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Numerical Solutions of Time Fractional Nonlinear Partial Differential Equations Using Yang Transform Combined with Variational Iteration Method

R.Aruldoss and G.Jasmine
Department of Mathematics, Government Arts College (Autonomous),
(Affiliated to Bharathidasan University,Tiruchirapalli)
Kumbakonam - 612 002, TamilNadu, India
Email:krvarul@gmail.com, asjasmine94@gmail.com

Abstract

In this article, we combine Yang transform with a semi-analytical method, namely variational iteration method to arrive numerical solution of time fractional nonlinear partial differential equations. This new analytical method provides a solution as a more realistic series which converges rapidly to the exact solution.

1. INTRODUCTION

Fractional calculus is the field of mathematical analysis which deals with the investigation and applications of integrals and derivatives of arbitrary order. Let us consider a general nonlinear partial differential equation with time fractional derivatives

$$^{c}D_{\tau}^{\beta}U(x,\tau) + RU(x,\tau) + NU(x,\tau) = g(x,\tau)$$

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[2]Key Words: Variational iteration method, Yang transform, Caputo fractional derivatives,Fractional Fornberg Whitham equation

with initial condition

$$\left[\frac{\partial^{m-1}U(x,\tau)}{\partial\tau^{m-1}}\right]_{\tau=0} = g_{m-1}(x)$$

where R and N are linear and non linear differential operators. There are many semi-analytic methods such as Homotopy analysis method, Adomian decomposition method, Variational iteration method and Homotopy perturbation method to find exact solution of fractional differential equations. Comparing with these methods, Variational iteration method is fast convergence to get solution. A new option emerged recently, that the semi analytic methods are combined with Laplace transform, Sumudu transform, Natural transform or Elzaki transform, among which are Laplace homotopy analysis method, Homotopy analysis sumudu transform method, Modified fractional homotopy analysis transform method, Adomian decomposition method coupled with Laplace transform method, Sumudu decomposition method for nonlinear equations, Elzaki transform decomposition algorithm, Natural decomposition method, Variational iteration method coupled with laplace transform method, Variational iteration sumudu transform method, Elzaki variational iteration method, Homotopy perturbation transform method, Homotopy perturbation sumudu transform method, Homotopy perturbation elzaki transform method. The aim of this article is to combine variational iteration method with Yang transform. The motivation of this article is to make a change on the method proposed by Kharde Uttan Dattu [4], and then extend it to solve nonlinear partial differential equations with time-fractional derivative.

The present article has been organized as follows: In Section 2 some basic definitions and properties of Yang transform are discussed. In section 3 we propose fractional yang variational iteration method. In section 4 we illustrate some examples by using proposed fractional yang variational iteration method.

2. PRELIMINARIES

Definition 2.1. *Let* $\Omega = [x, y]$, $(-\infty < x < y < +\infty)$, *be a finite interval on the real axis* $\Re$. *The Riemann-Liouville fractional integral of order* $\beta > 0$, *of the function* $g(\tau)$, *denoted by* $(I_0^\beta g)(\tau)$, *is given by*

$$(I_0^\beta g)(\tau) = \frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\xi)^{\beta-1} f(\xi)d\xi, \qquad \tau > 0$$

where $\Gamma(x) = \int_0^\infty \tau^{x-1}e^{-\tau}d\tau$, $x > 0$ *is gamma function of Euler.*

Definition 2.2. *Suppose that* $\beta > 0$, $\tau > 0$,$\beta, \tau \in \Re$.*The fractional operator*

$$ {}^{c}D_{\tau}^{\beta} g(\tau) = \begin{cases} \frac{1}{\Gamma(m-\beta)} \int_0^{\tau} (\tau - \xi)^{m-\beta-1} g^{(m)}(\xi) d\xi, & m-1 < \beta < m \\ \\ \frac{d^m}{d\tau^m} g(\tau), & \beta = m \end{cases} $$

where $m \in N^*$, *is called the Caputo fractional derivative or Caputo fractional differential operator of order* β *of the function* $g(\tau)$, *which was introduced by italian mathematician, Caputo in 1967.*

