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Some Results Associated with Jacobi Matrix Polynomials

Mosaed M. Makky

Department of Mathematics, Faculty of Science, South Valley University (Qena-Egypt)

Email: mosaed_makky11@yahoo.com, mosaed_makky@sci.svu.edu.eg

Abstract

In this paper the main idea is to obtain some recurrence relations and generating matrix functions for Jacobi matrix polynomials of two complex variables by acting of differential operator.

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1. INTRODUCTION

The study of special matrix functions found in previous studies and both the Lie group theory, number theory is defined in [1, 2]. A polynomials matrix has also appeared in recent studies in [3, 4, 5, 6]. The authors presented and studied the polynomials of the Jacobi matrix in [7, 8, 9, 10, 11, 12].

The theory of orthogonal polynomials extends, as in papers [13, 14, 15], to a polynomials matrix.

Some results in the classical polynomials orthogonal theory extended in [4, 16] to orthogonal polynomials.

Our main aim in this paper is to demonstrate new Jacobi matrix polynomials properties under the use of certain differential operators.

For a matrix $A \in \mathbb{C}^{NxN}$, its spectrum $\sigma(A)$ represents the set of all eigenvalues of matrix A.

If f (z) and g(z) are holomorphic functions of the one complex variable z, which are defined in an open set Φ of the complex plane and A, B are matrices in $\mathbb{C}^{NxN}$ with $\sigma(A) \subset \Phi$ and $\sigma(B) \subset \Phi$; then from the properties of the matrix functional calculus (see e.g. [17]), it follows that

$$f(A)\,g(B) = g(B)\,f(A). \tag{1.1}$$

If $A \in \mathbb{C}^{NxN}$ is a matrix such that $A + nI$ is an invertible matrix for all integer $n \geq 0$ we have the matrix version of the pochhammer symbol is

$$(A)_n = A(A+I)(A+2I)\ldots(A+(n-1)I)\ ;\ \ n \geq 1;\ \ (A)_0 \equiv I\ . \tag{1.2}$$

The classical Jacobi matrix polynomials $P_n^{(A,B)}(x)$ of one variable as given

$$P_n^{(A,B)}(z) = \frac{(I+A)_n}{n!}\, {}_2F_1\left(-nI, I+A+B+nI; I+A; \frac{1-z}{2}\right) \tag{1.3}$$

where the hypergeometric function write as follows (e.g. [1]):

$${}_2F_1(A;B,C;z) = \sum_{n=0}^{\infty} \frac{(A,n)(B,n)(C,n)^{-1}}{(1,n)} z^n$$

for matrices $A, B, C \in \mathbb{C}^{NxN}$ such that $C + nI$ is invertible for all integer $n \geq 0$ and for |z| < 1.

For A is an arbitrary matrix in $\mathbb{C}^{NxN}$ and using (1.2), we have the following relations (e.g. [18])

$$(A)_{n+k} = A_n(A+nI)_k = A_k(A+kI)_n$$

$$(A)_{n+2k} = A_n(A+nI)_{2k} = A_{2k}(A+2kI)_n$$

and

$$(A)_{2k} = 2^{2k}\left(\frac{1}{2}A\right)_k\left(\frac{1}{2}(A+I)\right)_k .$$

Cekim B. and etc. (e.g. [19]) some recurrence relations for Jacobi matrix polynomials (JMP's) are given and a generating matrix functions for JMP's is also obtained in this article furthermore, we show the integral representations for JMP's are given it, for A and B be matrices in $\mathbb{C}^{NxN}$ whose eigenvalues, z, all satisfy $\mathrm{Re}(z) > -1$, Jacobi matrix polynomials satisfies some recurrence relations which satisfied by Jacobi matrix polynomials (JMP's) of one variable (e.g. [19]) are given in the following:

Theorem (1.1)

Let A and B be matrices in $\mathbb{C}^{NxN}$ whose eigenvalues z, all satisfy $\mathrm{Re}(z)>-1$ (JMP's) then we have

i. $$\frac{d}{dx}P_n^{(A,B)}(x)=\sum_{k=0}^{n}\frac{\left(n+1\right)I+A+B}{2}P_{n-1}^{(A+I,B+I)}(x)$$

ii. $$\frac{d^s}{dx^s}P_n^{(A,B)}(x)=\frac{\left(\left(n+1\right)I+A+B\right)_s}{2^s}P_{n-s}^{(A+sI,B+sI)}(x) \qquad \text{for} \quad 0\le s\le n$$

iii. $P_n^{(A,B)}(-x)=(-1)^n P_n^{(A,B)}(x)$.

