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Aboodh Transform for Solving Fractional Differential Equations

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Abstract

In this paper , Aboodh transform is applied to solve some fractional differential equations. The methodology presented here is based on some general theorems on particular solutions of some fractional differential equations with Aboodh transform and the expansion coefficients of binomial series.

1. INTRODUCTION

Fractional Calculus is the branch of mathematics which deals with the investigation and applications of integrals and derivatives of arbitrary order. Owing to the increasing applications, there has been significant interest in developing transforms for the solutions of fractional differential equations.
Integral transforms aret the most useful techniques of the mathematics which are used to find the solutions of differential equations, partial differential equations,integro-differential equations, partial integro-differential equations,delay differential equations and population growth.
Aboodh transform is derived from the classical Fourier integral. Because of the simplicity of the Aboodh transform and its mathematical properties, Aboodh transform was introduced by Khalid Aboodh to demonstrate the process of solving some ordinary differential equations in the time domain. Typically, Fourier, Laplace, Elzaki and Sumudu transforms are the essential mathematical tools for solving differential equations.

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[2]Key Words: Integral transforms, fractional derivative and integrals,fractional differential equations

Also Aboodh transform and some of its fundamental properties are used to solve differential equations. The Aboodh transform is defined for functions of exponential order, we consider functions in the set A given by

$$A = \{f(t) : \exists M, k_1, k_2 > 0, |f(t)| < Me^{-vt}\} \quad (1)$$

For a given function in the set A, the constant M must be finite number,k_1, k_2 may be finite or infinite.
The Aboodh transform denoted by the operator $A(.)$ is defined by the integral equation

$$A[f(t)](v) = K(v) = \frac{1}{v}\int_0^\infty f(t)e^{-vt}dt, \qquad t \geq 0, k_1 \leq v \leq k_2 \quad (2)$$

The variable v in this transform is used to factor the variable t in the argument of the function f. This transform has deeper connection with the Laplace transform.

In this paper, we apply the aboodh transform of the fractional derivative and the expansion co-efficients of binomial series to derive solutions of some familes of fractional differential equations.

We present here some useful definitions and Preliminaries as follows:

Definition 1.1. *The fractional derivative of a casual function f(t) is defined by*

$$\frac{d^\alpha f(t)}{dt^\alpha} = \begin{cases} f^{(n)}(t) & if \ \alpha = n \epsilon N \\ \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(t)}{(t-x)^{(\alpha-n+1)}}dt & if \ n-1 < \alpha < n \end{cases} \quad (3)$$

where the Euler gamma function $\Gamma(.)$ *is defined by*

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt, \quad (Re(z) > 0) \quad (4)$$

Definition 1.2. *The inverse Aboodh transform of a function f(t),t*$\epsilon(0, +\infty)$ *if A[f(t)]=k(v) then*

$$f(t) = A^{-1}[k(v)] \quad (5)$$

Definition 1.3. *Two parameters Mittag-Leffler function is defined by*

$$E_{\alpha,\beta} = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad (z, \alpha, \beta \epsilon C, Re(\alpha) > 0) \quad (6)$$

Definition 1.4. *The simplest wright function is defined by*

$$\phi(\alpha, \beta; z) = \sum_{k=0}^\infty \frac{1}{\Gamma(\alpha k + \beta)}\frac{z^k}{k!}, \qquad (z, \alpha, \beta \epsilon C) \quad (7)$$

Definition 1.5. *The Riemann-Liouville fractional derivatives $D^{\alpha}_{a+}y$ and $D^{\alpha}_{b-}y$ of order $\alpha\epsilon C$ are defined by*

$$\begin{aligned}(D^{\alpha}_{a+}y)(x) &= \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^{n}\int_{a}^{x}\frac{y(t)dt}{(x-t)^{(\alpha-n+1)}} \quad (n=[\Re(\alpha)]+1; x>a)\\ (D^{\alpha}_{b-}y)(x) &= \frac{1}{\Gamma(n-\alpha)}\left(\frac{-d}{dx}\right)^{n}\int_{x}^{b}\frac{y(t)dt}{(t-x)^{(\alpha-n+1)}} \quad (n=[\Re(\alpha)]+1; x<b),\end{aligned} \tag{8}$$

respectively, where $[\Re(\alpha)]$ means the integral part of $\Re(\alpha)$

2. PRELIMINARIES

1.$A[f^{(n)}(t)](v) = v^{n}k(v) - \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{v^{2-n+k}}$

