ u_x(0,t) = 0, u_x(1,t) = -u^{-q}(1,t), \quad 0 < t < T, \\ u(x,0) = u_0(x), \quad 0 \leq x \leq 1, \end{cases}$$

where $p, q > 0, T \leq \infty$.
They showed $x = 0$ is the unique quenching point in finite time if u_0 satisfies $u_{xx}(x,0) + (1 - u(x,0))^{-p} \geq 0, u_x(x,0) \leq 0$,and u_t blow up at quenching time.Finally,they obtained a lower bound for the quenching time.

Ozalp and Selcuk [11] considered the problem

$$\begin{cases} u_t = u_{xx} + (1-u)^{-p}, \quad 0 < x < 1, 0 < t < T, \\ u_x(0,t) = 0, u_x(1,t) = (1-u(1,t))^{-p}, \quad 0 < t < T, \\ u(x,0) = u_0(x), \quad 0 \le x \le 1, \end{cases}$$

where $p, q > 0, T \le \infty$.
They showed $x = 1$ is the unique quenching point in finite time if u_0 satisfies $u_{xx}(x,0) + (1-u(x,0))^{-p} \ge 0, u_x(x,0) \ge 0$,and u_t blow up at quenching time.Finally,they obtained a quenching rate and a lower bound for the quenching time.

Lin and Wang [13] considered the problem

$$\begin{cases} u_t = u_{xx} + (1-u)^{-p}, \quad 0 < x < 1, 0 < t < T, \\ u_x(0,t) = 0, u_x(1,t) = -u^{-q}(1,t), \quad 0 < t < T, \\ u(x,0) = u_0(x), \quad 0 \le x \le 1, \end{cases}$$

where $p, q > 0, T \le \infty$.
they showed that the solution have a finite time blow-up and obtained the exact blow-up rates for the necessary and sufficient conditions.They also proved that the blow-up will occur only at the boundary $x = 1$.Then,by applying the well-known method of Giga-Kohn,they derived the time asymptotic of solutions near the blow-up time.Finally,they proved that the blow-up was complete.
Yuanhong Zhi [16]consider the problem

$$\begin{cases} u_t = u_{xx} + f(x)(1-u)^{-p}, \quad 0 < x < 1, 0 < t < T, \\ u_x(0,t) = u^{-q}(0,t), u_x(1,t) = 0, \quad 0 < t < T, \\ u(x,0) = u_0(x), \quad 0 \le x \le 1, \end{cases}$$

where $p, q > 0, T \le \infty$,$f(x)$ is a non-negative function.
They showed $x = 0$ is the unique quenching point in finite time if u_0 satisfies $u_{xx}(x,0) + f(x)(1-u(x,0))^{-p} \le 0, u_x(x,0) \ge 0, f'(x) \ge 0$,further,they obtained lower and upper bounds for the quenching time.
In this paper,a quenching problem with two types of singularity terms,namely,a source term $(1-u)^{-p}$,the boundary outflux term $-u^{-q}$,and the effect of $f(x)$,are considered.In Section 2,we first show that quenching occurs in finite time under condition (1.2),Then we show that the only quenching point $x = 0$ under condition (1.2) and(1.3) ,Further we show that u_t blow up at the quenching point $x = 0$,In Section 3,we get a quenching rate and a lower bound for quenching time.

2. QUENCHING ON THE BOUNDARY AND BLOW-UP OF U_T

Definition 1 A solution u(x,t) of problem (1.1) is said to quench if there exists a finite time T such that
$\lim_{t\to T^-} max\{u(x,t) : 0 \le x \le 1\} \to 1$ or $\lim_{t\to T^-} min\{u(x,t) : 0 \le x \le 1\} \to 0$
From now on,we denote the quenching time of problem (1.1) with T.
The main results of this paper read as follows.

lemma 1 if u_0 satisfies (1.3),then $u_x < 0$ in $[0, 1] \times [0, T)$.
proof let $w = u_x$,then w satisfies

$$\begin{cases} w_t = w_{xx} + f'(x)(1-u)^{-p} + f(x)p(1-u)^{-p-1}w, & 0 < x < 1, 0 < t < T, \\ w(0,t) = 0, w(1,t) = -u^{-q}(1,t), & 0 < t < T, \\ w(x,0) = u_0'(x) & 0 \le x \le 1, \end{cases}$$

since $u_0'(x) \le 0$,then $w(x, 0) \le 0$.besides,$w(0, t) = 0, w(1, t) = -u^{-q}(1, t) < 0$,thus by the maximum principle $u_x(x, t) < 0$ in $(0, 1] \times (0, T)$

lemma 2 If u_0 satisfies (1.2),then $u_t \ge 0$ in $[0, 1] \times [0, T)$.
proof we give the proof by utilizing Lemma 3.1 in [10].Let $v = u_t(x, t)$.Then $v(x, t)$ satisfies

$$\begin{cases} v_t = v_{xx} + f(x)p(1-u)^{-p-1}v, & 0 < x < 1, 0 < t < T, \\ v_x(0,t) = 0, v_x(1,t) = qu^{-q-1}(1,t)v(1,t), & 0 < t < T, \\ v(x,0) = u_{xx}(x,0) + f(x)(1-u(x,0))^{-p} \ge 0, & 0 \le x \le 1, \end{cases}$$