Theorem (1.2)

Let A and B be matrices in $\mathbb{C}^{NxN}$ whose eigenvalues z, all satisfy $\mathrm{Re}(z)>-1$. For the Jacobi matrix polynomials (JMP's), the following recurrence relations are achieved:

$$(x+1)\frac{d}{dx}P_n^{(A,B)}(x)=n\,P_n^{(A,B)}(x)+\left(B+nI\right)P_{n-1}^{(A+I,B)}(x)$$

and

$$(x-1)\frac{d}{dx}P_n^{(A,B)}(x)=n\,P_n^{(A,B)}(x)-\left(A+nI\right)P_{n-1}^{(A,B+I)}(x).$$

From above theorem we can write

$$2\frac{d}{dx}P_n^{(A,B)}(x)=\left(B+nI\right)P_{n-1}^{(A+I,B)}(x)+\left(A+nI\right)P_{n-1}^{(A,B+I)}(x)$$

and

$$\left(A+B+(n+2)I\right)P_n^{(A+I,B+I)}(x)=\left(B+(n+1)I\right)P_n^{(A+I,B)}(x)+\left(A+(n+1)I\right)P_n^{(A,B+I)}(x).$$

In addition to many other results studied in article [20].

The Jacobi matrix polynomials of two complex variables $P_n^{(A,B)}(z,w)$ is defined as follows:

$$P_n^{(A,B)}(z,w)=\sum_{k=0}^{n}\frac{\left(I+A\right)_n\left(I+A+B\right)_{n+k}\left(I+A\right)_k^{-1}\left(I+A+B\right)_n^{-1}}{k!(n-k)!}\left(\frac{z-\sqrt{w}}{2}\right)^k$$

where A,B be a positive stable matrices in $\mathbb{C}^{NxN}$, $z,w\in(-1,1)$, $(A\,B=B\,A)$ which can be simplified to obtain the generating matrix polynomials as follows:

$$P_n^{(A,B)}(z,w)=\sum_{k=0}^{n}\frac{(-1)^{n-k}(I+B)_n(I+A+B)_{n+k}(I+B)_k^{-1}(I+A+B)_n^{-1}}{k!(n-k)!}\left(\frac{z+\sqrt{w+1}}{2\sqrt{w+1}}\right)^k$$

$$P_n^{(A,B)}(z,w)=\sum_{n,k=0}^{\infty}\frac{(I+A)_n(I+B)_n(I+A)_k^{-1}(I+B)_{n-k}^{-1}}{k!(n-k)!}\left(\frac{z-\sqrt{w}}{2}\right)^k\left(\frac{z+\sqrt{w}}{2}\right)^{n-k}$$

$$P_n^{(A,B)}(z,w)=\frac{(I+A)_n}{n!}\left(\frac{z+\sqrt{w}}{2}\right)^n {}_2F_1\left(-nI,-B-nI,I+A;\frac{z-\sqrt{w}}{z+\sqrt{w}}\right)$$

and

$$P_n^{(A,B)}(z,w)=\frac{(I+A)_n}{n!}\,{}_2F_1\left(-nI,I+A+B+nI,I+A;\frac{\sqrt{w}-z}{2}\right).$$

2. MAIN RESULTS

In this section we will study the effect of the differential operator on the Jacobi matrix polynomials to identify the results that can be obtained for this study.

