Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 16, Number 2 (2020), pp. 167-186

http://www.ripublication.com/gjpam.htm

QUASI BI-NORMAL BI-MATRIX

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Abstract

In this paper, the concept of quasi Bi-normal Bi-matrices as a generalization of quasi normal matrices and bimatrices are studied. Some of the properties of quasi normal matrices are extended to quasi Bi-normal Bimatrices. Also, some results of quasi bi-normal bimatrices are obtained.

Keywords: Normal Bi-matrix, Quasi Normal Matrix, Quasi Bi-Normal Bi-Matrix.

AMS Classification: 15A09,15A12, 15A15, 15B99.

1. Introduction

In 1918, the concept of a normal matrix with entries from complex field was introduced by Toeplitz [2] who gave a necessary and sufficient condition for a complex matrix to be normal. A matrix A is said to be normal if $AA^* = A^*A$, that is a normal matrix is one which commutes with its conjugate transpose.

In [1], K.Morita has defined the concept of quasi normal matrix, that is, a complex matrix A which is such that $AA^{CT} = A^TA^C$.

In [3,4], W.B.Vasantha kandasamy et.al., introduced the notion of bimatrices that is bimatrix is of the form $A_B = A_1 \cup A_2$ ('$\cup$' is not an operation only a symbol).

In this paper, the notion of quasi Bi-normal Bi-matrices is introduced. Also, some results of quasi Bi-normal Bi-matrices are discussed.

2. Quasi-bi-Normal-bi-Matrix

In this section, we have to develop the concept of quasi Bi-normal Bi-matrices as a generalization of quasi normal matrices. Some results of quasi normal matrices are extended to quasi Bi-normal Bi-matrices.

Definition 2.1

$$\left[A_B A_B^{CT}\right]\left[A_B^{CT} A_B\right]=\left[A_B^{CT} A_B\right]\left[A_B A_B^{CT}\right]$$

Where $A_B^{CT}=A_B^{-T}=A_B^{*}$

$$A_B^{CT} A_B = A_B^C A_B^T = A_B^T A_B^C = A_B A_B^{CT}$$

$$\left[A_B A_B^{*}\right]\left[A_B^{*} A_B\right]=\left[A_B^{T} A_B^{C}\right]\left[A_B^{C} A_B^{T}\right]$$

$$\Rightarrow\left[(A_1\cup A_2)(A_1\cup A_2)^{*}\right]\left[(A_1\cup A_2)^{*}(A_1\cup A_2)\right]=\left[(A_1\cup A_2)^{T}(A_1\cup A_2)^{C}\right]\left[(A_1\cup A_2)^{C}(A_1\cup A_2)^{T}\right]$$

$$\Rightarrow\left[(A_1\cup A_2)(A_1^{*}\cup A_2^{*})\right]\left[(A_1^{*}\cup A_2^{*})(A_1\cup A_2)\right]=\left[(A_1^{T}\cup A_2^{T})(A_1^{C}\cup A_2^{C})\right]\left[(A_1^{C}\cup A_2^{C})(A_1^{T}\cup A_2^{T})\right]$$

$$\Rightarrow\left[A_1A_1^{*}A_1^{*}A_1\right]\cup\left[A_2A_2^{*}A_2^{*}A_2\right]=\left[A_1^{T}A_1^{C}A_1^{C}A_1^{T}\right]\cup\left[A_2^{T}A_2^{C}A_2^{C}A_2^{T}\right]$$

Now,

$$A_1A_1^{*}A_1^{*}A_1 = A_1^{T}A_1^{C}A_1^{C}A_1^{T}$$

$$A_2A_2^{*}A_2^{*}A_2 = A_2^{T}A_2^{C}A_2^{C}A_2^{T}$$

Example 2.2

$$A_1=\begin{bmatrix}1 & -i & 1+i\\ i & 1 & 1-i\\ 1-i & 1+i & 0\end{bmatrix},\ A_2=\begin{bmatrix}1 & -2i & 1+2i\\ 2i & 1 & 1-2i\\ 1-2i & 1+2i & 0\end{bmatrix},$$

We find that A_1^{*}, A_2^{*}

$$\overline{A_1}=\begin{bmatrix}1 & i & 1-i\\ -i & 1 & 1+i\\ 1+i & 1-i & 0\end{bmatrix}\quad \overline{A_2}=\begin{bmatrix}1 & 2i & 1-2i\\ -2i & 1 & 1+2i\\ 1+2i & 1-2i & 0\end{bmatrix}$$

$$\overline{A_1}^{T}=\begin{bmatrix}1 & -i & 1+i\\ i & 1 & 1-i\\ 1-i & 1+i & 0\end{bmatrix}=A_1^{*}\qquad \overline{A_2}^{T}=\begin{bmatrix}1 & -2i & 1+2i\\ 2i & 1 & 1-2i\\ 1-2i & 1+2i & 0\end{bmatrix}=A_2^{*}$$

$$A_1^{\mathrm{T}} = \begin{bmatrix} 1 & i & 1-i \\ -i & 1 & 1+i \\ 1+i & 1-i & 0 \end{bmatrix} \quad A_2^{\mathrm{T}} = \begin{bmatrix} 1 & 2i & 1-2i \\ -2i & 1 & 1+2i \\ 1+2i & 1-2i & 0 \end{bmatrix}$$

Quasi-bi-Normal-bi-matrix Example for A1 matrix

$$\left[A_1A_1^*A_1^*A_1\right] \cup \left[A_2A_2^*A_2^*A_2\right] = \left[A_1^{\mathrm{T}}A_1^{C}A_1^{C}A_1^{\mathrm{T}}\right] \cup \left[A_2^{\mathrm{T}}A_2^{C}A_2^{C}A_2^{\mathrm{T}}\right]$$

$$\left[A_1A_1^*A_1^*A_1\right] = \left[A_1^{\mathrm{T}}A_1^{C}A_1^{C}A_1^{\mathrm{T}}\right]$$

$$\left[A_2A_2^*A_2^*A_2\right] = \left[A_2^{\mathrm{T}}A_2^{C}A_2^{C}A_2^{\mathrm{T}}\right]$$

L.H.S

$$A_1A_1^*A_1^*A_1 \Rightarrow \begin{bmatrix} 1 & -i & 1+i \\ i & 1 & 1-i \\ 1-i & 1+i & 0 \end{bmatrix}\begin{bmatrix} 1 & -i & 1+i \\ i & 1 & 1-i \\ 1-i & 1+i & 0 \end{bmatrix}\begin{bmatrix} 1 & -i & 1+i \\ i & 1 & 1-i \\ 1-i & 1+i & 0 \end{bmatrix}\begin{bmatrix} 1 & -i & 1+i \\ i & 1 & 1-i \\ 1-i & 1+i & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 16 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 16 \end{bmatrix} \to (1)$$

R.H.S

$$A_1^{\mathrm{T}}A_1^{C}A_1^{C}A_1^{\mathrm{T}}$$

$$\Rightarrow \begin{bmatrix} 1 & i & 1-i \\ -i & 1 & 1+i \\ 1+i & 1-i & 0 \end{bmatrix}\begin{bmatrix} 1 & i & 1-i \\ -i & 1 & 1+i \\ 1+i & 1-i & 0 \end{bmatrix}\begin{bmatrix} 1 & i & 1-i \\ -i & 1 & 1+i \\ 1+i & 1-i & 0 \end{bmatrix}\begin{bmatrix} 1 & i & 1-i \\ -i & 1 & 1+i \\ 1+i & 1-i & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 16 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 16 \end{bmatrix} \to (2)$$

L.H.S=R.H.S

Quasi-bi-Normal-bi-Matrix A2 Matrix

$$A_2 = \begin{bmatrix} 1 & -2i & 1+2i \\ 2i & 1 & 1-2i \\ 1-2i & 1+2i & 0 \end{bmatrix}$$

$$A_2A_2^*A_2^*A_2 = A_2^{\mathrm{T}}A_2^{C}A_2^{C}A_2^{\mathrm{T}}$$

L.H.S

$$A_2 A_2^* A_2^* A_2 \Rightarrow \begin{bmatrix} 1 & -2i & 1+2i \\ 2i & 1 & 1-2i \\ 1-2i & 1+2i & 0 \end{bmatrix}\begin{bmatrix} 1 & -2i & 1+2i \\ 2i & 1 & 1-2i \\ 1-2i & 1+2i & 0 \end{bmatrix}\begin{bmatrix} 1 & -2i & 1+2i \\ 2i & 1 & 1-2i \\ 1-2i & 1+2i & 0 \end{bmatrix}\begin{bmatrix} 1 & -2i & 1+2i \\ 2i & 1 & 1-2i \\ 1-2i & 1+2i & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (7-12i-52i^2+7+26i-14i-52i^2) & (-14i-6-26i+7+26i+14i+52i^2) & (7+14i-6-26i+12i+52i^2) \\ (-6+26i+14i+7-26i-14i+52i^2) & (12i-52i^2+7+7-26i+14i-52i^2) & (-6+26i-12i+52i^2+7-14i) \\ (7-26i+14i+52i^2-6+12i) & (-14i+52i^2+7+26i-6-12i) & (7-26i+14i-52i^2+7+26i-14i-52i^2) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 118 & -51 & -51 \\ -51 & 118 & -51 \\ -51 & -51 & 118 \end{bmatrix} \rightarrow (1)$$