For any fixed $\tau \in (0, T)$,let

$$L = \max_{0\le x\le 1, 0\le t\le \tau}\left(\frac{1}{2}qu^{-q-1}(x,t)\right), M = 2L + 4L^2 + \max_{0\le x\le 1, 0\le t\le \tau}\left(f(x)p(1-u(x,t))^{-p-1}\right). \quad (2.1)$$

Set $\omega(x, t) = e^{-Mt-Lx^2}v(x, t)$.Then ω satisfies

$$\begin{cases} \omega_t = \omega_{xx} + 4Lx\omega_x + c\omega, & 0 < x < 1, 0 < t < T, \\ \omega_x(0,t) = 0, \omega_x(1,t) = d(t)(1,t)\omega(1,t), & 0 < t < T, \\ \omega(x,0) \ge 0, & 0 \le x \le 1, \end{cases}$$

where

$$c = c(x,t) = 4L^2(x^2-1) + f(x)p(1-u(x,t))^{-p-1} - \max_{0\le x\le 1, 0\le t\le \tau}\left(f(x)p(1-u(x,t))^{-p-1}\right) \le 0. \quad (2.2)$$

and

$$d(t) = -\max_{0\le x\le 1, 0\le t\le \tau}(qu^{-q-1}(x,t)) + qu^{-q-1}(1,t) \le 0 \tag{2.3}$$

By the maximum principle and the Hopf lemma,we obtained that $\omega \ge 0$ in $[0,1]\times[0,\tau]$.Thus $u_t(x,t)\ge 0$ in $[0,1]\times[0,T)$.

Theorem 1 If u_0 satisfies (1.2),then there exists a finite time T such that the solution u of problem (1.1) quenching at time T.

proof We prove this result by contradiction. Assume on the contrary u can not quenches at all time.Set

$$\gamma := -u^{-q}(1,0) + \int_0^1 f(x)(1-u(x,0))^{-p}dx \ge 0, \tag{2.4}$$

Introduce a mass function :$m(t) = \int_0^1 (1-u(x,t))dx, 0 < t < T$.Then

$$m'(t) = u^{-q}(1,0) - \int_0^1 f(x)(1-u(x,t))^{-p}dx \le -\gamma. \tag{2.5}$$

by Lemma 2.Thus,$m(t)\le m(0)-\gamma t$,which means that $m(T_0)=0$ for some $T_0(0<T\le T_0)$,which means that u quenches in a finite time.

Theorem 2 If u_0 satisfies (1.2)(1.3) and (1.4),then $x=0$ is the only quenching point.

proof Define

$$K(x,t) = u_x + \epsilon(b_2 - x) \;\; in\; [b_1,b_2]\times[\tau,T), \tag{2.6}$$

where $b_2\in(0,1], b_1\in(0,b_2), \tau\in[0,T)$ and ϵ is a positive constant to be specified later.Then K(x,t) satisfies

$$K_t - K_{xx} = f'(x)(1-u)^{-p} + f(x)p(1-u)^{-p-1}u_x. \tag{2.7}$$

since $u_x(x,t)<0, f'(x)\le 0$ in $(0,1]\times[0,T)$.Thus,$K(x,t)-K_{xx}<0$ in $(b_1,b_2)\times(\tau,T)$ .Further,if ϵ is small enough,

$$K(b_1,t) = u_x(b_1,t) + \epsilon(b_2-b_1) < 0, \tag{2.8}$$

$$K(b_2,t) = u_x(b_2,t) < 0, \tag{2.9}$$

for $t \in (\tau, T)$.By the maximum principle,we obtain that $K(x,t) < 0$ in $[b_1, b_2] \times [\tau, T)$,i.e,$u_x < -\epsilon(b_2 - x)$ for $(x,t) \in [b_1, b_2] \times [\tau, T)$.Integrating this with respect to x from b_1 to b_2 ,we have

$$u(b_2,t) < u(b_1,t) - \frac{\epsilon(b_2-b_1)^2}{2} < 1 - \frac{\epsilon(b_2-b_1)^2}{2} < 1. \tag{2.10}$$

So u does not quench in $(0, 1]$.The theorem is proved.

