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Berwald Connection of the Finsler Space with an (α, β)-Metric

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ABSTRACT

In the present paper, we find the condition that a Finsler space F^n with the metric $L(\alpha,\beta) = \alpha + \frac{\beta^{m+1}}{\alpha^m}$, where $m \neq 0,\ m \neq -1$ to be Berwald space, where α is a Riemannian metric and β is differential 1-form and also, we find the vector field $B^i(x, y)$ on the Finsler space.

Key Words: Finsler space, Berwald space, Berwald connection, (α, β)-metric.
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1. INTRODUCTION

The concept of (α, β)-metric in Finsler space was introduced by M. Matsumoto in 1972. L. Berwald introduced a connection and two curvature tensors in 1926. The Berwald connection of a Finsler space with various special (α, β)-metrics have been studied by many authors ([1], [4], [5], [6]). An important result concerned with the Berwald space was given by M. Matsumoto [3]. He gave the necessary and sufficient condition for a Berwald space. In 1997, Hong-Suh Park, Ha-Yong Lee and Byung-Doo Kim [5] studied the Berwald connection of a Finsler space with a special (α, β)-metric. The authors M. Hashiguchi and Y. Ichijyo [1] studied on some special (α, β)-metric in 1975.

In this paper, we find the necessary and sufficient condition for the Finsler spaces with the metric $L(\alpha, \beta) = \alpha + \frac{\beta^{m+1}}{\alpha^m}$ to be a Berwald space and its vector field $B^i(x, y)$ on the Finsler space.

2. PRELIMINARIES

Definition 2.1: Let M^n be an n-dimensional smooth manifold and *L(x, y)* be a fundamental function which satisfies the following conditions,

i) $L(x, y) > 0$, for $(x, y) \in D$,

ii) $L(x, \lambda y) = |\lambda| \, L(x, y)$, for any $(x, y) \in D$ and $\lambda \in R$ such that $(x, \lambda y) \in D$,

iii) The d−tensor field $g_{ij}(x,y) = \frac{1}{2} \dot{\partial}^i \dot{\partial}^j L^2$, $(x, y) \in D$,

then $F^n = (M^n, L)$ is called Finsler space equipped with a fundamental function *L(x, y)* on M^n, where $\dot{\partial}^i = \partial / \partial y^i$ is non-degenerate on D.

Let $F^n = (M^n, L(\alpha, \beta))$ be an n-dimensional Finsler space with an *(α, β)*-metric,

$$L(\alpha, \beta) = \alpha + \frac{\beta^{m+1}}{\alpha^m}, \quad where\ m \neq 0, m \neq -1 \tag{2.1}$$

where α is a Riemannian metric $\alpha^2 = a_{ij}\, y^i\, y^j$ and β is a differential 1-form $\beta = b_i(x)\, y^i$.

The Riemannian space $R^n = (M^n, \alpha)$ is called associated Riemannian space with F^n and the Christoffel symbols of $R^n = (M^n, \alpha)$ are indicated by $\gamma_j{}^i{}_k$. Then the Riemannian connection $(\gamma_j{}^i{}_k)$ gives rise to the linear Finsler connection $F\Gamma = (\gamma_j{}^i{}_k, \gamma_0{}^i{}_j, 0)$, where the subscript 0 denotes a contraction by y^i.

The Berwald connection $B\Gamma = (G_j{}^i{}_k, G_0{}^i{}_j, 0)$ is a Finsler connection which is uniquely determined from the fundamental function L(x, y) by the following Okada's axiomatic system [2]:

(a) L – metric: $L_{|i} = 0$,

(b) *(h)h* – torsion tensor $T_j{}^i{}_k = G_j{}^i{}_k - G_k{}^i{}_j = 0$,

(c) Deflection tensor $D^i_j = y^k G_k{}^i{}_j - G^i{}_j = 0,$

(d) $(v)hv$ – torsion tensor $P_j{}^i{}_k = \dot{\partial}_k G^i_j - G_k{}^i{}_j = 0,$

(e) $(h)hv$ – torsion tensor $C_j{}^i{}_k = 0,$

where the symbol (\) in (a) denotes the h- covariant derivative with respect to the Finsler connection.