The Caputo fractional derivative of order β *of the function* $\gamma(x, \tau)$, *is given by*

$$ {}_{0}^{c}D_{\tau}^{\beta} \gamma(x,\tau) = \begin{cases} \frac{1}{\Gamma(m-\beta)} \int_0^{\tau} (\tau - \xi)^{m-\beta-1} \frac{\partial^m \gamma(x,\xi)}{\partial \xi^m} d\xi, & m-1 < \beta < m \\ \\ \frac{\partial^m \gamma(x,\xi)}{\partial \xi^m}, & \beta = m \end{cases} $$

Definition 2.3. *Let* $g(\tau)$ *be a function belonging to a class T,where*

$$ T = \{g(\tau); \exists\, M, q_1 \; and/or \; q_2 > 0 \; such \; that \; |g(\tau)| < M e^{\frac{|\tau|}{q_j}}, if \; \tau \in \; (-1)^j \times [0, \infty)\} $$

The Yang transform of the function $g(\tau)$, *denoted by* $Y\{g(\tau); u\}$ *or T(u), is defined as*

$$ Y\{g(\tau); u\} = T(u) = \int_0^{\infty} e^{\frac{-\tau}{u}} g(\tau) d\tau, \quad \tau > 0 $$

provided the integral exists for some u, where u$\in (-\tau_1, \tau_2)$.

Definition 2.4. *The Yang transform of the function* $g(x, \tau)$, *denoted by* $Y\{g(x, \tau); u\}$ *or* $T(x, u)$, *is defined as*

$$ Y\{g(x,\tau); u\} = T(x,u) = \int_0^{\infty} e^{\frac{-\tau}{u}} g(x,\tau) d\tau, \quad \tau > 0 $$

Theorem 2.5. *Suppose* $T(u)$ *is the Yang transform of the function* $g(\tau)$. *Then*

$$ Y\{{}^{c}D_{\tau}^{\beta} g(\tau); u\} = \frac{T(u)}{u^{\beta}} - \sum_{k=0}^{m-1} u^{k-\beta+1} g^{(k)}(0) $$

Proof. By Laplace-Yang duality property, we have

$$ Y\{{}^{c}D_{\tau}^{\beta} g(\tau); u\} = L\left\{{}^{c}D_{\tau}^{\beta} g(\tau); \frac{1}{u}\right\} $$

Using the Laplace transform formula for the Caputo fractional derivative $({}^cD_\tau^\beta g)(\tau)$ that is.,

$$L\{({}^cD_\tau^\beta g)(\tau); s\} = s^\beta F(s) - \sum_{k=0}^{m-1} s^{\beta-k-1} g^{(k)}(0),$$

we have

$$Y\{{}^cD_\tau^\beta g(\tau); u\} = \left(\frac{1}{u}\right)^\beta F\left(\frac{1}{u}\right) - \sum_{k=0}^{m-1} \left(\frac{1}{u}\right)^{(\beta-k-1)} g^{(k)}(0) \quad \text{and so}$$

$$Y\{{}^cD_\tau^\beta g(\tau); u\} = \frac{T(u)}{u^\beta} - \sum_{k=0}^{m-1} u^{k-\beta+1} g^{(k)}(0)$$

□

Remark 2.6. *The Yang transform of Caputo fractional derivative of order β of the function $g(x,\tau)$ with respect to τ, is given by*

$$Y\{{}^cD_\tau^\beta g(x,\tau); u\} = \frac{T(x,u)}{u^\beta} - \sum_{k=0}^{m-1} u^{k-\beta+1} g^{(k)}(x,0)$$

