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Basic Reproductive Number and Bifurcation of infectious Disease Model with Carriers

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Abstract

In the present paper we consider an $S - I_c - I - R$ epidemic model of infectious disease that transmitted through carriers. Then we find out basic reproductive number for a compartmental disease transmission model based on a system of ODE. Also Bifurcation analysis is performed by applying the Sotomayer's theorem.

Keywords: Basic Reproductive Number. Sotomayer's Theorem. Transcritical Bifurcation. Saddle-node Bifurcation. Carriers. Infectious Disease.

1. INTRODUCTION

Study of epidemic modeling of infectious disease is an emergent field of current research. Carriers are those individuals who transmit the infectious disease but do not exhibit any symptom. There are two types of carriers one of them is genetic carrier and another is infectious disease carrier. Genetic carrier [7] transmit genetic diseases like diabetis,thalasamia etc. to their next generation by genetic combination. But the infectious disease[1-6] do not transmit genetically and they are exponent in nature. Initially they start their movements with in a small group of people and gradually spread. Some infectious diseases are like Typhoid, Tuberculosis,Cholera Hepatitis B, Epstein-Bar Virus(EBV), Clostridium difficile-associated disease (CDAD) etc.

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In the 20th century Typhoid, caused by Salomoneila Typhi, reached public noteriet. Even though in recent time its treatment and vaccination available but Typhoid infects 21 million people and kills 200000 world wide in every year. In the same way Hepatitis B, EBV etc, are transmitted and large outbreak caused by the viruses [1]. Although these types of infectious diseases are serious public health issues but the work on carrier transmission of such diseases have not received the sufficient attention by the researchers[1-8] in the field of epidemiological mathematical modeling. Modeling the effects of carriers on transmission dynamics of infectious diseases [7] was reported when the expression of basic reproductive number was obtained by usual method considering constant birth rate of total population . In this present paper we have found the basic reproductive number of a $S - I - R$ model including the effect of infectious disease carrier by compartmental method[10,11]. We have also investigated the transcritical and saddle-node bifurcation of the $S - I_c - I - R$ model with the help of the Sotomayer's theorem[12-15]. We have also derived a critical value of the probability that an infected individual remains carrier. Below the critical value the disease free equilibrium point is stable and unstable otherwise. This probability is thus considered as a bifurcation parameter. We identified certain parameter value based on which transcritical and saddle-node bifurcation have been investigated.This paper we have organized as follows: Section-1 describes the formulation of basic model and existence of the equilibrium points. In section-2 we find out basic reproductive number by compartmental method.In section-3 we investigate the condition of transcritical and saddle-node bifurcations and finally we summarize our results in section-4.

2. MATHEMATICAL FORMULATION

In the model considered in this paper we have divided the individuals into four compartment $S - I_c - I - R$ where S, I_c, I and R are susceptible, carrier, infected and recovered population and formulated the model under following assumptions :

1. The birth rate b of the susceptible population S is proportional to the total population N.
2. The susceptible individuals S can be infected by the contact with infectious individuals I or carriers I_c.
3. Infected individuals become carriers with probability p and infected with probability $(1 - p)$.
4. The rate of transmission from susceptible class to carrier β is higher than the rate transmission γ to infected class .

5. The carrier's transmission rate into the infected class is α.
6. The natural death rate of the population is μ.
7. The death rate for carrier and infected individuals due to infections individuals are d_c and d_i respectively.
8. The vaccination rate is θ i.e., susceptible population directly transfer to the recovery class R by the vaccination at this rate.
9. π is the recovery rate of infected individuals.

Considering the above assumption we formulate a model

$$\frac{dS}{dt} = bN - \mu S - \frac{S(\beta I_c + \gamma I)}{N} - \theta S \tag{2.1a}$$

$$\frac{dI_c}{dt} = p\frac{S(\beta I_c + \gamma I)}{N} - (\mu + d_c + \alpha)I_c \tag{2.1b}$$

$$\frac{dI}{dt} = (1-p)\frac{S(\beta I_c + \gamma I)}{N} - (\mu + d_i + \pi)I + \alpha I_c \tag{2.1c}$$