R.H.S

$$A_2^{\mathrm{T}} A_2^C A_2^C A_2^{\mathrm{T}} \Rightarrow \begin{bmatrix} 1 & 2i & 1-2i \\ -2i & 1 & 1+2i \\ 1+2i & 1-2i & 0 \end{bmatrix}\begin{bmatrix} 1 & 2i & 1-2i \\ -2i & 1 & 1+2i \\ 1+2i & 1-2i & 0 \end{bmatrix}\begin{bmatrix} 1 & 2i & 1-2i \\ -2i & 1 & 1+2i \\ 1+2i & 1-2i & 0 \end{bmatrix}\begin{bmatrix} 1 & 2i & 1-2i \\ -2i & 1 & 1+2i \\ 1+2i & 1-2i & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} (7+12i-52i^2+7-26i+14i-52i^2) & (14i-6+26i-14i+52i^2) & (7-14i-6+26i-12i+52i^2) \\ (-6-26i-14i+7+26i+14i+52i^2) & (-12i-52i^2+7+7+26i-14i-52i^2) & (-6-26i+12i+52i^2+7+14i) \\ (7+26i-14i+52i^2-6+12i) & (14i+52i^2+7-26i-6+12i) & (7+26i-14i-52i^2+7-26i+14i-52i^2) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 118 & -51 & -51 \\ -51 & 118 & -51 \\ -51 & -51 & 118 \end{bmatrix} \rightarrow (2)$$

L.H.S=R.H.S

(1)=(2)

Theorem 2.3

If A_B is Quasi-bi-Normal-bi-Matrix then $\left[A_B^{C\mathrm{T}}\right]^{\mathrm{T}}$ is Quasi-bi-Normal-bi-Matrix.

Proof

Given A_B is Quasi-bi-Normal-bi-Matrix.

We have to prove that $\left[A_B^{C\mathrm{T}}\right]^{\mathrm{T}}$ is Quasi-bi-Normal-bi-Matrix

$$\left[\left(A_B^{C\mathrm{T}}\right)^{\mathrm{T}} A_B^{\mathrm{T}}\right]\left[A_B^{\mathrm{T}}\left(A_B^{C\mathrm{T}}\right)^{\mathrm{T}}\right] = \left[\left(A_B^{\mathrm{T}}\right)^{\mathrm{T}}\left(A_B^{\mathrm{T}}\right)^{C}\right]\left[\left(A_B^{\mathrm{T}}\right)^{C}\left(A_B^{\mathrm{T}}\right)^{\mathrm{T}}\right]$$

$$\Rightarrow\left[\left(A_1\cup A_2\right)^{CT}\right]^{T}\left[\left(A_1\cup A_2\right)^{T}\right]\left[\left(A_1\cup A_2\right)^{T}\left[\left(A_1\cup A_2\right)^{CT}\right]^{T}\right]=\left[\left(A_1\cup A_2\right)^{T}\right]^{T}\left[\left(A_1\cup A_2\right)^{T}\right]^{C}$$
$$\left[\left(A_1\cup A_2\right)^{T}\right]^{C}\left[\left(A_1\cup A_2\right)^{T}\right]^{T}$$

L.H.S

$$\Rightarrow\left[A_1^{CT}\cup A_2^{CT}\right]^{T}\left[A_1^{T}\cup A_2^{T}\right]\left[A_1^{T}\cup A_2^{T}\right]\left[A_1^{CT}\cup A_2^{CT}\right]^{T}$$

$$\Rightarrow\left[A_1^{C}\cup A_2^{C}\right]\left[A_1^{T}\cup A_2^{T}\right]\left[A_1^{T}\cup A_2^{T}\right]\left[A_1^{C}\cup A_2^{C}\right]$$

$$\Rightarrow\left[A_1^{C}A_1^{T}A_1^{T}A_1^{C}\right]\cup\left[A_2^{C}A_2^{T}A_2^{T}A_2^{C}\right]$$

$$\Rightarrow\left[\left(A_1^{CT}\right)^{T}A_1^{T}A_1^{T}\left(A_1^{CT}\right)^{T}\right]^{T}\cup\left[\left(A_2^{CT}\right)^{T}A_2^{T}A_2^{T}\left(A_2^{CT}\right)^{T}\right]^{T}\rightarrow(1)$$

R.H.S

$$\Rightarrow\left[\left(A_1^{T}\cup A_2^{T}\right)\right]^{T}\left[\left(A_1^{T}\cup A_2^{T}\right)\right]^{C}\left[\left(A_1^{T}\cup A_2^{T}\right)\right]^{C}\left[\left(A_1^{T}\cup A_2^{T}\right)\right]^{T}$$

$$\Rightarrow\left[\left(A_1^{T}\right)^{T}\cup\left(A_2^{T}\right)^{T}\right]\left[A_1^{CT}\cup A_2^{CT}\right]\left[A_1^{CT}\cup A_2^{CT}\right]\left[A_1^{T}\cup A_2^{T}\right]^{T}$$

$$\Rightarrow\left[A_1\cup A_2\right]\left[A_1^{CT}\cup A_2^{CT}\right]\left[A_1^{CT}\cup A_2^{CT}\right]\left[A_1\cup A_2\right]$$

$$\Rightarrow\left[A_1\cup A_2\right]\left[A_1^{CT}\cup A_2^{CT}\right]\left[A_1^{CT}\cup A_2^{CT}\right]\left[A_1\cup A_2\right]$$

$$\Rightarrow\left[A_1A_1^{CT}A_1^{CT}A_1\right]\cup\left[A_2A_2^{CT}A_2^{CT}A_2\right]$$

$$\Rightarrow\left[\left(A_1^{T}\right)^{T}A_1^{CT}A_1^{CT}\left(A_1^{T}\right)^{T}\right]\cup\left[\left(A_2^{T}\right)^{T}A_2^{CT}A_2^{CT}\left(A_2^{T}\right)^{T}\right]^{T}\rightarrow(2)$$

Where

$$\left[A_1^{C}A_1^{T}A_1^{T}A_1^{C}\right]\cup\left[A_2^{C}A_2^{T}A_2^{T}A_2^{C}\right]=\left[A_1A_1^{CT}A_1^{CT}A_1\right]\cup\left[A_2A_2^{CT}A_2^{CT}A_2\right]$$

$$\Rightarrow\left[A_1A_1^{*}A_1^{*}A_1\right]\cup\left[A_2A_2^{*}A_2^{*}A_2\right]=\left[A_1^{C}A_1^{T}A_1^{T}A_1^{C}\right]\cup\left[A_2^{C}A_2^{T}A_2^{T}A_2^{C}\right]$$

$$\Rightarrow\left[A_1A_1^{*}A_1^{*}A_1\right]\cup\left[A_2A_2^{*}A_2^{*}A_2\right]=\left[A_1^{T}A_1^{C}A_1^{C}A_1^{T}\right]\cup\left[A_2^{T}A_2^{C}A_2^{C}A_2^{T}\right]$$

$\therefore\left[A_B^{CT}\right]^{T}$ is Quasi-bi-Normal-bi-Matrix.

Example 2.4

$$\left[\left(A_1^{CT}\right)^{T}A_1^{T}A_1^{T}\left(A_1^{CT}\right)^{T}\right]\cup\left[\left(A_2^{CT}\right)^{T}A_2^{T}A_2^{T}\left(A_2^{CT}\right)^{T}\right]=\left[\left(A_1^{T}\right)^{T}A_1^{CT}A_1^{CT}\left(A_1^{T}\right)^{T}\right]\cup\left[\left(A_2^{T}\right)^{T}A_2^{CT}A_2^{CT}\left(A_2^{T}\right)^{T}\right]$$

$$A_1=\begin{bmatrix}1 & 1+i\\ 1-i & -1\end{bmatrix}\qquad A_2=\begin{bmatrix}0 & 1+2i\\ 1-2i & 0\end{bmatrix}$$

$$A_1^{T} = \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix} \qquad A_2^{T} = \begin{bmatrix} 0 & 1-2i \\ 1+2i & 0 \end{bmatrix}$$

$$\overline{A_1} = \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix} \qquad \overline{A_2} = \begin{bmatrix} 0 & 1-2i \\ 1+2i & 0 \end{bmatrix}$$

$$\overline{A_1}^{T} = \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \qquad \overline{A_2}^{T} = \begin{bmatrix} 0 & 1+2i \\ 1-2i & 0 \end{bmatrix}$$

$$\left(A_1^{C^{T}}\right)^{T} = \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix} \qquad \left(A_2^{C^{T}}\right)^{T} = \begin{bmatrix} 0 & 1-2i \\ 1+2i & 0 \end{bmatrix}$$