2.1 Variational iteration method

To illustrate the concept of He's variational iteration method, we consider the following general non-linear partial differential equation in the form

$$Lu(x,\tau) + Nu(x,\tau) = g(x,\tau), \tag{1}$$

where L is the linear operator, N is the nonlinear operator and $g(x,\tau)$ is a known analytical function. The correction functional according to the variational iteration method is given by

$$u_{m+1}(x,\tau) = u_m(x,\tau) + \int_0^\tau \lambda\{Lu_m(x,\xi) + N\tilde{u}_m(x,\xi) - g(x,\xi)\}d\xi, m \geq 0$$

where λ is Lagrange multiplier,which can be identified optimally via the variational theory,the subscript m denotes the m^{th} approximation,and $\tilde{u}_m$ is considered as restricted variation,that is $\delta\tilde{u}_m = 0$. The initial approximation u_0 can be chosen freely if it satisfies the given conditions. The solution of (1) is given by

$$u(x,\tau) = lim_{m\to\infty} u_m(x,\tau).$$

3. MAIN RESULT

3.1 Fractional Yang variational iteration method

To illustrate the basic idea of this method,we consider a general fractional order nonlinear partial differential equation with time fractional derivatives

$$ {}^{c}D_{\tau}^{\beta}U(x,\tau)+RU(x,\tau)+NU(x,\tau)=g(x,\tau,) \qquad m-1<\beta\leq m; \quad m=1,2,....... \tag{2}$$

subject to the initial condition

$$\left[\frac{\partial^{m-1}U(x,\tau)}{\partial\tau^{m-1}}\right]_{\tau=0}=g_{m-1}(x),$$

where ${}^{c}D_{\tau}^{\beta}=\frac{\partial^{\beta}}{\partial\tau^{\beta}}$ is the caputo fractional differential operator of order β, R is the linear differential operator, N is the general non linear differential operator and $g(x,\tau)$ is the source term.
Applying Yang transform on both sides of (2) and using the differentiation property of Yang transform, we have

$$\frac{T(x,u)}{u^{\beta}}-\sum_{k=0}^{m-1}u^{k-\beta+1}g^{(k)}(x,0)+Y\{RU(x,\tau)\}+Y\{NU(x,\tau)\}=Y\{g(x,\tau)\}$$

and so

$$T(x,u)=\sum_{k=0}^{m-1}u^{k+1}g^{(k)}(x,0)-u^{\beta}[Y\{RU(x,\tau)+NU(x,\tau)\}]+u^{\beta}Y\{g(x,\tau)\} \tag{3}$$

Operating with the inverse Yang transform on the both sides of (3), we obtain,

$$U(x,\tau)=k(x,\tau)-Y^{-1}\left\{u^{\beta}Y\left\{RU(x,\tau)+NU(x,\tau)\right\}\right\}$$

where $\sum_{k=0}^{m-1}u^{k+1}g^{(k)}(x,0)+u^{\beta}Y\{g(x,\tau)\}=k(x,\tau)$, and so

$$\frac{\partial U(x,\tau)}{\partial\tau}+\frac{\partial}{\partial\tau}Y^{-1}\left\{u^{\beta}Y\left\{RU(x,\tau)+NU(x,\tau)\right\}\right\}-\frac{\partial k(x,\tau)}{\partial\tau}=0$$

According to the variational iteration method, we can construct a correct functional as follows

$$U_{m+1}(x,\tau)=U_{m}(x,\tau)-\int_{0}^{\tau}\left[\frac{\partial U_{m}}{\partial\eta}+\frac{\partial}{\partial\eta}Y^{-1}\left\{u^{\beta}Y\left\{RU_{m}+NU_{m}\right\}\right\}-\frac{\partial k(x,\eta)}{\partial\eta}\right]d\eta \tag{4}$$

Alternately,

$$U_{m+1}(x,\tau) = k(x,\tau) - Y^{-1}\left\{u^{\beta}Y\left\{RU_m(x,\tau) + NU_m(x,\tau)\right\}\right\}$$

Hence $U(x,\tau) = \lim_{m\to\infty} U_m(x,\tau)$.
By the preceding limit value, we can obtain the exact solution if it exists, or we can use the $(m+1)^{th}$ approximation for numerical purpose.

4. APPLICATIONS

To illustrate the efficiency of the fractional yang variational iteration method, we solve some non linear time fractional partial differential equations with caputo fractional derivatives.