2.$A[t^{n}](v) = \frac{n!}{v^{n+2}}$

3.$A[x^{\beta}] = \frac{\Gamma(\beta+1)}{v^{\beta+2}}$

4.$A[f^{(n)}(x)] = v^{n}k(v) - v^{n-2}f(0) - v^{n-3}f^{(1)}(0) - ...... - \frac{f^{(n-1)}(0)}{v}$

5.$A[\int_{0}^{t} f(t)dt] = \frac{k(v)}{v}$

6.$A[\int_{0}^{x} f(x-t)g(t)dt] = vk(v)g(v)$

7.The binomial co-efficients are defined by $\binom{\lambda}{n} = \frac{\lambda!}{\lambda!(\lambda-n)!}$ where λ and n are integers.

Note that $0! = 1, then$

$$\binom{\lambda}{0} = 1, \binom{\lambda}{\lambda} = 1 \quad \text{and} \quad (1-z)^{-\lambda} = \sum_{r=0}^{\infty}\frac{(\lambda_r)}{r!}z^{r} = \sum_{r=0}^{\infty}\binom{\lambda+r-1}{r}z^{r}$$

Lemma 2.1. *Aboodh transformation of Riemann-Liouville fractional integral operator of order $\alpha > 0$ can be written in the form*

$$A[J^{\alpha}f(x)] = \frac{k(v)}{v^{\alpha}} \tag{9}$$

Proof. Aboodh transform of Riemann-Liouville fractional integral operator $\alpha > 0$ is:

$$\begin{aligned} A[J^{\alpha} f(x)] =& A[\frac{1}{\Gamma(\alpha)} \int_0^x (x-a)^{(\alpha-1)} f(t)dt] \\ =& \frac{1}{\Gamma(\alpha)} vk(v)g(v) \\ =& \frac{k(v)}{v^{\alpha}} \\ \text{where} \quad g(v) =& A[x^{\alpha-1}] = \frac{\Gamma(\alpha)}{v^{\alpha+1}} \end{aligned}$$

□

Lemma 2.2. *Aboodh transformation of Caputo fractional derivative for*

$$\alpha > 0, m-1 < \alpha \leq m, m\varepsilon N$$

$$A[D^{\alpha} f(x)] = \frac{1}{v^{m-\alpha}} [v^m K(v) - v^{m-2} f(0) - v^{m-3} f^{(1)}(0) - .... - \frac{f^{(m-1)(0)}}{v}] \quad (10)$$

Proof.

$$A[D^{\alpha} f(x)] = A[J^{m-\alpha} f^{(m)}(x)] = \frac{A[f^{(m)}(x)]}{v^{m-\alpha}} \quad (11)$$

By use of Preliminary (4), the desired result follows. □

3. SOLUTIONS OF FRACTIONAL DIFFERENTIAL EQUATIONS

Throughout this section, we let y(t) be such that for some value of the parameter v, the Aboodh transform A[f(t)](v) converges.

Theorem 3.1. *Let* $0 < \alpha \leq 1$ *and* $b\epsilon R$ *. Then the fractional differential equation*

$$y^{\alpha}(t) - by(t) = 0 \quad (12)$$

with the initial condition $y(0) = c_0$ *has its solution given by*

$$y(t) = c_0 \sum_{k=0}^{\infty} \frac{(bt^{\alpha})^k}{\Gamma(\alpha k + 1)} = E_{\alpha,1}(bt^{\alpha}) \quad (13)$$

Proof. Applying Aboodh transform to (12),

$$\begin{aligned}
[v^{\alpha}k(v)-\frac{y(0)}{v^{2-\alpha}}]-bk(v)=&0\\
k(v)[v^{\alpha}-b]-\frac{c_0}{v^{2-\alpha}}=&0\\
k(v)[v^{\alpha}-b]=&c_0v^{\alpha-2}\\
k(v)=&c_0\sum_{k=0}^{\infty}b^kv^{-\alpha k-2}
\end{aligned}$$