Now, we shall find the Berwald connection B$\boldsymbol{\Gamma}$ in F^n. Putting

$$2G^i = \gamma_0{}^i{}_0 + 2B^i, \tag{2.2}$$

Also, from (b), (c), (d), we have

$$G^i_j = \gamma_0{}^i{}_j + B^i_j,$$
$$G_j{}^i{}_k = \gamma_j{}^i{}_k + B_j{}^i{}_k, \tag{2.3}$$

where $B_j{}^i{}_k = \dot{\partial}_k B^i_j$ and $B^i_j = \dot{\partial}_j B^i$.

The axiom (a): $L_{|i} = \dot{\partial}_i L - G^r{}_i \dot{\partial}_r L = 0$ is written as

$$L_\alpha B_j{}^k{}_i y^j y_k + \alpha L_\beta (B_j{}^r{}_i b_r - \nabla_i b_j) y^j = 0, \tag{2.4}$$

where $y_k = a_{ki} y^i$ and ∇_i is the differentiation with respect to $\gamma_j{}^i{}_k$.

Definition 2.2: A Finsler space F^n is called Berwald space if and only if the Berwald connection B$\boldsymbol{\Gamma}$ = ($G_j{}^i{}_k$, $G_0{}^i{}_j$, 0) is linear.

3. BERWALD SPACE WITH SPECIAL (α, β)- METRIC

In this section, Matsumoto's method of [4] will be applied to find the condition that F^n to be a Berwald space.

Theorem 3.1: The Finsler space F^n with the metric $L(\alpha, \beta) = \alpha + \frac{\beta^{m+1}}{\alpha^m}$, $where\ m \neq 0\ , m \neq -1$, is a Berwald space iff $\nabla_i b_j = 0$ and the Berwald connection is given by ($\gamma_j{}^i{}_k$, $\gamma_0{}^i{}_j$, 0).

Proof: Consider an (α, β)-metric as in (2.1), we have

$$\left.\begin{aligned} L_\alpha &= 1 - \frac{m\beta^{m+1}}{\alpha^{m+1}}, \\ L_\beta &= \frac{(m+1)\,\beta^m}{\alpha^m}. \end{aligned}\right\} \tag{3.1}$$

Now substitute (3.1) in (2.4), gives

$$\left(1 - \frac{m\beta^{m+1}}{\alpha^{m+1}}\right) B_j{}^k{}_i\, y^j y_k + \alpha\, \frac{(m+1)\,\beta^m}{\alpha^m} (\, B_j{}^r{}_i\, b_r - \nabla_i b_j) y^j = 0,$$

$$\Rightarrow\ (\alpha^{m+1} - m\beta^{m+1})\, B_j{}^k{}_i\, y^j y_k + (m+1)\alpha^2\beta^m (\, B_j{}^r{}_i\, b_r - \nabla_i b_j) y^j = 0. \tag{3.2}$$

Assume that F^n is a Berwald space i.e., $B_j{}^i{}_k = B_j{}^i{}_k(x)$. Now in (3.2), separating rational and irrational terms of y^i as

$$\alpha\, \{\alpha^m B_j{}^k{}_i\, y^j y_k + \alpha\,(m+1)\beta^m \left(B_j{}^r{}_i\, b_r - \nabla_i b_j \right) y^j\} =$$

$m\beta^{m+1} B_j{}^k{}_i\, y^j y_k$,

which yields two equations,

$$m\beta^{m+1} B_j{}^k{}_i\, y^j y_k = 0, \tag{3.3}$$

and $$\alpha^m B_j{}^k{}_i\, y^j y_k + \alpha\,(m+1)\beta^m \left(B_j{}^r{}_i\, b_r - \nabla_i b_j \right) y^j = 0. \tag{3.4}$$

On simplifying (3.3) and (3.4), we get

$$\left(B_j{}^r{}_i\, b_r - \nabla_i b_j \right) = 0.$$

Thus equation (3.4) $\Rightarrow \alpha^m B_j{}^k{}_i\, y^j y_k = 0$,

$$\Rightarrow B_j{}^k{}_i\, y^j y_k = 0,$$

$$\Rightarrow B_j{}^k{}_i = 0,$$

$$\Rightarrow \nabla_i b_j = 0.$$

Thus, by M. Hashiguchi and Y. Ichijyo [1], if $\nabla_i b_j = 0$ then the Finsler space with the metric (2.1) is a Berwald Finsler space.