Example 4.1. *First we consider the nonlinear time fractional diffusion equation*

$$^{c}D_{\tau}^{\beta}U(x,\tau) = \frac{x^2}{2}U_{xx}(x,\tau), \quad 0 < \beta \leq 1 \tag{5}$$

subject to the initial condition

$$U(x,0) = x^2$$

and the boundary conditions $U(0,\tau) = 0$ *and* $U(1,\tau) = f(\tau)$

Applying Yang transform on both sides of (5) and using differentiation property, we have

$$\frac{T(x,u)}{u^{\beta}} - \sum_{k=0}^{m-1} u^{k-\beta+1}g^{(k)}(x,0) = Y\left\{\frac{x^2}{2}U_{xx}(x,\tau)\right\} \quad \text{and so}$$

$$Y\{U(x,\tau)\} = x^2 + u^{\beta}Y\left\{\frac{x^2}{2}U_{xx}(x,\tau)\right\} \tag{6}$$

Now,taking inverse Yang transform on both sides of (6), we get

$$U(x,\tau) = x^2 + Y^{-1}\left\{u^{\beta}Y\left\{\frac{x^2}{2}U_{xx}(x,\tau)\right\}\right\} \text{and so}$$

$$\frac{\partial U(x,\tau)}{\partial\tau} - \frac{\partial}{\partial\tau}Y^{-1}\left\{u^{\beta}Y\left\{\frac{x^2}{2}U_{xx}(x,\tau)\right\}\right\} = 0$$

According to variational iteration method, we have

$$U_{m+1}(x,\tau) = U_m(x,\tau) - \int_0^{\tau}\left\{\frac{\partial U_m(x,\eta)}{\partial\eta} - \frac{\partial}{\partial\eta}Y^{-1}\left\{u^{\beta}Y\left\{\frac{x^2}{2}(U_m)_{xx}(x,\eta)\right\}\right\}\right\}d\eta \tag{7}$$

By using iteration formula (7), we have

$$\begin{aligned}
U_0(x,\tau) =& x^2 \\
U_1(x,\tau) =& x^2 + Y^{-1}\left\{u^\beta Y\left\{\frac{x^2}{2}(U_0)_{xx}\right\}\right\} \\
=& x^2 + x^2\frac{\tau^\beta}{\Gamma(\beta+1)} \\
U_2(x,\tau) =& x^2 + x^2\frac{\tau^\beta}{\Gamma(\beta+1)} + x^2\frac{\tau^{2\beta}}{\Gamma(2\beta+1)} \\
U_3(x,\tau) =& x^2 + x^2\frac{\tau^\beta}{\Gamma(\beta+1)} + x^2\frac{\tau^{2\beta}}{\Gamma(2\beta+1)} + x^2\frac{\tau^{3\beta}}{\Gamma(3\beta+1)}
\end{aligned}$$

and so on..
Proceeding in this manner, we obtain

$$U_m(x,\tau) = \sum_{k=0}^{m} \frac{x^2\tau^{k\beta}}{\Gamma(k\beta+1)},$$

which we can assume as the m^{th} approximate solution of (5)
The exact solution of (5) is given by