Applying inverse Aboodh transform,

$$\begin{aligned}
=&c_0\sum_{k=0}^{\infty}\frac{b^kt^{\alpha k}}{\Gamma(\alpha k+1)}\\
y(t)=&c_0\sum_{k=0}^{\infty}\frac{(bt^{\alpha})^k}{\Gamma(\alpha k+1)}\\
=&c_0E_{\alpha,1}(bt^{\alpha})
\end{aligned}$$

□

Theorem 3.2. *Let* $1<\alpha<2$ *and* $a,b\varepsilon R$*. Then the fractional differential equation*

$$y''(t)+ay^{(\alpha)}(t)+by(t)=0 \tag{14}$$

with the initial conditions $y(0)=c_0$ *and* $y'(0)=c_1$ *has its solution given by*

$$\begin{aligned}
y(t)=c_0&\sum_{k=0}^{\infty}\frac{(-b)^kt^{(2k)}}{k!}\sum_{r=0}^{\infty}\frac{\Gamma(r+k+1)(-at^{(2-\alpha)})^r}{\Gamma[(2-\alpha)r+2k+1]r!}\\
+c_1&\sum_{k=0}^{\infty}\frac{(-b)^kt^{(2k+1)}}{k!}\sum_{r=0}^{\infty}\frac{\Gamma(r+k+1)(-at^{(2-\alpha)})^r}{\Gamma[(2-\alpha)r+2k+2]r!}\\
+ac_0&\sum_{k=0}^{\infty}\frac{(-b)^kt^{(2k-\alpha+2)}}{k!}\sum_{r=0}^{\infty}\frac{\Gamma(r+k+1)(-at^{(2-\alpha)})^r}{\Gamma[(2-\alpha)r+2k+3]r!}\\
+ac_1&\sum_{k=0}^{\infty}\frac{(-b)^kt^{(2k-\alpha+3)}}{k!}\sum_{r=0}^{\infty}\frac{\Gamma(r+k+1)(-at^{(2-\alpha)})^r}{\Gamma[(2-\alpha)r+2k+4]r!}
\end{aligned}\tag{15}$$

Proof. Applying Aboodh transform to equation (14),

$$\begin{aligned}
[v^2k(v)-y(0)-\frac{y'(0)}{v}]+a[v^{\alpha}k(v)-\frac{y(0)}{v^{2-\alpha}}-\frac{y'(0)}{v^{3-\alpha}}]+bk(v)=0\\
k(v)[v^2+av^{\alpha}+b]-c_0-\frac{c_1}{v}-\frac{ac_0}{v^{2-\alpha}}-\frac{ac_1}{v^{3-\alpha}}=0
\end{aligned}$$

$$k(v)=\frac{c_0+c_1v^{-1}+ac_0v^{\alpha-2}+ac_1v^{\alpha-3}}{v^2+av^{\alpha}+b} \tag{16}$$

Now
$$\frac{1}{v^2+av^\alpha+b} = \frac{1}{(v^2+av^\alpha)[1+\frac{b}{v^2+av^\alpha}]}$$
$$= \frac{1}{v^2+av^\alpha}[1+\frac{b}{v^2+av^\alpha}]^{-1}$$
$$= \frac{1}{v^2+av^\alpha}\sum_{k=0}^{\infty}\left(\frac{-b}{v^2+av^\alpha}\right)^k$$
$$= \sum_{k=0}^{\infty}\frac{(-b)^k}{(v^2+av^\alpha)^{k+1}}$$
$$= \sum_{k=0}^{\infty}\frac{(-b)^k}{(v^2)^{k+1}[1+av^{\alpha-2}]^{k+1}}$$
$$= \sum_{k=0}^{\infty}(-b)^k v^{-2k-2}\sum_{r=0}^{\infty}\binom{k+r}{r}(-av^{\alpha-2})^r$$
$$= \sum_{k=0}^{\infty}(-b)^k\sum_{r=0}^{\infty}\binom{k+r}{r}(-a)^r v^{(\alpha-2)r-2k-2}$$