$$\frac{dR}{dt} = \pi I + \theta S - \mu R \tag{2.1d}$$

$$N = S + I_c + I + R$$

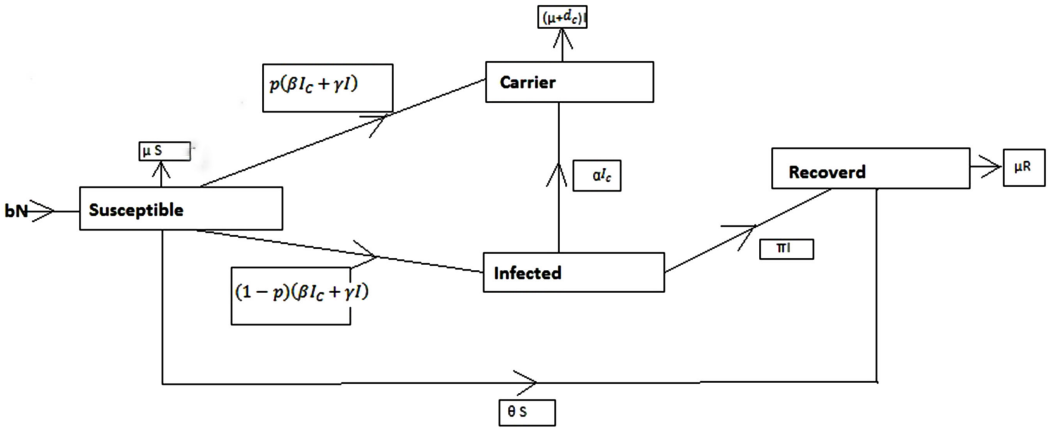

Figure 1: Schematic Diagram of the model

We express the model (2.1) in dimensionless form by choosing the variables as $x = \frac{S}{N}$, $y_c = \frac{I_c}{N}$,$y = \frac{I}{N}$ and $z = \frac{R}{N}$. Then we get

$$\frac{dN}{dt} = (b-\mu)N - (d_c I_c + d_i I) \tag{2.2}$$

$$\frac{dx}{dt} = b - (b+\theta)x - x(\beta y_c + \gamma y) + x(d_c y_c + d_i y) \tag{2.3a}$$

$$\frac{dy_c}{dt} = px(\beta y_c + \gamma y) - (b + d_c + \alpha)y_c + (d_c y_c + d_i y)y_c \tag{2.3b}$$

$$\frac{dy}{dt} = (1-p)x(\beta y_c + \gamma y) - (b + d_i + \pi)y + \alpha y_c + (d_c y_c + d_i y)y \tag{2.3c}$$

$$\frac{dz}{dt} = \pi y + \theta x - bz + (d_c y_c + d_i y)z \tag{2.3d}$$

and also

$$x + y_c + y + z = 1 \tag{2.4}$$

From (2.2) we get

$$\frac{1}{N}\frac{dN}{dt} = (b - \mu) - (d_c y_c + d_i y) \tag{2.5}$$

For a large population we choose that the rate of change of population as per population is equal to a constant(as we choose Malthusian growth rate here). i.e.,

$$\frac{1}{N}\frac{dN}{dt} = \epsilon \tag{2.6}$$

so from (2.2) we get $d_c y_c + d_i y = b - \mu - \epsilon$. From the system (2.3) we get using above relation

$$\frac{dx}{dt} = b - (\mu + \theta + \epsilon)x - x(\beta y_c + \gamma y) \tag{2.7a}$$

$$\frac{dy_c}{dt} = px(\beta y_c + \gamma y) - (d_c + \mu + \alpha + \epsilon)y_c \tag{2.7b}$$

$$\frac{dy}{dt} = (1-p)x(\beta y_c + \gamma y) - (d_i + \mu + \pi + \epsilon)y + \alpha y_c \tag{2.7c}$$

$$\frac{dz}{dt} = \pi y + \theta x - (\mu + \epsilon)z \tag{2.7d}$$

The equation (2.7d) is redundant as z can be determined from (2.4).

Lemma: Let $\mho = \{(x, y_c, y) : \ x,\ y_c,\ y \geqslant 0,\ \ 0 \leqslant x \leqslant \frac{b}{\mu+\theta+\epsilon},\ \ x + y_c + y \leqslant 1\}$. Then $\mho$ is a positive invariant under the flow induced equation (2.7).