$$U(x,\tau) = \lim_{m\to\infty} U_m(x,\tau) \; = x^2 e^\tau$$

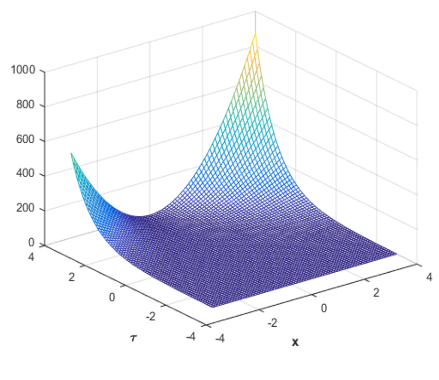

Figure 1: The exact solution for $\beta = 1$ of example 4.1

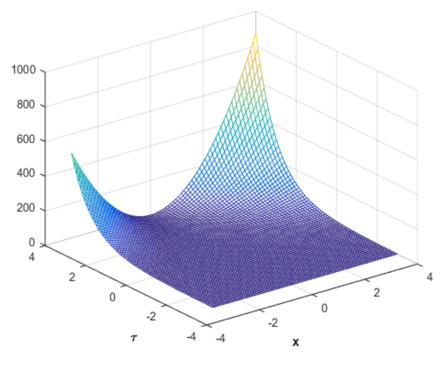

Figure 2: The approximate solution for $\beta = 1$ of example 4.1

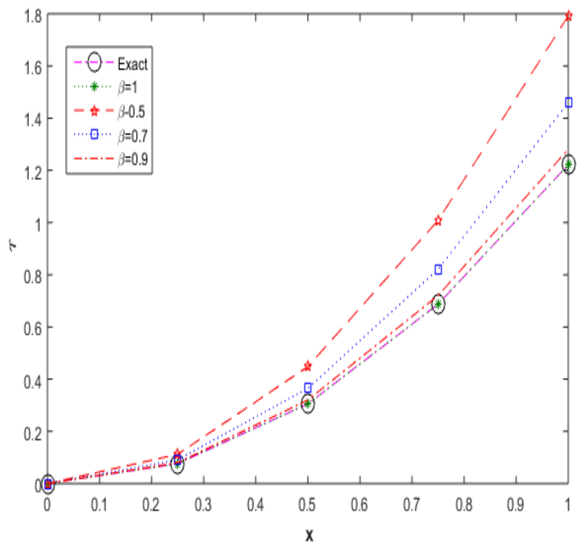

Figure 3: Approximate solutions obtained for different values of β and exact solution for $\beta = 1$ of example 4.1

x	τ	$\beta = 0.5$	$\beta = 0.7$	$\beta = 0.9$
0.25	0.2	0.111994377395286	0.091196729856063	0.080032422294882
0.50		0.447977509581142	0.364786919424253	0.320129689179529
0.75		1.007949396557571	0.820770568704570	0.720291800653941
1.0		1.791910038324569	1.459147677697013	1.280518756718117
0.25	0.4	0.148997263678171	0.117844278265281	0.099587715983651
0.50		0.595989054712684	0.471377113061125	0.398350863934606
0.75		1.340975373103540	1.060598504387530	0.896289443852863
1.0		2.383956218850738	1.885508452244499	1.593403455738424
0.25	0.6	0.187728390141446	0.148129189247630	0.122914336592625
0.50		0.750913560565782	0.592516756990521	0.491657346370500
0.75		1.689555511273010	1.333162703228671	1.106229029333625
1.0		3.003654242263129	2.370067027962082	1.966629385482000

Table 1: Approximate solutions for different values for β of example 4.1

Example 4.2. *Now we consider the time fractional Fornberg-Whitham equation*

$$^{c}D_{\tau}^{\beta}U(x,\tau) = U_{xx\tau} - U_x + UU_{xxx} - UU_x + 3U_xU_{xx} \quad 0 < \beta \leq 1, \quad \tau > 0 \tag{8}$$

with initial condition

$$U(x,0) = e^{\frac{x}{2}}$$

Applying Yang transform on both sides of (8) and using differentiation property of Yang transform,we have

$$\frac{T(x,u)}{u^{\beta}} - \sum_{k=0}^{m-1} u^{k-\beta+1}g^{(k)}(x,0) = Y\{U_{xx\tau} - U_x + UU_{xxx} - UU_x + 3U_xU_{xx}\} \text{ and so}$$

x	τ	Approximate for $\beta = 1$	Exact for $\beta = 1$	Absolute error
0.25	0.2	0.076337500000000	0.076337672385011	1.7239e-07
0.50		0.305350000000000	0.305350689540042	6.8954e-07
0.75		0.687037500000000	0.687039051465096	1.5515e-06
1.0		1.221400000000000	1.221402758160170	2.7582e-06
0.25	0.4	0.093233333333333	0.093239043602579	5.7103e-06
0.50		0.372933333333333	0.372956174410318	2.2841e-05
0.75		0.839100000000000	0.839151392423215	5.1392e-05
1.0		1.491733333333333	1.491824697641270	9.1364e-05
0.25	0.6	0.113837500000000	0.113882425024407	4.4925e-05
0.50		0.455350000000000	0.455529700097627	1.7970e-04
0.75		1.024537500000000	1.024941825219661	4.0433e-04
1.0		1.821400000000000	1.822118800390509	7.1880e-04