L.H.S

$$\left[\left(A_1^{C^{T}}\right)^{T} A_1^{T} A_1^{T} \left(A_1^{C^{T}}\right)^{T}\right]$$

$$\Rightarrow \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}\begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}\begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}\begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3-3i \\ 3+3i & -3 \end{bmatrix}\begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} \left(3+3-3i+3i-3i^2\right) & \left(3-3i-3+3i\right) \\ \left(3+3i-3-3i\right) & \left(3+3i-3i-3i^2+3\right) \end{bmatrix}$$

$$\begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} \rightarrow (1)$$

$$\left[\left(A_1^{T}\right)^{T} A_1^{C^{T}} A_1^{C^{T}} \left(A_1^{T}\right)^{T}\right]$$

$$\Rightarrow \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} (1+2) & (1+i-1-i) \\ (1-i-1+i) & (2+1) \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 3+3i \\ 3-3i & -3 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3+3i \\ 3-3i & -3 \end{bmatrix}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 9 & 3+3i-3-3i \\ 3+3i-3+3i & 9 \end{bmatrix}$$

$$\begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} \rightarrow (2)$$

R.H.S

$$\left[\left(A_2^{C^T}\right)^T A_2^T A_2^T \left(A_2^{C^T}\right)^T\right]$$

$$\Rightarrow \begin{bmatrix} 0 & 1-2i \\ 1+2i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1-2i \\ 1+2i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1-2i \\ 1+2i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1-2i \\ 1+2i & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 0 & 1-2i \\ 1+2i & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 5-10i \\ 5+10i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1-2i \\ 1+2i & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix} \rightarrow (2)$$

$$\left[\left(A_2^T\right)^T A_2^{C^T} A_2^{C^T} \left(A_2^T\right)^T\right]$$

$$\Rightarrow \begin{bmatrix} 0 & 1+2i \\ 1-2i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1+2i \\ 1-2i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1+2i \\ 1-2i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1+2i \\ 1-2i & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 0 & 1+2i \\ 1-2i & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 5+10i \\ 5-10i & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 5+10i \\ 5-10i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1+2i \\ 1-2i & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix} \rightarrow (2)$$

$$\left[\left(A_1^{C^T}\right)^T A_1^T A_1^T \left(A_1^{C^T}\right)^T\right] \cup \left[\left(A_2^{C^T}\right)^T A_2^T A_2^T \left(A_2^{C^T}\right)^T\right] = \left[\left(A_1^T\right)^T A_1^{C^T} A_1^{C^T} \left(A_1^T\right)^T\right] \cup \left[\left(A_2^T\right)^T A_2^{C^T} A_2^{C^T} \left(A_2^T\right)^T\right]$$

$$\begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} \cup \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} \cup \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix}$$

Theorem 2.5

If A_B is Quasi-bi-Normal-bi-Matrix then $\overline{A}_B^T$ is Quasi-bi-Normal-bi-Matrix for any conjugate of transpose.

Proof

Given A_B is Quasi-bi-Normal-bi-Matrix.

We have to prove that, $\left[\overline{A}_B^T\right]$ is Quasi-bi-Normal-bi-Matrix for any conjugate of transpose.

$$\overline{A}_B^T \left(\overline{A}_B^T\right)^{-T} = \left[A_B^T A_B^-\right]^{-T}$$

$$\Rightarrow \left[\left(A_B^{-T}\right)\left(A_B^{-T}\right)^{-T}\right]\left[\left(A_B^{-T}\right)^{-T}\left(A_B^{-T}\right)\right] = \left[\left(A_B^T\right)^{-T}\left(A_B^-\right)^{-T}\right]\left[\left(A_B^-\right)^{-T}\left(A_B^T\right)^{-T}\right]$$

$$\Rightarrow\left[A_B^{-\mathrm{T}}A_B\right]\left[A_BA_B^{-\mathrm{T}}\right]=\left[\left(\overline{A}_B\right)\left(A_B\right)^{\mathrm{T}}\right]\left[\left(A_B\right)^{\mathrm{T}}\left(\overline{A}_B\right)\right]$$

$$\Rightarrow\left[\left(A_1\cup A_2\right)^{-\mathrm{T}}\left(A_1\cup A_2\right)\right]\left[\left(A_1\cup A_2\right)\left(A_1\cup A_2\right)^{-\mathrm{T}}\right]=\left[\left(\overline{A_1\cup A_2}\right)\left(A_1\cup A_2\right)^{\mathrm{T}}\right]\left[\left(A_1\cup A_2\right)^{\mathrm{T}}\left(\overline{A_1\cup A_2}\right)\right]$$

L.H.S

$$\Rightarrow\left[\left(A_1^{-\mathrm{T}}\cup A_2^{-\mathrm{T}}\right)\left(A_1\cup A_2\right)\right]\left[\left(A_1\cup A_2\right)\left(A_1^{-\mathrm{T}}\cup A_2^{-\mathrm{T}}\right)\right]$$

$$\Rightarrow\left[A_1^{-\mathrm{T}}A_1A_1A_1^{-\mathrm{T}}\right]\cup\left[A_2^{-\mathrm{T}}A_2A_2A_2^{-\mathrm{T}}\right]$$

Where $A^{-\mathrm{T}}=A^{C\mathrm{T}}=A^{*},\ A^{C\mathrm{T}}AAA^{C\mathrm{T}}=AA^{C\mathrm{T}}A^{C\mathrm{T}}A$

$$\Rightarrow\left[A_1^{C\mathrm{T}}A_1A_1A_1^{C\mathrm{T}}\right]\cup\left[A_2^{C\mathrm{T}}A_2A_2A_2^{C\mathrm{T}}\right]=\left[A_1A_1^{C\mathrm{T}}A_1^{C\mathrm{T}}A_1\right]\cup\left[A_2A_2^{C\mathrm{T}}A_2^{C\mathrm{T}}A_2\right]$$

L.H.S $\Rightarrow\left[A_1A_1^{*}A_1^{*}A_1\right]\cup\left[A_2A_2^{*}A_2^{*}A_2\right]\rightarrow(1)$

R.H.S

$$\Rightarrow\left[\left(\overline{A}_1\cup\overline{A}_2\right)\left(A_1^{\mathrm{T}}\cup A_2^{\mathrm{T}}\right)\right]\left[\left(A_1^{\mathrm{T}}\cup A_2^{\mathrm{T}}\right)\left(\overline{A}_1\cup\overline{A}_2\right)\right]$$

$$\left[\overline{A}_1A_1^{\mathrm{T}}A_1^{\mathrm{T}}\overline{A}_1\right]\cup\left[\overline{A}_2A_2^{\mathrm{T}}A_2^{\mathrm{T}}\overline{A}_2\right]$$

Where, $\overline{A}=A^{C}$, $\quad A^{C}A^{\mathrm{T}}A^{\mathrm{T}}A^{C}=A^{\mathrm{T}}A^{C}A^{C}A^{\mathrm{T}}$

R.H.S $\Rightarrow\left[A_1^{\mathrm{T}}A_1^{C}A_1^{C}A_1^{\mathrm{T}}\right]\cup\left[A_2^{\mathrm{T}}A_2^{C}A_2^{C}A_2^{\mathrm{T}}\right]\rightarrow(2)$

L.H.S=R.H.S

$$\left[A_1A_1^{*}A_1^{*}A_1\right]\cup\left[A_2A_2^{*}A_2^{*}A_2\right]=\left[A_1^{\mathrm{T}}A_1^{C}A_1^{C}A_1^{\mathrm{T}}\right]\cup\left[A_2^{\mathrm{T}}A_2^{C}A_2^{C}A_2^{\mathrm{T}}\right]$$

Example 2.6

$$\left[\left(A_B^{-\mathrm{T}}\right)\left(B_B^{-\mathrm{T}}\right)^{-\mathrm{T}}\right]\left[\left(A_B^{-\mathrm{T}}\right)^{-\mathrm{T}}\left(B_B^{-\mathrm{T}}\right)^{-\mathrm{T}}\right]=\left[\left(A_B^{\mathrm{T}}\right)^{-\mathrm{T}}\left(A_B^{-}\right)^{-\mathrm{T}}\right]\left[\left(A_B^{-}\right)^{-\mathrm{T}}\left(A_B^{\mathrm{T}}\right)^{-\mathrm{T}}\right]$$

$$\Rightarrow\left[\left[\left(A_1\cup A_2\right)^{-\mathrm{T}}\right]\left[\left(A_1\cup A_2\right)^{-\mathrm{T}}\right]^{-\mathrm{T}}\right]\left[\left[\left(A_1\cup A_2\right)^{-\mathrm{T}}\right]^{-\mathrm{T}}\left[\left(A_1\cup A_2\right)^{-\mathrm{T}}\right]\right]$$

$$=\left[\left(A_1\cup A_2\right)^{\mathrm{T}}\right]^{-\mathrm{T}}\left[\left(A_1\cup A_2\right)^{-}\right]^{-\mathrm{T}}\left[\left(A_1\cup A_2\right)^{-}\left(A_1\cup A_2\right)^{\mathrm{T}}\right]^{-\mathrm{T}}$$

$$\left[A_1^{-\mathrm{T}}A_1A_1A_1^{-\mathrm{T}}\right]\cup\left[A_2^{-\mathrm{T}}A_2A_2A_2^{-\mathrm{T}}\right]=\left[A_1^{-}A_1^{\mathrm{T}}A_1^{\mathrm{T}}A_1^{-}\right]\cup\left[A_2^{-}A_2^{\mathrm{T}}A_2^{\mathrm{T}}A_2^{-}\right]$$

$$A_1=\begin{bmatrix} i & 1\\ 1 & i\end{bmatrix} A_2=\begin{bmatrix} 1+i & 1-i\\ 1-i & 1+i\end{bmatrix}$$

$$A_1^-=\begin{bmatrix} -i & 1\\ 1 & -i\end{bmatrix} A_2^-=\begin{bmatrix} 1-i & 1+i\\ 1+i & 1-i\end{bmatrix}$$

$$A_2^{-\mathrm{T}}=\begin{bmatrix} -i & 1\\ 1 & -i\end{bmatrix} A_2^{-\mathrm{T}}=\begin{bmatrix} 1-i & 1+i\\ 1+i & 1-i\end{bmatrix}$$