For this propose, let $A\in\mathbb{C}^{NxN}$ be a matrix version of the pochhammer symbol is

$$(A)_n=A(A+I)(A+2I)\dots(A+(n-1)I)\ ;\quad n\geq 1;\quad (A)_0\equiv I\ .$$

Now, we write the conjugate relations that we can use in the following theorem as follows:

$$(I+A+B)_{n+1}=(I+A+B)_n(I+A+B+nI) \tag{2.1}$$

$$(I+A)_n=(I+A)[I+(I+A)]_{n-1} \tag{2.2}$$

$$(I+A)_{k+1}=(I+A)[I+(I+A)]_k \tag{2.3}$$

$$(I+A+B)_{n+k+1}=(I+A+B)(2I+A+B)[I+(I+A)+(I+B)]_{(n-1)+k}\ . \tag{2.4}$$

This is what we will introduced in the following theorem:

Theorem (2.1):

Suppose that the Jacobi matrix polynomials of two complex variables given as follows:

$$P_n^{(A,B)}(z,w)=\sum_{k=0}^{n}\frac{\left(I+A\right)_n\left(I+A+B\right)_{n+k}\left(I+A\right)_k^{-1}\left(I+A+B\right)_n^{-1}}{k!(n-k)!}\left(\frac{z-\sqrt{w}}{2}\right)^k \quad (2.5)$$

where A,B be a positive stable matrices in $\mathbb{C}^{NxN}$, $z,w\in(-1,1)$ and given the differential operator

$$\Lambda=\left(\frac{\partial}{\partial z}+\sqrt{w}\,\frac{\partial}{\partial w}\right). \quad (2.6)$$

Then Jacobi matrix polynomials of two complex variables satisfies the matrix differential equation:

$$\Lambda\, P_n^{A,B}(z,w)-\frac{\left(I+A+B+nI\right)}{4}P_{n-1}^{A+I,B+I}(z,w)=0. \quad (2.7)$$

Proof:

By the differential operator (2.6) and using the relations (2.1), (2.2), (2.3) and (2.4) for the Jacobi matrix polynomials of two complex variables $P_n^{(\alpha,\beta)}(z,w)$ we see that

$$\left(\frac{\partial}{\partial z}+\sqrt{w}\,\frac{\partial}{\partial w}\right)P_n^{(A,B)}(z,w)$$

$$=\left(\frac{\partial}{\partial z}+\sqrt{w}\,\frac{\partial}{\partial w}\right)\sum_{k=0}^{n}\frac{\left(I+A\right)_n\left(I+A+B\right)_{n+k}}{k!(n-k)!}\left(I+A\right)_k^{-1}\left(I+A+B\right)_n^{-1}\left(\frac{z-\sqrt{w}}{2}\right)^k$$

$$=\sum_{k=0}^{n}\frac{k\left(I+A\right)_n\left(I+A+B\right)_{n+k}}{k!(n-k)!}\left(I+A\right)_k^{-1}\left(I+A+B\right)_n^{-1}\left(\frac{z-\sqrt{w}}{2}\right)^{k-1}\left(\frac{1}{2}-\frac{1}{4}\right)$$

$$=\frac{1}{4}\sum_{k=0}^{n}\frac{k\left(I+A\right)_n\left(I+A+B\right)_{n+k}}{k!(n-k)!}\left(I+A\right)_k^{-1}\left(I+A+B\right)_n^{-1}\left(\frac{z-\sqrt{w}}{2}\right)^{k-1}$$

$$=\frac{1}{4}\sum_{k=1}^{n}\frac{\left(I+A\right)_n\left(I+A+B\right)_{n+k+1}}{k!(n-(k+1))!}\left(I+A\right)_{k+1}^{-1}\left(I+A+B\right)_n^{-1}\left(\frac{z-\sqrt{w}}{2}\right)^k.$$

Using this relationship we get:

$$\Lambda P_n^{(A,B)}(z,w)=\frac{1}{4}\sum_{k=1}^{n}\frac{\left(I+A\right)_n\left(I+A+B\right)_{n+k+1}}{k!(n-(k+1))!\left(I+A\right)_{k+1}\left(I+A+B\right)_n}\left(\frac{z-\sqrt{w}}{2}\right)^k$$

$$=\frac{1}{4}\sum_{k=1}^{n}\left[\begin{array}{c}\dfrac{\left(I+A\right)\left[I+(I+A)\right]_{n-1}\left(I+A+B\right)\left(2I+A+B\right)\left[I+(I+A)+(I+B)\right]_{(n-1)+k}}{k!(n-(k+1))!\left(I+A\right)\left[I+(I+A)\right]_k}\times\\ \dfrac{\left(I+A+B+nI\right)}{\left(I+A+B\right)\left(2I+A+B\right)\left[I+(I+A)+(I+B)\right]_{n-1}}\left(\dfrac{z-\sqrt{w}}{2}\right)^k\end{array}\right]$$