Table 2: Approximate and exact solutions for $\beta = 1$ of example 4.1

$$T(x,u) = \sum_{k=0}^{m-1} u^{k+1} g^{(k)}(x,0) + u^{\beta} Y\{U_{xx\tau} - U_x + UU_{xxx} - UU_x + 3U_x U_{xx}\} \quad (9)$$

Now, taking inverse Yang transform on both sides of (9), we get

$$U(x,\tau) = e^{\frac{x}{2}} + Y^{-1}\{u^{\beta} Y\{U_{xx\tau} - U_x + UU_{xxx} - UU_x + 3U_x U_{xx}\}\} \text{ and so}$$

$$\frac{\partial U(x,\tau)}{\partial \tau} - \frac{\partial}{\partial \tau} Y^{-1}\{u^{\beta} Y\{U_{xx\tau} - U_x + UU_{xxx} - UU_x + 3U_x U_{xx}\} = 0$$

According to the variational iteration method, we have

$$U_{m+1}(x,\tau) = U_m(x,\tau) - \int_0^{\tau} \left\{ \frac{\partial U_m(x,\eta)}{\partial \eta} - \frac{\partial}{\partial \eta} Y^{-1}\{u^{\beta} Y\{(U_m)_{xx\tau} - (U_m)_x + (U_m)(U_m)_{xxx} - (U_m)(U_m)_x + 3(U_m)_x(U_m)_{xx}\} \right\} d\eta \quad (10)$$

By using iteration formula (10), we have

$$U_0(x,\tau) = e^{\frac{x}{2}}$$

$$U_1(x,\tau) = e^{\frac{x}{2}} + Y^{-1} u^{\beta} \left\{ -\frac{1}{2} e^{\frac{x}{2}} u + \frac{1}{8} e^{x} u - \frac{1}{2} e^{x} u + \frac{3}{8} e^{x} u \right\}$$

$$= e^{\frac{x}{2}} \left\{ 1 - \frac{\tau^{\beta}}{2\Gamma(\beta+1)} \right\}$$

$$U_2(x,\tau) = e^{\frac{x}{2}} \left\{ 1 - \frac{\tau^{\beta}}{2\Gamma(\beta+1)} - \frac{\tau^{2\beta-1}}{8\Gamma(\beta+)} + \frac{\tau^{2\beta}}{4\Gamma(2\beta+1)} \right\}$$

and so on

Proceeding in this manner,we can obtain the m^{th} approximate solution of(8)
The exact solution of (8) is given by