Conversely, if $\nabla_i b_j = 0$ then $B_{j\,i}^{\,k} = 0$ are uniquely determined from (2.3) and hence a Finsler space F^n with an (α, β)-metric is a Berwald space.

Thus $$G_{j\,k}^{\,i} = \gamma_{j\,k}^{\,i} + B_{j\,k}^{\,i},$$

$$\Rightarrow G_{jk}^{i} = \gamma_{j\,k}^{\,i}.$$

Therefore, the Berwald connection is given by $(\gamma_{j\,k}^{\,i}\ ,\ \gamma_{0\,j}^{\,i}\ , 0)$. Hence the proof.

4 . VECTOR FIELD ON THE BERWALD FINSLER SPACE

In this section, we shall find the vector field on the Berwald Finsler space with an (α, β)-metric given by (2.1). The Berwald connection is determined by $B_{j\,k}^{\,i}$ in the equation (2.4) uniquely.

Theorem 4.2: Let F^n = (M^n, L(α, β)) be the Finsler space with an *(α, β)*-metric, *L(α, β)*= $\alpha + \frac{\beta^{m+1}}{\alpha^m}$, $\quad where\ m \neq 0\ , m \neq -1$. Then the vector field B^i(x, y) in (2.2) is given by (4.6) and (4.7), where the quantities D and E are determined from (4.8) and (4.9).

Proof: Consider the equation (3.2)

$$(\alpha^{m+1} - m\beta^{m+1})\, B_{j\,i}^{\,k}\, y^j y_k + \alpha^2(m+1)\beta^m(\, B_{j\,i}^{\,r}\, b_r - \nabla_i b_j)y^j = 0. \tag{4.1}$$

By homogeneity, (4.1) can be written as

$$(m+1)\alpha^2\, \beta^m\, \nabla_i b_j y^j = (\alpha^{m+1} - m\beta^{m+1})\, B_{j\,i}^{\,k}\, y^j y_k + (m+1)\alpha^2\, \beta^m\, B_{j\,i}^{\,k}\, b_k y^j.$$

Put $y_k = e_k\, \alpha$, $B_{j\,i}^{\,k}\, y^j = B_i^k$

$$(m+1)\alpha^2\, \beta^m\, \nabla_i b_j y^j = \{(\alpha^{m+1} - m\beta^{m+1})\alpha\, e_k + (m+1)\alpha^2\, \beta^m b_k\}B_i^k. \tag{4.2}$$

Transvecting (4.2) by y^i and by using homogeneity, we have

$$(m+1)\alpha^2\, \beta^m\, r_{00} = 2\,\{\, (\alpha^{m+1} - m\beta^{m+1})\alpha\, e_k + (m+1)\alpha^2\, \beta^m b_k\ \}\, B^k. \tag{4.3}$$

Now differentiating equation (4.3) by y^i and by the virtue of $\dot{\partial}_i\alpha = e_i$,

$\dot{\partial}_i e_k = (a_{ki} - e_i e_k)/\alpha$, we get

$$2(m+1)\alpha^2 \beta^m r_{0i} + (m+1)(2\alpha\, e_i\beta^m + m\,\alpha^2\beta^{m-1}\, b_i)r_{00}$$

$$= 2\{[(\alpha^{m+1} - m\beta^{m+1})\alpha\, e_k + (m+1)\alpha^2 \beta^m b_k]\, B^k_i$$

$$+ B^k[(\alpha^{m+1} - m\beta^{m+1})\,(a_{ki} - e_i e_k)$$

$$+ e_k\,((m+2)\,\alpha^{m+1}\, e_i - m(m+1)\beta^m\alpha\, b_i$$

$$-m\beta^{m+1}e_i) + b_k(m+1)(2\alpha\, e_i\beta^m + m\,\alpha^2\beta^{m-1}\, b_i)]\}\,. \qquad (4.4)$$