$$=\frac{1}{4}\sum_{k=1}^{n}\frac{\left(I+A+B+nI\right)\left[I+(I+A)\right]_{n-1}\left[I+(I+A)+(I+B)\right]_{(n-1)+k}}{k!(n-(k+1))!\left[I+(I+A)\right]_k\left[I+(I+A)+(I+B)\right]_{n-1}}\left(\frac{z-\sqrt{w}}{2}\right)^k$$

$$=\frac{1}{4}\sum_{k=1}^{n}\left[\begin{array}{c}\dfrac{\left(I+A+B+nI\right)\left[I+(I+A)\right]_{n-1}\left[I+(I+A)+(I+B)\right]_{(n-1)+k}}{k!\left[(n-1)-k\right]!}\times\\ \left[I+(I+A)\right]_k^{-1}\left[I+(I+A)+(I+B)\right]_{n-1}^{-1}\left(\dfrac{z-\sqrt{w}}{2}\right)^k\end{array}\right]$$

$$\Lambda P_n^{A,B}(z,w)=\frac{\left(I+A+B+nI\right)}{4}P_{n-1}^{A+I,B+I}(z,w).$$

Then the Jacobi matrix polynomials (2.5) satisfies matrix differential equation (2.7).

In the same way as the differential operator on the Jacobi matrix polynomials and using differential operator suitable on different images of the Jacobi matrix polynomials we get the similar results:

$$P_n^{(A,B)}(z,w)=\sum_{k=0}^{n}\frac{\left(I+A\right)_n\left(I+A+B\right)_{n+k}\left(I+A\right)_k^{-1}\left(I+A+B\right)_n^{-1}}{k!(n-k)!}\left(\frac{z-\sqrt{w}}{2\sqrt{w+1}}\right)^k$$

$$P_n^{(A,B)}(z,w)=\sum_{k=0}^{n}\frac{\left(I+A\right)_n\left(I+B\right)_n\left(I+A\right)_k^{-1}\left(I+B\right)_{n-k}^{-1}}{k!(n-k)!}\left(\frac{z-\sqrt{w+1}}{2\sqrt{w+1}}\right)^k\left(\frac{z+\sqrt{w+1}}{2\sqrt{w+1}}\right)^{n-k}$$

and

$$P_n^{(A,B)}(z,w)=\sum_{k=0}^{n}\frac{\left(-1\right)^{n-k}\left(I+B\right)_n\left(I+A+B\right)_{n+k}\left(I+B\right)_k^{-1}\left(I+A+B\right)_n^{-1}}{k!(n-k)!}\left(\frac{z+\sqrt{w+1}}{2\sqrt{w+1}}\right)^k.$$

Recurrence relations for Jacobi matrix polynomials

Some recurrence relations have been deduced for the Jacobi matrix polynomials, as follows:

1- At first let $A, B \in \mathbb{C}^{N\,x\,N}$ where $0 < \Re(\xi) < 1$, for all $\xi \in \sigma(A)$, $\xi \in \sigma(B)$ and put w=1 in the Jacobi matrix polynomials (2.5) we get

$$P_n^{(A,B)}(z,1) = \sum_{k=0}^{n} \frac{(I+A)_n (I+A+B)_{n+k} (I+A)_k^{-1} (I+A+B)_n^{-1}}{k!(n-k)!} \left(\frac{z-1}{2}\right)^k \qquad (2.8)$$

where $z \in (-1,1)$.