Proof: From the (2.7a) we get,

$$\frac{dx}{dt} = b - (\mu + \theta + \epsilon)x - x(\beta y_c + \gamma y) \tag{2.8}$$

$$Hence, \quad \frac{dx}{dt} \leqslant b - (\mu + \theta + \epsilon)x \tag{2.9}$$

consequently $\limsup_{t\longrightarrow\infty} x(t) \leqslant \frac{b}{(\mu+\theta+\epsilon)}$

The relation (2.4) implies,

$x + y_c + y \leqslant 1$, since $z \geqslant 0$.

Therefore, $\mho$ is positive invariant.

Theorem For every positive solution (x, y_c, y), the system (2.7) is uniformly persistent.
Proof: From the equation (2.7a) we get,
$\frac{dx}{dt} \geqslant -(\mu + \theta + \epsilon)$
which clearly implies that $\limsup_{t\longrightarrow\infty} x(t) \geqslant x(0) \geqslant 0$.
Similarly from the equations (2.7b)and (2.7c) we can show that
$\limsup_{t\longrightarrow\infty} y_c(t) \geqslant y_c(0) \geqslant 0$ and $\limsup_{t\longrightarrow\infty} y(t) \geqslant y(0) \geqslant 0$.
Hence the system is uniformly persistent for every positive solution.

Equilibrium Points:
The system (2.7) has one axial and one interior equilibrium points exist.
$E_1 : (x_0, 0, 0) = (\frac{b}{(\mu+\theta+\epsilon)}, 0, 0)$
$E^* = (x^*, y_c^*, y^*)$
where $x^* = \frac{(d_c+\mu+\alpha+\epsilon)(d_i+\mu+\pi+\epsilon)}{\beta p(d_i+\mu+\pi+\epsilon)+\gamma[(1-p)(d_c+\mu+\alpha+\epsilon)+\alpha p]}$
$y_c^* = \frac{p(b-(\mu+\alpha+\epsilon)x^*)}{(d_c+\mu+\alpha+\epsilon)}$
$y^* = \frac{(1-p)(b-(\mu+\alpha+\epsilon)x^*)-\alpha y_c^*}{(d_i+\mu+\pi+\epsilon)}$.
Therefore the interior equilibrium point exist if $x^* \leqslant \frac{b}{(\mu+\alpha+\epsilon)}$ and $p \leqslant \frac{(d_c+\mu+\alpha+\epsilon)}{(d_c+\mu+2\alpha+\epsilon)}$

3. BASIC REPRODUCTIVE NUMBER

The basic reproductive number (R_0) represents the average number of secondary infections caused by a single infectious in an entirely susceptible population during entire infection period. If $R_0 < 1$, the number of infectives caused by a single infective less than 1 . Then the infectious disease gradually dies out from the population . On the other hand if $R_0 > 1$, the number of infectives caused by a single infective is greater than 1. Then there is a large outbreak of the infectious disease. In this paper we shall find out the basic reproductive number by the general compartment method[10] and also using the notation of P.van den Driessche and James Watmough[10] . The matrices $\Im$ and ν are respectively the rate of appearance of new infections and the rate of transfer from the group of infections. Also the matrices U and V stand for the new infection terms and the remaining terms respectively. The basic reproductive number is found by $R_0 = \rho(UV^{-1})$ where $\rho(UV^{-1})$ is the spectral radius of the matrix UV^{-1}. We recast

the system of equations (2.7) as

$$\frac{dx}{dt} = b - (\mu + \theta + \epsilon)x - x(\beta y_c + \gamma y) = f_1 \tag{3.1a}$$

$$\frac{dy_c}{dt} = px(\beta y_c + \gamma y) - (d_c + \mu + \alpha + \epsilon)y_c = f_2 \tag{3.1b}$$

$$\frac{dy}{dt} = (1-p)x(\beta y_c + \gamma y) - (d_i + \mu + \pi + \epsilon)y + \alpha y_c = f_3 \tag{3.1c}$$

$$\frac{dz}{dt} = \pi y + \theta x - (\mu + \epsilon)z = f_4 \tag{3.1d}$$

At the initial time we set $E_1 = (x^0, y_c^0, y^0, z^0) = (\frac{b}{(\mu+\theta+\epsilon)}, 0, 0, 0)$ is the equilibrium point.