L.H.S

$$A_1^{-\mathrm{T}}A_1A_1A_1^{-\mathrm{T}}$$

$$\Rightarrow\begin{bmatrix} -i & 1\\ 1 & -i\end{bmatrix}\begin{bmatrix} i & 1\\ 1 & i\end{bmatrix}\begin{bmatrix} i & 1\\ 1 & i\end{bmatrix}\begin{bmatrix} -i & 1\\ 1 & -i\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix} -i^2+1 & -i+i\\ i-i & 1-i^2\end{bmatrix}\begin{bmatrix} i & 1\\ 1 & i\end{bmatrix} \qquad \left\{\begin{matrix} \because A_1^{-\mathrm{T}}A_1A_1A_2^{-\mathrm{T}}=A_1^-A_1^{\mathrm{T}}A_1^{-\mathrm{T}}A_1\\ A_1^-A_1^{\mathrm{T}}A_1^{-\mathrm{T}}A_1^-=A_1^{\mathrm{T}}A_1^-A_1^-A_1^{\mathrm{T}}\end{matrix}\right\}$$

$$\begin{bmatrix} 2 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix} i & 1\\ 1 & i\end{bmatrix}\Rightarrow\begin{bmatrix} 2i & 2\\ 2 & 2i\end{bmatrix}\begin{bmatrix} -i & 1\\ 1 & -i\end{bmatrix}\Rightarrow\begin{bmatrix} 4 & 0\\ 0 & 4\end{bmatrix}\to(A)$$

$$A_1^-A_1^{\mathrm{T}}A_1^{-\mathrm{T}}A_1^-$$

$$\Rightarrow\begin{bmatrix} -i & 1\\ 1 & -i\end{bmatrix}\begin{bmatrix} i & 1\\ 1 & i\end{bmatrix}\begin{bmatrix} i & 1\\ 1 & i\end{bmatrix}\begin{bmatrix} -i & 1\\ 1 & -i\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix} 4 & 0\\ 0 & 4\end{bmatrix}\to(A)$$

$$A_2^{-\mathrm{T}}A_2A_2A_2^{-\mathrm{T}}$$

$$\Rightarrow\begin{bmatrix} 1-i & 1+i\\ 1+i & 1-i\end{bmatrix}\begin{bmatrix} 1+i & 1-i\\ 1-i & 1+i\end{bmatrix}\begin{bmatrix} 1+i & 1-i\\ 1-i & 1+i\end{bmatrix}\begin{bmatrix} 1-i & 1+i\\ 1+i & 1-i\end{bmatrix}$$

$$\begin{bmatrix} 2+2 & -2i+2i\\ 2i-2i & 2+2\end{bmatrix}\Rightarrow\begin{bmatrix} 4 & 0\\ 0 & 4\end{bmatrix}\begin{bmatrix} 1+i & 1-i\\ 1-i & 1+i\end{bmatrix}$$

$$\begin{bmatrix} 4+4i & 4-4i\\ 4-4i & 4+4i\end{bmatrix}\begin{bmatrix} 1-i & 1+i\\ 1+i & 1-i\end{bmatrix}\Rightarrow\begin{bmatrix} 16 & 0\\ 0 & 16\end{bmatrix}\to(B)$$

$$A_2^-A_2^{\mathrm{T}}A_2^{-\mathrm{T}}A_2^-$$

$$\Rightarrow\begin{bmatrix} 1-i & 1+i\\ 1+i & 1-i\end{bmatrix}\begin{bmatrix} 1+i & 1-i\\ 1-i & 1+i\end{bmatrix}\begin{bmatrix} 1+i & 1-i\\ 1-i & 1+i\end{bmatrix}\begin{bmatrix} 1-i & 1+i\\ 1+i & 1-i\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix} 16 & 0\\ 0 & 16\end{bmatrix}\to(2)$$

$$\left[A_1^{-T}A_1A_1A_2^{-T}\right]\cup\left[A_2^{-T}A_2A_2A_2^{-T}\right]=\left[A_1^{-}A_1^{T}A_1^{-T}A_1^{-}\right]\cup\left[A_2^{-}A_2^{T}A_2^{-T}A_2^{-}\right]$$

$$\begin{bmatrix}4&0\\0&4\end{bmatrix}\cup\begin{bmatrix}16&0\\0&16\end{bmatrix}=\begin{bmatrix}4&0\\0&4\end{bmatrix}\cup\begin{bmatrix}16&0\\0&16\end{bmatrix}$$

Theorem 2.7

If A_B is Quasi-bi-Normal-bi-Matrix then $\left[A_B^{CT}\right]^C$ is Quasi-bi-Normal-bi-Matrix.

Proof

$$\left[A_B^C\left[A_B^{CT}\right]^C\right]\left[\left[A_B^{CT}\right]^C A_B^C\right]=\left[\left(A_B^T\right)^T\left(A_B^T\right)^C\right]\left[\left(A_B^T\right)^C\left(A_B^T\right)^T\right]$$

$$\Rightarrow\left[\left(A_1\cup A_2\right)^C\left[\left(A_1\cup A_2\right)^{CT}\right]^C\right]\left[\left[\left(A_1\cup A_2\right)^{CT}\right]^C\left(A_1\cup A_2\right)^C\right]$$

$$=\left[\left(A_1\cup A_2\right)^T\right]^T\left[\left(A_1\cup A_2\right)^T\right]^C\left[\left(A_1^C\cup A_2^C\right)^T\right]\left[\left(A_1\cup A_2\right)^T\right]$$

$$\Rightarrow\left[A_1^C\cup A_2^C\right]\left[A_1^{CT}\cup A_2^{CT}\right]^C\left[A_1^{CT}\cup A_2^{CT}\right]^C\left[A_1^C\cup A_2^C\right]$$

$$=\left(A_1^T\cup A_2^T\right)^T\left[\left(A_1^T\cup A_2^T\right)\right]^C\left[\left(A_1^{CT}\cup A_2^{CT}\right)\right]\left[\left[A_1\cup A_2\right]^T\right]^T$$

$$\Rightarrow\left[A_1^C\cup A_2^C\right]\left[\left(A_1^{CT}\right)^C\cup\left(A_2^{CT}\right)^C\right]\left[A_1^{CT}\cup A_2^{CT}\right]^C\left[A_1^C\cup A_2^C\right]$$

$$=\left[\left(A_1^T\right)^T\cup\left(A_2^T\right)^T\right]\left[A_1^{CT}\cup A_2^{CT}\right]\left[A_1^{CT}\cup A_2^{CT}\right]\left[A_1\cup A_2\right]$$

L.H.S

$$\Rightarrow\left[A_1^C\cup A_2^C\right]\left[A_1^T\cup A_2^T\right]\left[A_1^T\cup A_2^T\right]\left[A_1^C\cup A_2^C\right]$$

$$\Rightarrow\left[A_1^CA_1^TA_1^TA_1^C\right]\cup\left[A_2^CA_2^TA_2^TA_2^C\right]$$

Where $A^CA^TA^TA^C=A^TA^CA^CA^T$

$$\left[A_1^TA_1^CA_1^CA_1^T\right]\cup\left[A_2^TA_2^CA_2^CA_2^T\right]\to(1)$$

R.H.S

$$\left[\left(A_1^T\right)^T\cup\left(A_2^T\right)^T\right]\left[A_1^{CT}\cup A_2^{CT}\right]\left[A_1^{CT}\cup A_2^{CT}\right]\left[A_1^T\cup A_2^T\right]^T$$

$$\Rightarrow\left[A_1\cup A_2\right]\left[A_1^{CT}\cup A_2^{CT}\right]\left[A_1^{CT}\cup A_2^{CT}\right]\left[A_1\cup A_2\right]$$

$$\Rightarrow\left[A_1A_1^{CT}A_1^{CT}A_1\right]\cup\left[A_2A_2^{CT}A_2^{CT}A_2\right]$$

Where $A_1^{CT}=A_1^*,\quad A_2^{CT}=A_2^*$

$\Rightarrow\left[A_1A_1^*A_1^*A_1\right]\cup\left[A_2A_2^*A_2^*A_2\right]\rightarrow(2)$

L.H.S=R.H.S

$\therefore\left[A_B^{CT}\right]^C$ is Quasi-bi-Normal-bi-Matrix.

Example 2.8

$$\left[A_B^C\left(A_B^{CT}\right)^C\right]\left[\left(A_B^{CT}\right)^C A_B^C\right]=\left[\left(A_B^{T}\right)^T\left(A_B^{T}\right)^C\right]\left[\left(A_B^{T}\right)^C\left(A_B^{T}\right)^T\right]$$

$$\Rightarrow\left[\left(A_1\cup A_2\right)^C\left(\left(A_1\cup A_2\right)^{CT}\right)^C\right]\left[\left(\left(A_1\cup A_2\right)^{CT}\right)^C\left(A_1\cup A_2\right)^C\right]$$

$$=\left[\left(A_1\cup A_2\right)^T\right]^T\left[\left(A_1\cup A_2\right)^T\right]^C\left[\left(A_1\cup A_2\right)^T\right]^C\left[\left(A_1\cup A_2\right)^T\right]^T$$

$$\left[A_1^C\left(A_1^{CT}\right)^C\left(A_1^{CT}\right)^C A_1^C\right]\cup\left[A_2^C\left(A_2^{CT}\right)^C\left(A_2^{CT}\right)^C A_2^C\right]$$

$$=\left[\left(A_1^T\right)^T\left(A_1^T\right)^C\left(A_1^T\right)^C\left(A_1^T\right)^T\right]\cup\left[\left(A_2^T\right)^T\left(A_2^T\right)^C\left(A_2^T\right)^C\left(A_2^T\right)^T\right]$$

$$\Rightarrow\left[A_1^C\left(A_1^{CT}\right)^C\left(A_1^{CT}\right)^C A_1^C\right]$$

$$A_1=\begin{bmatrix}1 & +i\\ -i & -1\end{bmatrix}, A_2=\begin{bmatrix}1 & -2i\\ 2i & -1\end{bmatrix}$$