$$U(x,\tau) = \lim_{m\to\infty} U_m(x,\tau) = e^{x/2-2\tau/3}$$

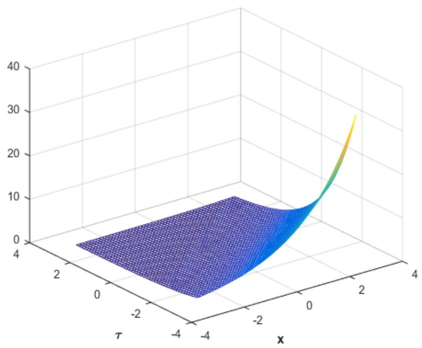

Figure 4: The exact solution for $\beta = 1$ of example 4.2

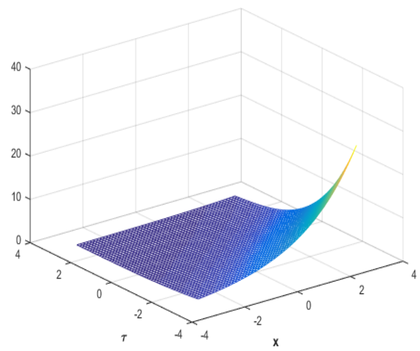

Figure 5 :The approximate solution for $\beta = 1$ of example 4.2

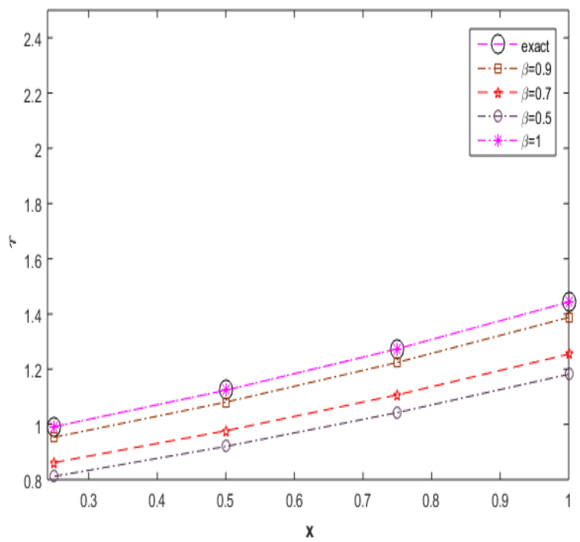

Figure 6: Approximate solutions obtained for different values of β and exact solution for $\beta = 1$ of example 4.2

5. CONCLUSION

In the article, Yang transform and Variational iteration method are successfully combined to form a powerful analytical method called fractional Yang variational iteration method, for solving nonlinear time fractional partial differential equations with caputo fractional derivatives.Numerical results reveal that this new analytical method

x	τ	$\beta = 0.5$	$\beta = 0.7$	$\beta = 0.9$
0.25	0.2	0.812105596464862	0.862133560651307	0.953617521798849
0.50		0.920236200361070	0.976925310589023	1.080590219643787
0.75		1.042764226895241	1.107001404455781	1.224469135788499
1.0		1.181606670619768	1.254396929001872	1.387505307046812
0.25	0.4	0.756479846850793	0.770094225269300	0.830554792223812
0.50		0.857203968235205	0.872631080079603	0.941141877995652
0.75		0.971339350568468	0.988820558490236	1.066453463167181
1.0		1.100671682499595	1.120480486213887	1.208450092055651
0.25	0.6	0.710509131675113	0.704238604586185	0.732112997872035
0.50		0.805112323447509	0.798006885376776	0.829592711008814
0.75		0.912311783859583	0.904260267701370	0.940051697155152
1.0		1.033784686595123	1.024661123515602	1.065218126434205

Table 3: Approximate solutions for different values for β of example 4.2

x	τ	Approximate for $\beta = 1$	Exact for $\beta = 1$	Absolute error
0.25	0.2	0.991884973310439	0.991701292638876	1.8368e-04
0.50		1.123952923126954	1.123744785658114	2.0814e-04
0.75		1.273605516161246	1.273369665510405	2.3585e-04
1.0		1.443184120455493	1.442916866655337	2.6725e-04
0.25	0.4	0.867793414069902	0.867910511777946	1.1710e-04
0.50		0.983338764734890	0.983471453821617	1.3269e-04
0.75		1.114268800099984	1.114419156533349	1.5036e-04
1.0		1.262631967133926	1.262802343293801	1.7038e-04
0.25	0.6	0.758975751686329	0.759572123224968	5.9637e-04
0.50		0.860032198938595	0.860707976425058	6.7578e-04
0.75		0.974544155814930	0.975309912028333	7.6576e-04
1.0		1.104303202607004	1.105170918075648	8.6772e-04

Table 4: Approximate and exact solutions for $\beta = 1$ of example 4.2

is very effective, simple and gives a series solution which converges rapidly to the exact solution.The simplicity and high precision of new analytical method are clearly illustrated by solving some non-linear time fractional partial differential equations.

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