We can write from the differential operator (2.6) that

$$\Lambda_z = \left(\frac{\partial}{\partial z}\right).$$

Therefore

$$\frac{\partial}{\partial z} P_n^{(A,B)}(z,1) = \frac{\partial}{\partial z} \sum_{k=0}^{n} \frac{(I+A)_n (I+A+B)_{n+k}}{k!(n-k)!} (I+A)_k^{-1} (I+A+B)_n^{-1} \left(\frac{z-1}{2}\right)^k$$

$$= \sum_{k=0}^{n} \frac{k\,(I+A)_n (I+A+B)_{n+k}}{k!(n-k)!} (I+A)_k^{-1} (I+A+B)_n^{-1} \left(\frac{z-1}{2}\right)^{k-1} \left(\frac{1}{2}\right)$$

$$= \frac{1}{2} \sum_{k=0}^{n} \frac{k\,(I+A)_n (I+A+B)_{n+k}}{k!(n-k)!} (I+A)_k^{-1} (I+A+B)_n^{-1} \left(\frac{z-1}{2}\right)^{k-1}$$

$$= \frac{1}{2} \sum_{k=1}^{n} \frac{(I+A)_n (I+A+B)_{n+k+1}}{k!(n-(k+1))!} (I+A)_{k+1}^{-1} (I+A+B)_n^{-1} \left(\frac{z-1}{2}\right)^k .$$

From which and relation (2.8) we get:

$$\Lambda_z\, P_n^{(A,B)}(z,1) = \frac{1}{2} \sum_{k=1}^{n} \frac{(I+A)_n (I+A+B)_{n+k+1}}{k!(n-(k+1))!(I+A)_{k+1}(I+A+B)_n} \left(\frac{z-1}{2}\right)^k$$

$$= \frac{1}{2} \sum_{k=1}^{n} \frac{(I+A+B+nI)[I+(I+A)]_{n-1} [I+(I+A)+(I+B)]_{(n-1)+k}}{k!(n-(k+1))![I+(I+A)]_k\, [I+(I+A)+(I+B)]_{n-1}} \left(\frac{z-1}{2}\right)^k$$

$$= \frac{1}{2} \sum_{k=1}^{n} \left[\frac{(I+A+B+nI)[I+(I+A)]_{n-1} [I+(I+A)+(I+B)]_{(n-1)+k}}{k!\,[(n-1)-k]!} \times [I+(I+A)]_k^{-1} [I+(I+A)+(I+B)]_{n-1}^{-1} \left(\frac{z-1}{2}\right)^k \right]$$

$$\Lambda_z\, P_n^{A,B}(z,1) = \frac{(I+A+B+nI)}{2} P_{n-1}^{A+I,B+I}(z,1).$$

Then Jacobi matrix polynomials satisfies the matrix differential equation:

$$\Lambda_z \, P_n^{A,B}(z,1) - \frac{\left(I+A+B+nI\right)}{2} P_{n-1}^{A+I,B+I}(z,1) = 0 . \tag{2.9}$$

Now by putting w=1 in the different images of the Jacobi matrix polynomials and with appropriate differential operators we get similar results:

$$P_n^{(A,B)}(z,1) = \sum_{k=0}^{n} \frac{\left(I+A\right)_n \left(I+A+B\right)_{n+k} \left(I+A\right)_k^{-1} \left(I+A+B\right)_n^{-1}}{k!(n-k)!} \left(\frac{z-\sqrt{2}}{2\sqrt{2}}\right)^k ,$$

$$P_n^{(A,B)}(z,1) = \sum_{k=0}^{n} \frac{\left(I+A\right)_n \left(I+B\right)_n \left(I+A\right)_k^{-1} \left(I+B\right)_{n-k}^{-1}}{k!(n-k)!} \left(\frac{z-\sqrt{2}}{2\sqrt{2}}\right)^k \left(\frac{z+\sqrt{2}}{2\sqrt{2}}\right)^{n-k}$$

and

$$P_n^{(A,B)}(z,1) = \sum_{k=0}^{n} \frac{\left(-1\right)^{n-k} \left(I+B\right)_n \left(I+A+B\right)_{n+k} \left(I+B\right)_k^{-1} \left(I+A+B\right)_n^{-1}}{k!(n-k)!} \left(\frac{z+\sqrt{2}}{2\sqrt{2}}\right)^k .$$

2- In the same way the following results can be obtained by put z=1 in (2.5) we get:

$$P_n^{(A,B)}(1,w) = \sum_{k=0}^{n} \frac{\left(I+A\right)_n \left(I+A+B\right)_{n+k} \left(I+A\right)_k^{-1} \left(I+A+B\right)_n^{-1}}{k!(n-k)!} \left(\frac{1-\sqrt{w}}{2}\right)^k$$

where $w \in (-1,1)$.