$$\overline{A}_1=\begin{bmatrix}1 & -i\\ +i & -1\end{bmatrix}\overline{A}_2=\begin{bmatrix}1 & 2i\\ -2i & -1\end{bmatrix}$$

$$\overline{A}_1^T=\begin{bmatrix}1 & +i\\ -i & -1\end{bmatrix}\overline{A}_2^T=\begin{bmatrix}1 & -2i\\ 2i & -1\end{bmatrix}$$

$$\left[\overline{A}_1^T\right]^C=\begin{bmatrix}1 & -i\\ +i & -1\end{bmatrix}\left[\overline{A}_2^T\right]^C=\begin{bmatrix}1 & 2i\\ -2i & -1\end{bmatrix}$$

$$\Rightarrow\left[A_1^C\left(A_1^{CT}\right)^C\left(A_1^{CT}\right)^C A_1^C\right]$$

$$\Rightarrow\begin{bmatrix}1 & -i\\ +i & -1\end{bmatrix}\begin{bmatrix}1 & -i\\ +i & -1\end{bmatrix}\begin{bmatrix}1 & -i\\ +i & -1\end{bmatrix}\begin{bmatrix}1 & -i\\ +i & -1\end{bmatrix}$$

$$\begin{bmatrix}2 & -2i\\ +2i & -2\end{bmatrix}\begin{bmatrix}1 & -i\\ +i & -1\end{bmatrix}\Rightarrow\begin{bmatrix}4 & 0\\ 0 & 4\end{bmatrix}\rightarrow(1)$$

$$\left[\left(A_1^T\right)^T\left(A_1^T\right)^C\left(A_1^T\right)^C\left(A_1^T\right)^T\right]$$

$$\Rightarrow \begin{bmatrix} 1 & -i \\ i & -1 \end{bmatrix} \begin{bmatrix} 1 & -i \\ i & -1 \end{bmatrix} \begin{bmatrix} 1 & -i \\ i & -1 \end{bmatrix} \begin{bmatrix} 1 & -i \\ i & -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} (2) & (-2i) \\ (2i) & (-2) \end{bmatrix} \begin{bmatrix} 1 & -i \\ i & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \rightarrow (2)$$

$$\left[A_2^C \left(A_2^{CT} \right)^C \left(A_2^{CT} \right)^C A_2^C \right]$$

$$\Rightarrow \begin{bmatrix} 1 & +2i \\ -2i & -1 \end{bmatrix} \begin{bmatrix} 1 & 2i \\ -2i & -1 \end{bmatrix} \begin{bmatrix} 1 & 2i \\ -2i & -1 \end{bmatrix} \begin{bmatrix} 1 & 2i \\ -2i & -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \left(1-4i^2\right) & \left(2i-2i\right) \\ \left(-2i+2i\right) & \left(-4i^2+1\right) \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2i \\ -2i & -1 \end{bmatrix}$$

$$\begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix} \rightarrow (1)$$

$$\left[\left(A_2^T \right)^T \left(A_2^T \right)^C \left(A_2^T \right)^C \left(A_2^T \right)^T \right]$$

$$\Rightarrow \begin{bmatrix} 1 & -2i \\ 2i & -1 \end{bmatrix} \begin{bmatrix} 1 & -2i \\ 2i & -1 \end{bmatrix} \begin{bmatrix} 1 & -2i \\ 2i & -1 \end{bmatrix} \begin{bmatrix} 1 & -2i \\ 2i & -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5 & -10i \\ +10i & -5 \end{bmatrix} \begin{bmatrix} 5 & -10i \\ +10i & -5 \end{bmatrix} \begin{bmatrix} 1 & -2i \\ 2i & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 5-20i^2 & -10i+10i \\ 10i-10i & 20i^2+5 \end{bmatrix}$$

$$\begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix} \rightarrow (2)$$

$$\left[A_1^C \left(A_1^{CT} \right)^C \left(A_1^{CT} \right)^C A_1^C \right] \cup \left[A_2^C \left(A_2^{CT} \right)^C \left(A_2^{CT} \right)^C \left(A_2^C \right) \right]$$

$$= \left[\left(A_1^T \right)^T \left(A_1^T \right)^C \left(A_1^T \right)^C \left(A_1^T \right)^T \right] \cup \left[\left(A_2^T \right)^T \left(A_2^T \right)^C \left(A_2^T \right)^C \left(A_2^T \right)^T \right]$$

$$\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \cup \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \cup \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix}$$

L.H.S=R.H.S

Theorem 2.9

If A_B and B_B are Quasi-bi-Normal-bi-Matrix then the product of $A_B B_B$ is also a Quasi-bi-Normal-bi-Matrix.

Proof

Given A_B and B_B are Quasi-bi-Normal-bi-Matrix.

$$\mathbf{[A_BB_B]\ [A_BB_B]^{CT}\ [B_BA_B]^{CT}\ [B_BA_B] = [A_BB_B]^T\ [A_BB_B]^C\ [B_BA_B]^C\ [B_BA_B]^T}$$

L.H.S

$[A_BB_B]\ [A_BB_B]^{CT}\ [B_BA_B]^{CT}\ [B_BA_B]$

$[(A_1UA_2)\ (B_1UB_2)]\ [(A_1UA_2)\ (B_1UB_2)]^{CT}\ [(B_1UB_2)\ (A_1UA_2)]^{CT}\ [(B_1UB_2)\ (A_1UA_2)]$

$[(A_1UA_2)\ (B_1UB_2)]\ [(B_1UB_2)^{CT}\ (A_1UA_2)^{CT}]\ [(A_1UA_2)^{CT}\ (B_1UB_2)^{CT}]\ [(B_1UB_2)\ (A_1UA_2)]$

$\mathbf{[(A_1B_1)\ (A_1B_1)^{CT}\ (B_1A_1)^{CT}\ (B_1A_1)]\ U\ [(A_2B_2)\ (A_2B_2)^{CT}\ (B_2A_2)^{CT}\ (B_2A_2)]}$

R.H.S

$[A_BB_B]^T\ [A_BB_B]^C\ [B_BA_B]^C\ [B_BA_B]^T$

$[(A_1UA_2)\ (B_1UB_2)]^T\ [(A_1UA_2)\ (B_1UB_2)]^C\ [(B_1UB_2)\ (A_1UA_2)]^C\ [(B_1UB_2)\ (A_1UA_2)]^T$

$[(B_1UB_2)^T\ (A_1UA_2)^T]\ [(A_1UA_2)^C\ (B_1UB_2)^C]\ [(B_1UB_2)^C\ (A_1UA_2)^C]\ [(A_1UA_2)^T\ (B_1UB_2)^T]$

$\mathbf{[(A_1B_1)^T\ (A_1B_1)^C\ (B_1A_1)^C\ (B_1A_1)^T]\ U\ [(A_2B_2)^T\ (A_2B_2)^C\ (B_2A_2)^C\ (B_2A_2)^T]}$

L.H.S=R.H.S

Therefore A_BB_B be a Quasi-bi-Normal-bi-Matrix.

Remark 2.10

Similarly, $(A_B)^T \& (A_B)^C$ are Quasi-bi-Normal-bi-Matrix.

Theorem 2.11

If A_B and B_B are Quasi-bi-Normal-bi-Matrix then $(A_BB_B)^T$ is also be Quasi-bi-Normal-bi-Matrix.

Proof

Let A_B £ $C_{n\times n}$

We have prove that

$$\mathbf{[(A_BB_B)^T]^T\ [(A_BB_B)^T]^C\ [(B_BA_B)^T]^C\ [(B_BA_B)^T]^T = [(A_BB_B)^T]\ [(A_BB_B)^T]^{CT}\ [(B_BA_B)^T]^{CT}\ [(B_BA_B)^T]}$$

L.H.S

$[(A_BB_B)^T]^T\ [(A_BB_B)^T]^C\ [(B_BA_B)^T]^C\ [(B_BA_B)^T]^T$

$[\,[(A_1UA_2)\ (B_1UB_2)]^T\,]^T\ [\,[(A_1UA_2)\ (B_1UB_2)]^T\,]^C\ [\,[(B_1UB_2)\ (A_1UA_2)]^T\,]^C\ [\,[(B_1UB_2)\ (A_1UA_2)]^T\,]^T$

$[(A_1UA_2)\,(B_1UB_2)\,]\ [(B_1UB_2)^T\ (A_1UA_2)^T]^C\ [(A_1UA_2)^T\ (B_1UB_2)^T]^C\ [(B_1UB_2)\ (A_1UA_2)\,]$

$\mathbf{[(A_1B_1)]\ [(A_1B_1)^T]^C\ [(B_1A_1)^T]^C\ [(B_1A_1)]\ U\ [(A_2B_2)]\ [(A_2B_2)^T]^C\ [(B_2A_2)^T]^C\ [(B_2A_2)]}$

R.H.S

$[(A_BB_B)^T]\ [(A_BB_B)^T]^{CT}\ [(B_BA_B)^T]^{CT}\ [(B_BA_B)^T]$

$[(A_1UA_2)\ (B_1UB_2)]^T\ [\,[(A_1UA_2)\ (B_1UB_2)]^T]^{CT}\ [\,[(B_1UB_2)\ (A_1UA_2)]^T]\ [(B_1UB_2)\ (A_1UA_2)]^T$

$[(B_1UB_2)^T\ (A_1UA_2)^T]\ [(A_1UA_2)\ (B_1UB_2)]^C\ [(B_1UB_2)\ (A_1UA_2)]^C\ [(A_1UA_2)^T\ (B_1UB_2)^T]$

$\mathbf{[(A_1B_1)]^T\ [(A_1B_1)]^C\ [(B_1A_1)]^C\ [(B_1A_1)]^T\ U\ [(A_2B_2)]^T\ [(A_2B_2)]^C\ [(B_2A_2)]^C\ [(B_2A_2)]^T}$

L.H.S=R.H.S

Therefore $(A_BB_B)^T$ is also be a Quasi-bi-Normal-bi-Matrix.