Since $r_{ij} = \frac{\nabla_j b_i + \nabla_i b_j}{2}$, $s_{ij} = \frac{\nabla_j b_i - \nabla_i b_j}{2}$ therefore put $\nabla_i b_j = r_{ij} - s_{ij}$, then the above equation can be written as

$$2\, a_{ki}B^k\,(\alpha^{m+1} - m\beta^{m+1})$$

$$= 2(m+1)\alpha^2 \beta^m r_{0i} + (m+1)(2\alpha\, e_i\beta^m + m\,\alpha^2\beta^{m-1}\, b_i)r_{00}$$

$$-2(m+1)\alpha^2 \beta^m (r_{ij} - s_{ij})\, y^j + 2\,(\alpha^{m+1} - m\beta^{m+1})e_k\, e_i\, B^k$$

$$-2\, e_k\, B^k\{(m+2)\,\alpha^{m+1}\, e_i - m(m+1)\beta^m\alpha\, b_i - m\beta^{m+1}e_i\}$$

$$-2(m+1)(2\alpha\, e_i\beta^m + m\,\alpha^2\beta^{m-1}\, b_i)\, b_k\, B^k,$$

which gives,

$$\Rightarrow 2\, a_{ki}B^k\,(\alpha^{m+1} - m\beta^{m+1})$$

$$= (m+1)\,(2\alpha e_i\beta^m + m\alpha^2\beta^{m-1}\, b_i)r_{00}$$

$$+ 2(m+1)\alpha^2\beta^m s_{i0} + 2\,(\alpha^{m+1} - m\beta^{m+1})e_k\, e_i B^k$$

$$-2\, e_k\, B^k\{(m+2)\,\alpha^{m+1}\, e_i - m(m+1)\beta^m\alpha b_i - m\,\beta^{m+1}e_i\}$$

$$-2(m+1)(2\alpha\, e_i\beta^m + m\,\alpha^2\beta^{m-1}\, b_i)\, b_k\, B^k. \qquad (4.5)$$

Put $e_k\,B^k$= E, $b_k B^k = D$, divide the equation (4.5) throughout by $2(\alpha^{m+1} - m\beta^{m+1})$ and contract with a^{ij} , we get

$$B^i = P_1 e^i + P_2\, s_0^i + P_3 b^i, \tag{4.6}$$

where

$$\left.\begin{aligned} P_1 &= E + \frac{\{(m+1)\alpha\beta^m\,(r_{00}-2D)+E(m\,\beta^{m+1}-(m+2)\,\alpha^{m+1})\}}{(\alpha^{m+1}-m\beta^{m+1})} \\ P_2 &= \frac{(m+1)\alpha^2\,\beta^m}{(\alpha^{m+1}-m\beta^{m+1})} \\ P_3 &= \frac{(m+1)m\alpha^2\beta^{m-1}\,(r_{00}-2D)+2m(m+1)\beta^m\alpha E}{2\,(\alpha^{m+1}-m\beta^{m+1})} \end{aligned}\right\} \tag{4.7}$$

Now to find E and D,

Consider the equation (4.3),

$$(m+1)\alpha^2\,\beta^m\, r_{00} = 2\ (\alpha^{m+1} - m\beta^{m+1})\alpha\, E + 2(m+1)\alpha^2\,\beta^m D, \tag{4.8}$$

by the virtue of $e_k\,B^k$ = E, $b_k B^k = D$, $b_i b^i = b^2$ and $e^i b_i = \beta/\alpha$.

Transvecting (4.6) by b_i , we get

$$\begin{aligned} r_{00}\{2(m+1)\beta^{m+1} + m(m+1)\alpha^2\,\beta^{m-1}\,b^2\} + 2(m+1)\alpha^2\,\beta^m s_0 &= E\{2(m+1)\alpha^m\,\beta - 2m(m+1)\alpha\beta^m\,b^2\} \\ &\quad + 2D\{\,\alpha^{m+1} + m(m+1)\alpha^2\,\beta^{m-1}b^2 \\ &\quad +(m+2)\beta^{m+1}\}. \end{aligned} \tag{4.9}$$

Thus, on solving (4.8) and (4.9), we get D and E. Hence the proof.

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