If the above system of equations are rearranged as with non-negative initial conditions

$$\frac{dy_c}{dt} = px(\beta y_c + \gamma y) - (d_c + \mu + \alpha + \epsilon)y_c = f_2 \tag{3.2a}$$

$$\frac{dy}{dt} = (1-p)x(\beta y_c + \gamma y) - (d_i + \mu + \pi + \epsilon)y + \alpha y_c = f_3 \tag{3.2b}$$

$$\frac{dx}{dt} = b - (\mu + \theta + \epsilon)x - x(\beta y_c + \gamma y) = f_1 \tag{3.2c}$$

$$\frac{dz}{dt} = \pi y + \theta x - (\mu + \epsilon)z = f_4 \tag{3.2d}$$

The corresponding equilibrium will be $E_1 = (0, 0, \frac{b}{(\mu+\theta+\epsilon)}, 0)$ We construct two matrices $\Im$,ν in such a way that $(f_2, f_3, f_1, f_4)^T$ can be written as $\Im - \nu$ [9,10] where the infected compartment are y_c and y . An equilibrium solution with $y_c = y = 0$ has the form $E_1 = (0, 0, x^0, 0)$,where x^0 is a positive quantity. Using the notation [9,10], the matrices U and V, for the new infection terms and the remaining transfer terms respectively, are given by.

$$U = \begin{pmatrix} p\beta x^0 & p\gamma x^0 \\ (1-p)\beta x^0 & (1-p)\gamma x^0 \end{pmatrix} \tag{3.3}$$

and

$$V = \begin{pmatrix} (d_c + \mu + \alpha + \epsilon) & 0 \\ -\alpha & (d_i + \pi + \mu + \epsilon) \end{pmatrix} \tag{3.4}$$

Then we calculate

$$V^{-1} = \frac{1}{(d_c + \mu + \alpha + \epsilon)(d_i + \pi + \mu + \epsilon)} \begin{pmatrix} (d_i + \pi + \mu + \epsilon) & 0 \\ \alpha & (d_c + \mu + \alpha + \epsilon) \end{pmatrix} \tag{3.5}$$

and,

$$UV^{-1} = \frac{x^0}{(d_c+\mu+\alpha+\epsilon)(d_i+\pi+\mu+\epsilon)} \begin{pmatrix} p\beta & p\gamma \\ (1-p)\beta & (1-p)\gamma \end{pmatrix} \begin{pmatrix} (d_i+\pi+\mu+\epsilon) & 0 \\ \alpha & (d_c+\mu+\alpha+\epsilon) \end{pmatrix}$$

$$= \frac{x^0}{(d_c+\mu+\alpha+\epsilon)(d_i+\pi+\mu+\epsilon)} \begin{pmatrix} p\beta(d_i+\pi+\mu+\epsilon)+p\alpha\gamma & p\gamma(d_c+\mu+\alpha+\epsilon) \\ (1-p)\beta(d_i+\pi+\mu+\epsilon)+p\alpha\gamma & (1-p)\gamma(d_c+\mu+\alpha+\epsilon) \end{pmatrix} \quad (3.6)$$

Then the basic reproductive number R_0 which is the spectral density of the Matrix UV^{-1} is obtained as

$$R_0 = \frac{x^0}{(d_c+\mu+\alpha+\epsilon)(d_i+\pi+\mu+\epsilon)}[p\beta(d_i+\pi+\mu+\epsilon)+(1-p)(d_c+\mu+\alpha+\epsilon)+p\alpha\gamma] \quad (3.7)$$

We see that $R_0 = 1$ is the critical value at which the system changes its behaviour. Since $R_0 = 1$ implies

$$p = \frac{(d_c+\mu+\alpha+\epsilon)(d_i+\pi+\mu+\epsilon)-\gamma(d_c+\mu+\alpha+\epsilon)x^0}{x^0[\beta d_i+\pi+\mu+\epsilon)-\gamma(d_c+\mu+\epsilon)]} = p^*(say) \quad (3.8)$$

we may consider p as a critical parameter of the system under consideration.