Theorem 2.12

If A_B and B_B are Quasi-bi-Normal-bi-Matrix then $(A_BB_B)^c$ is also be Quasi-bi-Normal-bi-Matrix.

Proof

$\mathbf{[(A_BB_B)^c]^c\ [(A_BB_B)^c]^T\ [(B_BA_B)^C]^T\ [(B_BA_B)^C]^C = [(A_BB_B)^C]^{CT}\ [(A_BB_B)^C]\ [(B_BA_B)^C]\ [(B_BA_B)^C]^{CT}}$

L.H.S

$[(A_BB_B)^C]^C\ [(A_BB_B)^C]^T\ [(B_BA_B)^C]^T\ [(B_BA_B)^C]^C$

$[\,[(A_1UA_2)\ (B_1UB_2)]^C\,]^C\ [\,[(A_1UA_2)\ (B_1UB_2)]^C\,]^T\ [\,[(B_1UB_2)\ (A_1UA_2)]^C\,]^T\ [\,[(B_1UB_2)\ (A_1UA_2)]^C\,]^C$

$[(A_1UA_2)\ (B_1UB_2)\,]\ [(B_1UB_2)^{CT}\ (A_1UA_2)^{CT}\,]\ [(A_1UA_2)^{CT}\ (B_1UB_2)^{CT}]\ [(B_1UB_2)\ (A_1UA_2)]$

$[(A_1B_1)]\ [(A_1B_1)^{CT}\,]\ [(B_1A_1)^{CT}\,]\ [(B_1A_1)]\ U\ [(A_2B_2)]\ [(A_2B_2)^{CT}\,]\ [(B_2A_2)^{CT}]\ [(B_2A_2)]$

R.H.S

$[(A_BB_B)^C]^{CT}\ [(A_BB_B)^C]\ [(B_BA_B)^C]\ [(B_BA_B)^C]^{CT}$

$[\,[(A_1UA_2)\ (B_1UB_2)]^C\,]^{CT}\ [(A_1UA_2)\ (B_1UB_2)]^C\ [(B_1UB_2)\ (A_1UA_2)]^C\ [\,[(B_1UB_2)\ (A_1UA_2)]^C]^{CT}$

$[(B_1UB_2)^T\ (A_1UA_2)^T]\ [(A_1UA_2)^C\ (B_1UB_2)^C]\ [(B_1UB_2)^C\ (A_1UA_2)^C]\ [(A_1UA_2)^T\ (B_1UB_2)^T]$

$[(A_1B_1)]^T\ [(A_1B_1)]^C\ [(B_1A_1)]^C[(B_1A_1)]^T \cup [(A_2B_2)]^T\ [(A_2B_2)]^C\ [(B_2A_2)]^C\ [(B_2A_2)]^T$

L.H.S=R.H.S

- Therefore $(A_BB_B)^C$ is also be a Quasi-bi-Normal-bi-Matrix.

Theorem 2.13

If the Sum of Quasi-bi-Normal-bi-Matrix A_B and B_B are Quasi-bi-Normal-bi-Matrix then

Proof

Let $A, B \in C_{nxn}$

We have prove that is the Quasi-bi-Normal-bi-Matrix.

$$\left[(A_B+B_B)\left[(A_B+B_B)\right]^{CT}\right]\left[\left[(A_B+B_B)\right]^{CT}(A_B+B_B)\right]=(A_B+B_B)^{T}(A_B+B_B)^{C}(A_B+B_B)^{C}(A_B+B_B)^{T}$$

L.H.S

$$\Rightarrow\left[(A_B+B_B)(A_B+B_B)^{CT}\right]\left[(A_B+B_B)^{CT}(A_B+B_B)\right]$$

$$\Rightarrow\left[(A_1\cup A_2)+(B_1\cup B_2)\right]\left[(A_1\cup A_2)^{CT}+(B_1\cup B_2)^{CT}\right]\left[(A_1\cup A_2)^{CT}+(B_1\cup B_2)^{CT}\right]\left[(A_1\cup A_2)+(B_1\cup B_2)\right]$$

$$\Rightarrow\left[(A_1\cup A_2)+(B_1\cup B_2)\right]\left[\left(A_1^{CT}\cup A_2^{CT}\right)+\left(B_1^{CT}\cup B_2^{CT}\right)\right]\left[\left(A_1^{CT}\cup A_2^{CT}\right)+\left(B_1^{CT}\cup B_2^{CT}\right)\right]\left[(A_1\cup A_2)+(B_1\cup B_2)\right]$$

$$\Rightarrow\left[\left[(A_1+B_1)\left(A_1^{CT}+B_1^{CT}\right)\left(A_1^{CT}+B_1^{CT}\right)(A_1+B_1)\right]\cup\left[(A_2+B_2)\left(A_2^{CT}+B_2^{CT}\right)\left(A_2^{CT}+B_2^{CT}\right)(A_2+B_2)\rightarrow(1)\right.\right.$$

R.H.S

$$\Rightarrow\left[(A_B+B_B)^{T}(A_B+B_B)^{C}\right]\left[(A_B+B_B)^{C}(A_B+B_B)^{T}\right]$$

$$\Rightarrow\left[(A_1\cup A_2)+(B_1\cup B_2)\right]^{T}\left[(A_1\cup A_2)+(B_1\cup B_2)\right]^{C}\left[(A_1\cup A_2)+(B_1\cup B_2)\right]^{C}\left[(A_1\cup A_2)+(B_1\cup B_2)\right]^{T}$$

$$\Rightarrow\left[(A_1\cup A_2)^{T}+(B_1\cup B_2)^{T}\right]\left[(A_1\cup A_2)^{C}+(B_1\cup B_2)^{C}\right]\left[(A_1\cup A_2)^{C}+(B_1\cup B_2)^{C}\right]\left[(A_1\cup A_2)^{T}+(B_1\cup B_2)^{T}\right]$$

$$\Rightarrow\left[\left(A_1^{T}\cup A_2^{T}\right)+\left(B_1^{T}\cup B_2^{T}\right)\right]\left[\left(A_1^{C}\cup A_2^{C}\right)+\left(B_1^{C}\cup B_2^{C}\right)\right]\left[\left(A_1^{C}\cup A_2^{C}\right)+\left(B_1^{C}\cup B_2^{C}\right)\right]\left[\left(A_1^{T}\cup A_2^{T}\right)+\left(B_1^{T}\cup B_2^{T}\right)\right]$$

$$\Rightarrow\left[\left(A_1^{T}+B_1^{T}\right)\left(A_1^{C}+B_1^{C}\right)\right]\left[\left(A_1^{C}+B_1^{C}\right)\left(A_1^{T}+B_1^{T}\right)\right]\cup\left[\left(A_2^{T}+B_2^{T}\right)\left(A_2^{C}+B_2^{C}\right)\right]\left[\left(A_2^{C}+B_2^{C}\right)\left(A_2^{T}+B_2^{T}\right)\right]\rightarrow(2)$$

(1)&(2)

$$\Rightarrow\left[\left(A_1+B_1\right)\left(A_1^{CT}+B_1^{CT}\right)\right]\left[\left(A_1^{CT}+B_1^{CT}\right)\left(A_1+B_1\right)\right]\cup\left[\left(A_2+B_2\right)\left(A_2^{CT}+B_2^{CT}\right)\right]\left[\left(A_2^{CT}+B_2^{CT}\right)\left(A_2+B_2\right)\right]$$
$$=\left[\left(A_1^{T}+B_1^{T}\right)\left(A_1^{C}+B_1^{C}\right)\right]\left[\left(A_1^{C}+B_1^{C}\right)\left(A_1^{T}+B_1^{T}\right)\right]\cup\left[\left(A_2^{T}+B_2^{T}\right)\left(A_2^{C}+B_2^{C}\right)\right]\left[\left(A_2^{C}+B_2^{C}\right)\left(A_2^{T}+B_2^{T}\right)\right]$$

Where

$$\left[\left(A_1+B_1\right)\left(A_1^{CT}+B_1^{CT}\right)\left(A_1^{CT}+B_1^{CT}\right)\left(A_1+B_1\right)\right]=\left[\left(A_1^{T}+B_1^{T}\right)\left(A_1^{C}+B_1^{C}\right)\left(A_1^{C}+B_1^{C}\right)\left(A_1^{T}+B_1^{T}\right)\right]$$
$$\left[\left(A_2+B_2\right)\left(A_2^{CT}+B_2^{CT}\right)\left(A_2^{CT}+B_2^{CT}\right)\left(A_2+B_2\right)\right]=\left[\left(A_2^{T}+B_2^{T}\right)\left(A_2^{C}+B_2^{C}\right)\left(A_2^{C}+B_2^{C}\right)\left(A_2^{T}+B_2^{T}\right)\right]$$

Therefore Sum of A and B are Quasi-bi-Normal-bi-Matrix.

Remark 2.14

The Sum of Quasi-bi-Normal-bi-Matrix need not be Quasi-bi-Normal-bi-Matrix.

Theorem 2.15

If $\left(A_B+B_B\right)$ is Quasi-bi-Normal-bi-Matrix then $\left(A_B+B_B\right)^{T}$ is Quasi-bi-Normal-bi-Matrix.