Remark 1 Our hypothesis is reasonable,in other words,we can make a initial function to satisfies (1.2), (1.3) and the compatibility conditions,for example,let $u_0(x) = 0.9 - \frac{2}{3}x^{4.5}$, when $p = 9$ and $q == \log_{30/7}(3)$,$u_0(x)$ satisfies (1.2), (1.3) and the compatibility conditions

Theorem 3 if $p \geq 1$,then u_t blows up at the quenching point $x = 0$.

proof Suppose that u_t is bound on$[0, 1] \times [0, T)$,Then,there exists a positive constant M,such that $u_t < M$,That is

$$u_{xx} + f(x)(1-u)^{-p} < M, \tag{2.11}$$

Since $f(x)$ is a continuous positive function,and $f'(x) \leq 0$,thus $f(x)$ is bounded .Then,there exists a positive constant a,such that $f(x) \geq a$,therefore the above inequality will be

$$u_{xx} + a(1-u)^{-p} < M, \tag{2.12}$$

Multiplying this inequality by u_x,and integrating with respect to x from 0 to x.we have

$$a\ln(1-u(0,t)) > -\frac{1}{2}u_x^2 + a\ln(1-u(x,t)) + M[u(x,t) - u(0,t)], \tag{2.13}$$

for $p = 1$ and

$$a\frac{(1-u(0,t))^{-p+1}}{-p+1} > -\frac{1}{2}u_x^2 + a\frac{(1-u(0,t))^{-p+1}}{-p+1} + M[u(x,t) - u(0,t)] \tag{2.14}$$

for $p \neq 1$.We have,as $t \to T^-$ and $p \geq 1$,that the left-hand side tends to negative infinity,while the right-hand side is finite.This contradiction shows that u_t blows up at the quenching point $x = 0$.

3. A QUENCHING RATE AND A LOW BOUND FOR THE QUENCHING TIME

In this section,we get a quenching rate and a lower bound for the quenching time.Throughout this section,we assume that

$$u_x(x,0) \le -xu^{-q}(x,0), \quad 0 \le x \le 1, \tag{3.1}$$

$$u_t(0,t) = u_{xx}(0,t) + f(0)(1-u(0,t))^{-p}, \quad 0 < t < T. \tag{3.2}$$

Theorem 4 If u_0 satisfies $(1.2), (1.3), (1.4)$, (3.1)and(3.2),then there exists a positive constant C_1 Such that

$$u(0,t) \ge 1 - C_1(T-t)^{1/(p+1)}. \tag{3.3}$$

for t sufficiently close to T.
proof Define $J(x,t) = u_x + xu^{-q}$ in $[0,1] \times [0,T)$ .Then,$J(x,t)$ satisfies

$$J_t - J_{xx} = fp(1-u)^{-p-1}u_x + f'(1-u)^{-p} - fqx(1-u)^{-p}u^{-q-1} + 2qu^{-q-1}u_x - q(q+1)xu^{-q-2}u_x^2, \tag{3.4}$$

since$u_x < 0, f'(x) \le 0, f(x) > 0, J_t - J_{xx} < 0 \; in(0,1) \times [0,T)$ .On the other hand,since

$$J(0,t) = 0, J(1,t) = 0, \tag{3.5}$$

for $t \in (0,t)$ and assumption (3.1),thus by the maximum principle,we obtained that $J(x,t) \le 0$ for$(x,t) \in [0,1] \times [0,T)$.Therefore

$$J_x(0,t) = \lim_{h \to 0^+} \frac{J(h,t) - J(0,t)}{h} = \lim_{h \to 0^+} \frac{J(h,t)}{h} \le 0, \tag{3.6}$$

From (3.2),we get

$$J_x(0,t) = u_{xx}(0,t) + u^{-q}(0,t) = u_t(0,t) - f(0)(1-u(0,t))^{-p} + u^{-q}(0,t) \le 0, \tag{3.7}$$

and

$$u_t(0,t) \le f(0)(1-u(0,t))^{-p} \tag{3.8}$$

Integrating for t from t to T we get

$$u(0,t) \ge 1 - C_1(T-t)^{1/(p+1)}, \tag{3.9}$$

where $C_1 = [f(0)(p+1)]^{1/p+1}$.

Remark 2 we can calculate a lower bound for the quenching time from Theorem 4,a lower bound is $(1 - u_0(0))^{p+1}/f(0)(p + 1)$ for quenching time T,If we choose,as in Remark 1,$u_0(x) = 0.9 - \frac{2}{3}x^{4.5}$, then we have $T = 10^{-11}$ for $p = 9$.

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