4. STABILITY AND BIFURCATION ANALYSIS

Bifurcation analysis is performed on the basis of the Sotomayor's theorem[12,15]. Under certain condition the system exhibit the different kind of bifurcation. The jacobian matrix of the system(2.7) is given by

$$J(E,p) = \begin{pmatrix} -(\mu+\theta+\epsilon)-(\beta y_c+\gamma y) & -\beta x & -\gamma x \\ p(\beta y_c+\gamma y) & p\beta x-(d_c+\mu+\alpha+\epsilon) & p\gamma x \\ (1-p)\beta y_c+\gamma y) & (1-p)\beta x+\alpha & (1-p)\gamma x-(d_i+\pi+\mu+\epsilon) \end{pmatrix} \quad (4.1)$$

Then the equilibrium point E_1 is locally asymptotically stable if $p < p^*$, otherwise it is unstable. For $p < p^*$ the endemic equilibrium point E^* is globally asymptotically stable [7]. So $p = p^*$ is a bifurcation point for the equilibrium point E_1. At the equilibrium

point E_1 jacobian matrix is,

$$J(E_1, p^*) = \begin{pmatrix} -(\mu+\theta+\epsilon) & -\beta x^0 & -\gamma x^0 \\ 0 & p^*\beta x^0 - (d_c+\mu+\alpha+\epsilon) & p^*\gamma x^0 \\ (0 & (1-p^*)\beta x^0 + \alpha & (1-p^*)\gamma x^0 - (d_i+\pi+\mu+\epsilon) \end{pmatrix} \tag{4.2}$$

The eigenvalues of this jacobian matrix are

$$\lambda_1 = -(\mu+\theta+\epsilon) \tag{4.3}$$

$$\lambda_2 = \{p^*\beta + (1-p^*)\gamma\}x^0 - (d_c+\mu+\alpha+\epsilon) - (d_i+\pi+\mu+\epsilon) \tag{4.4}$$

$$\lambda_3 = 0 \tag{4.5}$$

Now $\lambda_2 < 0$ if $p^* < \frac{(d_c+\mu+\alpha+\epsilon)+(d_i+\pi+\mu+\epsilon)-\gamma x^0}{(\beta-\gamma)x^0}$.
Thus $\lambda_3 = 0$ is a simple zero eigenvalue and the other two eigenvalues are real and negative under certain condition. So for $p = p^*$ the disease free equilibrium E_1 is a hyperbolic equilibrium point. Then we find out the left eigen-vector and right eigen-vector corresponding to the eigenvalue $\lambda_3 = 0$. Then we use the Sotomayor's theorem [12,15] for finding the local bifurcation under certain conditions.

Theorem The system (2.7) near the axial equilibrium point E_1 with the parameter p has no saddle -node bifurcation but transcritical bifurcation arises if $x^0 \neq \frac{(d_c+\mu+\epsilon)}{\beta}$.
Proof: Let $w = (w_1, w_2, w_3)^T$ be the right eigen vector of the jacobian matrix (4.2) of the system (2.7).Then

$$(\mu+\theta+\epsilon)w_1 + \beta x^0 w_2 + \gamma x^0 w_3 = 0 \tag{4.6a}$$

$$(p^*\beta x^0 - (d_c+\mu+\alpha+\epsilon))w_2 + p^*\gamma x^0 w_3 = 0 \tag{4.6b}$$

$$(1-p^*)\beta x^0 + \alpha)w_2 + ((1-p^*)\gamma x^0 - (d_i+\pi+\mu+\epsilon))w_3 = 0 \tag{4.6c}$$

Solving the equations (4.6) we get , if $w_3 = 1$ then $w_2 = -\frac{\gamma p^* x^0}{p^*\beta x^0 - (d_c+\mu+\alpha+\epsilon)}$
$w_1 = \frac{\gamma x^0(d_c+\mu+\alpha+\epsilon)}{(p^*\beta x^0 - (d_c+\mu+\alpha+\epsilon))(\mu+\theta+\epsilon)}$
Let $v = (v_1, v_2, v_3)$ be the left eigen vector of the jacobian matrix (4.2) of the system (2.7). Then

$$(\mu+\theta+\epsilon)v_1 = 0 \tag{4.7a}$$

$$-\beta x^0 v_1 + (p^*\beta x^0 - (d_c+\mu+\alpha+\epsilon))v_2 + (1-p^*)\beta x^0 + \alpha)v_3 = 0 \tag{4.7b}$$

$$-\gamma x^0 v_1 + p^*\gamma x^0 v_2 + ((1-p^*)\gamma x^0 - (d_i+\pi+\mu+\epsilon))v_3 = 0 \tag{4.7c}$$