Proof

$$\left[\left(A_B+B_B\right)\right]^{T}\left[\left(A_B+B_B\right)^{T}\right]^{CT}\left[\left(A_B+B_B\right)^{T}\right]^{CT}\left[\left(A_B+B_B\right)\right]^{T}$$
$$=\left[\left(A_B+B_B\right)^{T}\right]^{T}\left[\left(A_B+B_B\right)^{T}\right]^{C}\left[\left(A_B+B_B\right)^{T}\right]^{C}\left[\left(A_B+B_B\right)^{T}\right]^{T}$$

L.H.S

$$\Rightarrow\left[\left(A_B+B_B\right)\right]^{T}\left[\left(A_B+B_B\right)^{T}\right]^{CT}\left[\left(A_B+B_B\right)^{T}\right]^{CT}\left[\left(A_B+B_B\right)\right]^{T}$$

$$\Rightarrow\left[\left(A_1\cup A_2\right)+\left(B_1\cup B_2\right)\right]^{T}\left[\left(\left(A_1\cup A_2\right)+\left(B_1\cup B_2\right)\right)^{T}\right]^{CT}\left[\left(\left(A_1\cup A_2\right)+\left(B_1\cup B_2\right)\right)^{T}\right]^{CT}\left[\left(A_1\cup A_2\right)+\left(B_1\cup B_2\right)\right]^{T}$$

$$\Rightarrow\left[\left(A_1\cup A_2\right)^{T}+\left(B_1\cup B_2\right)^{T}\right]\left[\left[\left(A_1\cup A_2\right)^{T}\right]^{CT}+\left[\left(B_1\cup B_2\right)^{T}\right]^{CT}\right]$$
$$\left[\left[\left(A_1\cup A_2\right)^{T}\right]^{CT}+\left[\left(B_1\cup B_2\right)^{T}\right]^{CT}\right]\left[\left(A_1\cup A_2\right)^{T}+\left(B_1\cup B_2\right)^{T}\right]$$

$$\Rightarrow\begin{matrix}\left[\left(A_1^{T}\cup A_2^{T}\right)+\left(B_1^{T}\cup B_2^{T}\right)\right]\left[\left(A_1^{T}\right)^{CT}\cup\left(A_2^{T}\right)^{CT}+\left(B_1^{T}\right)^{CT}\cup\left(B_2^{T}\right)^{CT}\right]\\ \left[\left(A_1^{T}\right)^{CT}\cup\left(A_2^{T}\right)^{CT}+\left(B_1^{T}\right)^{CT}\cup\left(B_2^{T}\right)^{CT}\right]\left[\left(A_1^{T}\cup A_2^{T}\right)+\left(B_1^{T}\cup B_2^{T}\right)\right]\rightarrow(1)\end{matrix}$$

$$\Rightarrow\left[\left(A_B+B_B\right)^{T}\right]^{T}\left[\left(A_B+B_B\right)^{T}\right]^{C}\left[\left(A_B+B_B\right)^{T}\right]^{C}\left[\left(A_B+B_B\right)^{T}\right]^{T}$$

$$\Rightarrow\left[\left(A_1^{T}\cup A_2^{T}\right)+\left(B_1^{T}\cup B_2^{T}\right)^{T}\right]^{T}\left[\left(A_1^{T}\cup A_2^{T}\right)+\left(B_1^{T}\cup B_2^{T}\right)^{T}\right]^{C}\left[\left(A_1^{T}\cup A_2^{T}\right)+\left(B_1^{T}\cup B_2^{T}\right)^{T}\right]^{C}\left[\left(A_1^{T}\cup A_2^{T}\right)+\left(B_1^{T}\cup B_2^{T}\right)^{T}\right]^{T}$$

$$\Rightarrow\left[\left(A_1^{T}\cup B_2^{T}\right)^{T}\left(A_1^{T}\cup B_2^{T}\right)^{C}\right]\left[\left(A_1^{T}\cup B_2^{T}\right)^{C}\left(A_1^{T}\cup B_2^{T}\right)^{T}\right]\cup\left[\left(A_1^{T}\cup B_2^{T}\right)^{T}\left(A_1^{T}\cup B_2^{T}\right)^{C}\left(A_1^{T}\cup B_2^{T}\right)^{C}\left(A_1^{T}\cup B_2^{T}\right)^{T}\right]\rightarrow(2)$$

Therefore,

$$\left[\left(A_1^{T}B_1^{T}\right)\left(A_1^{T}B_1^{T}\right)^{CT}\left(A_1^{T}B_1^{T}\right)^{CT}\left(A_1^{T}B_1^{T}\right)\right]\cup\left[\left(A_2^{T}B_2^{T}\right)\left(A_2^{T}B_2^{T}\right)^{CT}\left(A_2^{T}B_2^{T}\right)^{CT}\left(A_2^{T}B_2^{T}\right)\right]$$

$$=\left[\left(A_1^{T}B_1^{T}\right)^{T}\left(A_1^{T}B_1^{T}\right)^{C}\left(A_1^{T}B_1^{T}\right)^{C}\left(A_1^{T}B_1^{T}\right)^{T}\right]\cup\left[\left(A_2^{T}B_2^{T}\right)^{T}\left(A_2^{T}B_2^{T}\right)^{C}\left(A_2^{T}B_2^{T}\right)^{C}\left(A_2^{T}B_2^{T}\right)^{T}\right]$$

Where, $\left[\left(A_1^{T}B_1^{T}\right)\left(A_1^{T}B_1^{T}\right)^{CT}\left(A_1^{T}B_1^{T}\right)^{CT}\left(A_1^{T}B_1^{T}\right)\right]=\left[\left(A_1^{T}B_1^{T}\right)^{T}\left(A_1^{T}B_1^{T}\right)^{C}\left(A_1^{T}B_1^{T}\right)^{C}\left(A_1^{T}B_1^{T}\right)^{T}\right]$

$$\left[\left(A_2^{T}B_2^{T}\right)\left(A_2^{T}B_2^{T}\right)^{CT}\left(A_2^{T}B_2^{T}\right)^{CT}\left(A_2^{T}B_2^{T}\right)\right]=\left[\left(A_2^{T}B_2^{T}\right)^{T}\left(A_2^{T}B_2^{T}\right)^{C}\left(A_2^{T}B_2^{T}\right)^{C}\left(A_2^{T}B_2^{T}\right)^{T}\right]$$

L.H.S=R.H.S

$\therefore(A+B)^{T}$ is Quasi-bi-Normal-bi-Matrix.

Theorem 2.16

If A_B+B_B is Quasi-bi-Normal-bi-Matrix then $\left(A_B+B_B\right)^{C}$ is Quasi-bi-Normal-bi-Matrix for any conjugate.

Proof

$$\left[(A+B)^{C}\left[(A+B)^{C}\right]^{CT}\right]\left[(A+B)^{C}\right]^{CT}[A+B]^{C}=\left[(A+B)^{C}\right]^{T}\left[(A+B)^{C}\right]^{C}\left[(A+B)^{C}\right]^{C}\left[(A+B)^{C}\right]^{T}$$

$$\Rightarrow\left[\left(A_B+B_B\right)^{C}\right]\left[\left(A_B+B_B\right)^{C}\right]^{CT}\left[\left(A_B+B_B\right)^{C}\right]^{CT}\left[\left(A_B+B_B\right)^{C}\right]=\left[\left(A_B+B_B\right)^{C}\right]^{T}\left[\left(A_B+B_B\right)^{C}\right]^{C}\left[\left(A_B+B_B\right)^{C}\right]^{C}\left[\left(A_B+B_B\right)^{C}\right]^{T}$$

L.H.S

$$\Rightarrow\left[\left(A_B+B_B\right)^{C}\right]\left[\left(A_B+B_B\right)^{C}\right]^{CT}\left[\left(A_B+B_B\right)^{C}\right]^{CT}\left[\left(A_B+B_B\right)^{C}\right]$$

$$\Rightarrow\left[\left(A_1\cup A_2\right)+\left(B_1\cup B_2\right)\right]^{C}\left[\left[\left(A_1\cup A_2\right)+\left(B_1\cup B_2\right)\right]^{C}\right]^{CT}\left[\left[\left(A_1\cup A_2\right)+\left(B_1\cup B_2\right)\right]^{C}\right]^{CT}\left[\left(A_1\cup A_2\right)+\left(B_1\cup B_2\right)\right]^{C}$$

$$\Rightarrow\left[(A_1\cup A_2)^C+(B_1\cup B_2)^C\right]\left[\left[(A_1\cup A_2)+(B_1\cup B_2)\right]^C\right]^{CT}\left[\left[(A_1\cup A_2)+(B_1\cup B_2)\right]^C\right]^{CT}\left[(A_1\cup A_2)^C+(B_1\cup B_2)^C\right]$$

$$\Rightarrow\left[(A_1\cup A_2)^C+(B_1\cup B_2)^C\right]\left[(A_1\cup A_2)^C+(B_1\cup B_2)^C\right]^{CT}\left[(A_1\cup A_2)^C+(B_1\cup B_2)^C\right]^{CT}\left[(A_1\cup A_2)^C+(B_1\cup B_2)^C\right]$$

$$\Rightarrow\left[\left(A_1^C\cup A_2^C\right)+\left(B_1^C\cup B_2^C\right)\right]\left[\left(A_1^C\cup A_2^C\right)+\left(B_1^C\cup B_2^C\right)\right]^{CT}\left[\left(A_1^C\cup A_2^C\right)+\left(B_1^C\cup B_2^C\right)\right]^{CT}\left[\left(A_1^C\cup A_2^C\right)+\left(B_1^C\cup B_2^C\right)\right]$$

$$\Rightarrow\left[\left(A_1^C+B_1^C\right)\left(A_1^C+B_1^C\right)^{CT}\left(A_1^C+B_1^C\right)^{CT}\left(A_1^C+B_1^C\right)\right]\cup\left[\left(A_2^C+B_2^C\right)\left(A_2^C+B_2^C\right)^{CT}\left(A_2^C+B_2^C\right)^{CT}\left(A_2^C+B_2^C\right)\right]\rightarrow(1)$$