From (16)

$$k(v) = c_0\sum_{k=0}^{\infty}(-b)^k\sum_{k=0}^{\infty}\binom{k+r}{r}(-a)^r v^{[(2-\alpha)r+2k]-2}$$
$$+c_1\sum_{k=0}^{\infty}(-b)^k\sum_{k=0}^{\infty}\binom{k+r}{r}(-a)^r v^{[(2-\alpha)r+2k]-1-1+1}$$
$$+ac_0\sum_{k=0}^{\infty}(-b)^k\sum_{k=0}^{\infty}\binom{k+r}{r}(-a)^r v^{[(2-\alpha)r+2k]+\alpha-4}$$
$$+ac_1\sum_{k=0}^{\infty}(-b)^k\sum_{k=0}^{\infty}\binom{k+r}{r}(-a)^r v^{[(2-\alpha)r+2k]+\alpha-5}$$

Applying inverse Aboodh transform, we have

$$y(t) = c_0\sum_{r=0}^{\infty}\frac{(-b)^k}{k!}\frac{\Gamma(k+r+1)(-a)^r t^{(2-\alpha)r+2k}}{\Gamma[(2-\alpha)r+2k+1]r!}$$
$$+c_1\sum_{r=0}^{\infty}\frac{(-b)^k}{k!}\frac{\Gamma(k+r+1)(-a)^r t^{(2-\alpha)r+2k+1}}{\Gamma[(2-\alpha)r+2k+2]r!}$$
$$+ac_0\sum_{r=0}^{\infty}\frac{(-b)^k}{k!}\frac{\Gamma(k+r+1)(-a)^r t^{(2-\alpha)r+2k-\alpha+2}}{\Gamma[(2-\alpha)r+2k-\alpha+3]r!}$$
$$+ac_1\sum_{r=0}^{\infty}\frac{(-b)^k}{k!}\frac{\Gamma(k+r+1)(-a)^r t^{(2-\alpha)r+2k-\alpha+3}}{\Gamma[(2-\alpha)r+2k-\alpha+4]r!}$$

Thus we will get the desired solution (15). □

Theorem 3.3. *Let* $1 < \alpha < 2$ *and* $a, b \epsilon R$. *Then the fractional differential equation*

$$y^{(\alpha)}(t) + ay'(t) + by(t) = 0 \tag{17}$$

with the initial condition $y(0) = c_0$ *and* $y'(0) = c_1$ *has its solution given by*

$$\begin{aligned} y(t) = c_0 \sum_{k=0}^{\infty} \frac{(-b)^k}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)(-a)^r t^{(\alpha-1)r+\alpha k}}{\Gamma[(\alpha-1)r+\alpha k]r!} \\ +c_1 \sum_{k=0}^{\infty} \frac{(-b)^k}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)(-a)^r t^{(\alpha-1)r+\alpha k+1}}{\Gamma[(\alpha-1)r+\alpha k+2]r!} \\ +ac_0 \sum_{k=0}^{\infty} \frac{(-b)^k}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)(-a)^r t^{(\alpha-1)r+\alpha k+\alpha+1}}{\Gamma[(\alpha-1)r+\alpha k+\alpha]r!} \end{aligned} \tag{18}$$

Proof. Applying Aboodh transform to equation (17),

$$\begin{aligned} [v^\alpha k(v) - \frac{y(0)}{v^{2-\alpha}} - \frac{y'(0)}{v^{3-\alpha}}] + a[vk(v) - \frac{y(0)}{v}] + bk(v) = 0 \\ k(v)[v^\alpha + av + b] - \frac{c_0}{v^{2-\alpha}} - \frac{c_1}{v^{3-\alpha}} - \frac{ac_0}{v} = 0 \\ k(v)[v^\alpha + av + b] = c_0 v^{\alpha-2} + c_1 v^{\alpha-3} + ac_0 v^{-1} \end{aligned}$$

$$k(v) = \frac{c_0 v^{\alpha-2} + c_1 v^{\alpha-3} + ac_0 v^{-1}}{v^\alpha + av + b} \tag{19}$$

Since

$$\begin{aligned} \frac{1}{v^\alpha + av + b} &= \frac{1}{(v^\alpha + av)[1 + \frac{b}{v^\alpha + av}]} \\ &= \frac{1}{v^\alpha + av}[1 + \frac{b}{v^\alpha + av}]^{-1} \\ &= \frac{1}{v^\alpha + av} \sum_{k=0}^{\infty} (\frac{-b}{v^\alpha + av})^k \\ &= \sum_{k=0}^{\infty} (\frac{(-b)^k}{(v^\alpha + av)^{k+1}}) \\ &= \sum_{k=0}^{\infty} \frac{(-b)^k}{(v^\alpha)^{k+1}(1 + av^{1-\alpha})^{k+1}} \\ &= \sum_{k=0}^{\infty} (-b)^k v^{-\alpha k - \alpha} \sum_{r=0}^{\infty} \binom{k+r}{r} (-av^{1-\alpha})^r \\ &= \sum_{k=0}^{\infty} (-b)^k \sum_{r=0}^{\infty} \binom{k+r}{r} (-a)^r v^{-(\alpha-1)r - \alpha k - \alpha} \end{aligned}$$