Solving (4.7) we get $v_1 = 0 \ , \ if \ \ v_3 = 1 \ \ then \ \ v_2 = -\frac{(1-p^*)\beta x^0+\alpha)}{(p^*\beta x^0-(d_c+\mu+\alpha+\epsilon))}$.
Now we choose the column matrix F from (3.1)

$$F = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \tag{4.8}$$

then

$$F_p = \begin{pmatrix} 0 \\ x(\beta y_c + \gamma y) \\ -x(\beta y_c + \gamma y) \end{pmatrix} \tag{4.9}$$

then

$$F_p(E_1, p^*) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{4.10}$$

Now since $v^T.F_p(E_1, p^*) = 0$
Thus, according to Sotomayer's theorem [12,15] for local bifurcation, the saddle-node bifurcation can't occur. While the first condition of transcritical bifurcation is satisfied. Now

$$DF_p(E_1, p^*) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta & \gamma \\ 0 & -\beta & -\gamma \end{pmatrix} \tag{4.11}$$

Then we calculate

$$v^T.[DF_p(E_1, p^*).w] = x^0 \frac{((1-p^*)\beta x^0 + \gamma(d_c+\mu+\alpha+\epsilon))(\beta x^0 - (d_c+\mu+\epsilon))}{(p^*\beta x^0 - (d_c+\mu+\alpha+\epsilon))^2} \tag{4.12}$$

The above expression (4.12) not equal to zero provided $x^0 \neq \frac{(d_c+\mu+\epsilon)}{\beta}$. Then the partial derivative of the matrix F with respect to $x \ , y_c \ and \ y$ respectively we get

$$F_x = \begin{pmatrix} -(\mu+\theta+\epsilon) - (\beta y_c + \gamma y) \\ p(\beta y_c + \gamma y) \\ -(1-p)(\beta y_c + \gamma y) \end{pmatrix} \tag{4.13}$$

$$F_{y_c} = \begin{pmatrix} -\beta x \\ px\beta - (d_c+\mu+\alpha+\epsilon) \\ -(1-p)\beta x + \alpha) \end{pmatrix} \tag{4.14}$$

and

$$F_y = \begin{pmatrix} -\gamma x \\ px\gamma \\ -(1-p)\gamma x(d_i+\pi+\mu+\epsilon)) \end{pmatrix} \tag{4.15}$$

Then we calculate

$$(D^2F)(w,w) = 2w_1(\beta w_2 + \gamma w_3)\begin{pmatrix} -1 \\ p^* \\ (1-p^*) \end{pmatrix} \tag{4.16}$$

Therefore, $v^T[(D^2F)(w,w)] = 2w_1(\beta w_2 + \gamma w_3)[p^* v_2 + (1-p^*)]$

$= w_1 \frac{(d_c+\mu+\alpha+\epsilon)((1-p^*)(d_c+\mu+\alpha+\epsilon)+p^*\alpha)}{(p^*\beta x^0-(d_c+\mu+\alpha+\epsilon))^2} \neq 0$

Thus the system has a transcritical bifurcation at E_1 when the parameter passes through the bifurcation value p^*.

5. CONCLUSION

In the present chapter we first formulate an $S - I_c - I - R$ model based on different kind of infectious diseases under certain conditions. We see that every positive solution of the system is uniformly persistent on some domain $\mho = \{(x, y_c, y) : x,\ y_c,\ y \geqslant 0,\quad 0 \leqslant x \leqslant \frac{b}{\mu+\theta+\epsilon},\quad x + y_c + y \leqslant 1\}$ which is also a positive invariant under the flow induced system. Then we find out the basic reproductive number $R_0 = \frac{x^0}{(d_c+\mu+\alpha+\epsilon)(d_i+\pi+\mu+\epsilon)}[p\beta(d_i + \pi + \mu + \epsilon) + (1-p)(d_c + \mu + \alpha + \epsilon) + p\alpha\gamma]$ by general compartmental method. We also get a parameter p such that if the parameter passes through the point $p^* = \frac{(d_c+\mu+\alpha+\epsilon)(d_i+\pi+\mu+\epsilon)-\gamma(d_c+\mu+\alpha+\epsilon)x^0}{x^0[\beta d_i+\pi+\mu+\epsilon)-\gamma(d_c+\mu+\epsilon)]}$ then the Sotomayer's theorem states that the system has transcritical bifurcation but no saddle-node bifurcation under condition $x^0 \neq \frac{(d_c+\mu+\epsilon)}{\beta}$.

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