R.H.S

$$\Rightarrow\left[(A_B+B_B)^C\right]^{T}\left[(A_B+B_B)^C\right]^{C}\left[(A_B+B_B)^C\right]^{C}\left[(A_B+B_B)^C\right]^{T}$$

$$\Rightarrow\left[(A_1\cup A_2)^C+(B_1\cup B_2)^C\right]^{T}\left[(A_1\cup A_2)^C+(B_1\cup B_2)^C\right]^{C}\left[(A_1\cup A_2)^C+(B_1\cup B_2)^C\right]^{C}\left[(A_1\cup A_2)^C+(B_1\cup B_2)^C\right]^{T}$$

$$\Rightarrow\left[\left(A_1^C\cup A_2^C\right)+\left(B_1^C\cup B_2^C\right)\right]^{T}\left[\left(A_1^C\cup A_2^C\right)+\left(B_1^C\cup B_2^C\right)\right]^{C}\left[\left(A_1^C\cup A_2^C\right)+\left(B_1^C\cup B_2^C\right)\right]^{C}\left[\left(A_1^C\cup A_2^C\right)+\left(B_1^C\cup B_2^C\right)\right]^{T}$$

$$\Rightarrow\left[\left(A_1^C+B_1^C\right)^{T}\left(A_1^C+B_1^C\right)^{C}\left(A_1^C+B_1^C\right)^{C}\left(A_1^C+B_1^C\right)^{T}\right]\cup\left[\left(A_2^C+B_2^C\right)^{T}\left(A_2^C+B_2^C\right)^{C}\left(A_2^C+B_2^C\right)^{C}\left(A_2^C+B_2^C\right)^{T}\right]$$

Where

$$\left[\left(A_1^C+B_1^C\right)\left(A_1^C+B_1^C\right)^{CT}\left(A_1^C+B_1^C\right)^{CT}\left(A_1^C+B_1^C\right)\right]=\left[\left(A_1^C+B_1^C\right)^{T}\left(A_1^C+B_1^C\right)^{C}\left(A_1^C+B_1^C\right)^{C}\left(A_1^C+B_1^C\right)^{T}\right]$$

$$\left[\left(A_2^C+B_2^C\right)\left(A_2^C+B_2^C\right)^{CT}\left(A_2^C+B_2^C\right)^{CT}\left(A_1^C+B_1^C\right)\right]=\left[\left(A_2^C+B_2^C\right)^{T}\left(A_2^C+B_2^C\right)^{C}\left(A_2^C+B_2^C\right)^{C}\left(A_2^C+B_2^C\right)^{T}\right]$$

L.H.S=R.H.S

Example 2.17

$$\left[(A_1+B_1)(A_1+B_1)^{CT}(A_1+B_1)^{CT}(A_1+B_1)\right]\cup\left[(A_2+B_2)(A_2+B_2)^{CT}(A_2+B_2)^{CT}(A_2+B_2)\right]$$

$$=\left[(A_1+B_1)^{T}(A_1+B_1)^{C}(A_1+B_1)^{C}(A_1+B_1)^{T}\right]\cup\left[(A_2+B_2)^{T}(A_2+B_2)^{C}(A_2+B_2)^{C}(A_2+B_2)^{T}\right]$$

$$A_1=\begin{bmatrix}1 & 1\\ -i & 0\end{bmatrix},\quad B_1=\begin{bmatrix}i & -i\\ 1 & 0\end{bmatrix}$$

$$A_1 + B_1 = \begin{bmatrix} l+i & l-i \\ l-i & l+i \end{bmatrix}, \quad \left(\overline{A_1 + B_1}\right) = \begin{bmatrix} l-i & l+i \\ l+i & 0 \end{bmatrix}, \left(A_1 + B_1\right)^{\mathrm{T}} = \begin{bmatrix} l+i & l-i \\ l-i & 0 \end{bmatrix}$$

$$\left(A_1 + B_1\right)^{-\mathrm{T}} = \begin{bmatrix} 1-i & 1+i \\ 1+i & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 2 & 2 \\ -2i & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 2i & -2i \\ 2 & 0 \end{bmatrix}$$

$$A_2 + B_2 = \begin{bmatrix} 2+2i & 2-2i \\ 2-2i & 0 \end{bmatrix}, \qquad \left(A_2 + B_2\right)^{-} = \begin{bmatrix} 2-2i & 2+2i \\ 2+2i & 0 \end{bmatrix}, \qquad \left(A_2 + B_2\right)^{\mathrm{T}} = \begin{bmatrix} 2+2i & 2-2i \\ 3-2i & 0 \end{bmatrix}$$

$$\left(A_2 + B_2\right)^{-\mathrm{T}} = \begin{bmatrix} 2-2i & 2+2i \\ 2+2i & 0 \end{bmatrix}$$

$$\left[\left(A_1 + B_1\right)\left(A_1 + B_1\right)^{C\mathrm{T}}\left(A_1 + B_1\right)^{C\mathrm{T}}\left(A_1 + B_1\right)\right]$$

$$\Rightarrow \begin{bmatrix} 1+i & 1-i \\ 1-i & 0 \end{bmatrix}\begin{bmatrix} 1-i & 1+i \\ 1+i & 0 \end{bmatrix}\begin{bmatrix} 1-i & 1+i \\ 1+i & 0 \end{bmatrix}\begin{bmatrix} 1+i & 1-i \\ 1-i & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2-2i & 4+4i \\ 0 & 2-2i \end{bmatrix}\begin{bmatrix} 1+i & 1-i \\ 1-i & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 12 & -4i \\ -4i & 0 \end{bmatrix} \to (1)$$

$$\left(A_1 + B_1\right)^{\mathrm{T}}\left(A_1 + B_1\right)^{C}\left(A_1 + B_1\right)^{C}\left(A_1 + B_1\right)^{\mathrm{T}}$$

$$\Rightarrow \begin{bmatrix} 1+i & 1-i \\ 1-i & 0 \end{bmatrix}\begin{bmatrix} 1-i & 1+i \\ 1+i & 0 \end{bmatrix}\begin{bmatrix} 1-i & 1+i \\ 1+i & 0 \end{bmatrix}\begin{bmatrix} 1+i & 1-i \\ 1-i & 0 \end{bmatrix}$$

$$\begin{bmatrix} 12 & -4i \\ -4i & 0 \end{bmatrix} \to (2)$$

$$(1) = (2)$$

$$\left(A_2 + B_2\right)\left(A_2 + B_2\right)^{C\mathrm{T}}\left(A_2 + B_2\right)^{C\mathrm{T}}\left(A_2 + B_2\right)$$

$$\Rightarrow \begin{bmatrix} 2+2i & 2-2i \\ 2-2i & 0 \end{bmatrix}\begin{bmatrix} 2-2i & 2+2i \\ 2+2i & 0 \end{bmatrix}\begin{bmatrix} 2-2i & 2+2i \\ 2+2i & 0 \end{bmatrix}\begin{bmatrix} 2+2i & 2-2i \\ 2-2i & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -16i & 16+32i \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2+2i & 2-2i \\ 2-2i & 0 \end{bmatrix}$$

$$\begin{bmatrix} \left(-16-32i^2+32+64i-32i-64i^2\right) & \left(-32+32i^2\right) \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 128+16i & -64 \\ 0 & 0 \end{bmatrix} \to (1)$$

$$(A_2+B_2)^{T}(A_2+B_2)^{C}(A_2+B_2)^{C}(A_2+B_2)^{T}$$

$$\Rightarrow \begin{bmatrix} 2-2i & 2+2i \\ 2+2i & 0 \end{bmatrix}\begin{bmatrix} 2-2i & 2+2i \\ 2+2i & 0 \end{bmatrix}\begin{bmatrix} 2-2i & 2+2i \\ 2+2i & 0 \end{bmatrix}\begin{bmatrix} 2-2i & 2+2i \\ 2+2i & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 128+16i & -64 \\ 0 & 0 \end{bmatrix} \rightarrow (2)$$

$$(1)=(2)$$

$$\therefore \left[(A_1+B_1)(A_1+B_1)^{CT}(A_1+B_1)^{CT}(A_1+B_1)\right]\cup\left[(A_2+B_2)(A_2+B_2)^{CT}(A_2+B_2)^{CT}(A_2+B_2)\right]$$

$$=\left[(A_1+B_1)^{T}(A_1+B_1)^{C}(A_1+B_1)^{C}(A_1+B_1)^{T}\right]\cup\left[(A_2+B_2)^{T}(A_2+B_2)^{C}(A_2+B_2)^{C}(A_2+B_2)^{T}\right]$$

$$\begin{bmatrix} 12 & -4i \\ -4i & 0 \end{bmatrix}\cup\begin{bmatrix} 128+16i & -64 \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} 128+16i & -64 \\ 0 & 0 \end{bmatrix}\cup\begin{bmatrix} 12 & -4i \\ -4i & 0 \end{bmatrix}$$

L.H.S=R.H.S

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