From (19),

$$\begin{aligned} k(v) = c_0 \sum_{k=0}^{\infty}(-b)^k \sum_{k=0}^{\infty} \binom{k+r}{r} (-a)^r v^{-(\alpha-1)r-\alpha k-2} \\ +c_1 \sum_{k=0}^{\infty}(-b)^k \sum_{k=0}^{\infty} \binom{k+r}{r} (-a)^r v^{-(\alpha-1)r-\alpha k-3} \\ +ac_0 \sum_{k=0}^{\infty}(-b)^k \sum_{k=0}^{\infty} \binom{k+r}{r} (-a)^r v^{-(\alpha-1)r-\alpha k-\alpha-1} \end{aligned}$$

Applying inverse Aboodh transform, we have

$$\begin{aligned} k(v)) = c_0 \sum_{k=0}^{\infty} \frac{(-b)^k}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)(-a)^r t^{(\alpha-1)r+\alpha k}}{\Gamma[(\alpha-1)r+\alpha k]r!} \\ +c_1 \sum_{k=0}^{\infty} \frac{(-b)^k}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)(-a)^r t^{(\alpha-1)r+\alpha k+1}}{\Gamma[(\alpha-1)r+\alpha k+2]r!} \\ +ac_0 \sum_{k=0}^{\infty} \frac{(-b)^k}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)(-a)^r t^{(\alpha-1)r+\alpha k+\alpha+1}}{\Gamma[(\alpha-1)r+\alpha k+\alpha]r!} \end{aligned}$$

Thus we will get the desired solution (18).

□

Remark 3.4. *If a=0 in equation (17), then the equation*

$$y^{\alpha}(t) + by(t) = 0, 1 < \alpha \leq 2 \tag{20}$$

with the initial conditions $y(0) = c_0$ *and* $y'(0) = c_1$ *has its solution given by*

$$\begin{aligned} y(t) = c_0 \sum_{k=0}^{\infty} \frac{(-bt^{\alpha})^k}{\Gamma(\alpha k+1)} + c_1 t \sum_{k=0}^{\infty} \frac{(-bt^{\alpha})^k}{\Gamma(\alpha k+2)} \\ = c_0 E_{\alpha,1}(-bt^{\alpha}) + c_1 t E_{\alpha,2}(-bt^{\alpha}) \end{aligned} \tag{21}$$

4. CONCLUSION

In this paper, ”Aboodh transform” was applied to solve some fractional differential equations. The connection of Aboodh transform with the Laplace transform goes much deeper and we can find other relations of Aboodh transform by this connection. Aboodh transform is a convenient tool for solving fractional differential equations in the time domain.

REFERENCES

[1] I.Podlubny., *Fractional Differential Equation*, Academic Press, San Diego(1999)

[2] Khalid Sulliman Aboodh., *The New Integral transform "Aboodh transform"*, Global Journal of Pure and Applied Mathematics, Volume 9, No.1(2013), pp.35-43.

[3] S.Lin, C.Lu, *Laplace transform for solving some families of fractional differential equations and its applications*, Advances in Difference Equations(2013),1-9.

[4] Abdelbagy A. Alshikh, Mohand M.Abdelrahim Mahgob, *A Comparative study between Laplace transform and two new integrals "Elzaki" Transform and "Aboodh" Transform*, Pure and Applied Mathematics Journal, Vol 5, No.5, 2016, pp.145-150.

[5] A. Kashuri, A. Fundo, R.Liko, *New Integral transform for solving some fractional differential equations*, International Journal of Pure and Applied Mathematics, Volume 103, No.4,(2015), 675-682.

[6] Elsayed A.E. Mohamed, *Elzaki transformation for Linear Fractional Differential Equations*, Journal of Computational and Theoretical Nanoscience, Vol. 12, 2303-2305, 2015.