We can write from the differential operator (2.6) that

$$\Lambda_w = \left(\sqrt{w}\,\frac{\partial}{\partial w}\right).$$

Therefore

$$\left(\sqrt{w}\,\frac{\partial}{\partial w}\right) P_n^{(A,B)}(1,w)$$

$$= \left(\sqrt{w}\,\frac{\partial}{\partial w}\right) \sum_{k=0}^{n} \frac{\left(I+A\right)_n \left(I+A+B\right)_{n+k}}{k!(n-k)!} \left(I+A\right)_k^{-1} \left(I+A+B\right)_n^{-1} \left(\frac{1-\sqrt{w}}{2}\right)^k$$

$$= -\frac{1}{4} \sum_{k=0}^{n} \frac{k \left(I+A\right)_n \left(I+A+B\right)_{n+k}}{k!(n-k)!} \left(I+A\right)_k^{-1} \left(I+A+B\right)_n^{-1} \left(\frac{1-\sqrt{w}}{2}\right)^{k-1}$$

$$= -\frac{1}{4} \sum_{k=1}^{n} \frac{\left(I+A\right)_n \left(I+A+B\right)_{n+k+1} \left(I+A\right)_{k+1}^{-1} \left(I+A+B\right)_n^{-1}}{k!(n-(k+1))!} \left(\frac{1-\sqrt{w}}{2}\right)^k .$$

Then we get:

$$\Lambda_w\ P_n^{(A,B)}(1,w) = -\frac{1}{4}\sum_{k=1}^{n}\ \frac{(I+A)_n\,(I+A+B)_{n+k+1}}{k\,!(n-(k+1))!(I+A)_{k+1}(I+A+B)_n}\left(\frac{1-\sqrt{w}}{2}\right)^k$$

$$= -\frac{1}{4}\sum_{k=1}^{n}\ \frac{(I+A+B+nI)\big[I+(I+A)\big]_{n-1}\big[I+(I+A)+(I+B)\big]_{(n-1)+k}}{k\,!\big[(n-1)-k\big]!\big[I+(I+A)\big]_k\ \big[I+(I+A)+(I+B)\big]_{n-1}}\left(\frac{1-\sqrt{w}}{2}\right)^k$$

$$= -\frac{1}{4}\sum_{k=1}^{n}\ \frac{(I+A+B+nI)\big[I+(I+A)\big]_{n-1}\big[I+(I+A)+(I+B)\big]_{(n-1)+k}}{k\,!\big[(n-1)-k\big]!\big[I+(I+A)\big]_k\ \big[I+(I+A)+(I+B)\big]_{n-1}}\left(\frac{1-\sqrt{w}}{2}\right)^k$$

therefore

$$\Lambda_w\ P_n^{A,B}(1,w) = -\frac{(I+A+B+nI)}{4}\,P_{n-1}^{A+I,B+I}(1,w)\,.$$

Then the Jacobi matrix polynomials satisfies matrix differential equation:

$$\Lambda_w P_n^{A,B}(1,w) + \frac{(I+A+B+nI)}{4}\,P_{n-1}^{A+I,B+I}(1,w) = 0\,. \tag{2.10}$$

Again by putting z=1 in the different images of the Jacobi matrix polynomials and with appropriate differential operators we get similar results for the case w=1.

3- Now by putting A=I in the Jacobi matrix polynomials and using differential operator (2.6) we get:

$$\left(\frac{\partial}{\partial z} + \sqrt{w}\,\frac{\partial}{\partial w}\right)P_n^{(I,B)}(z,w)$$

$$= \left(\frac{\partial}{\partial z} + \sqrt{w}\,\frac{\partial}{\partial w}\right)\sum_{k=0}^{n}\ \frac{(2I)_n\,(2I+B)_{n+k}}{k\,!(n-k)!}(2I)_k^{-1}(2I+B)_n^{-1}\left(\frac{z-\sqrt{w}}{2}\right)^k$$

$$= \frac{1}{4}\sum_{k=0}^{n}\ \frac{k\,(2I)_n\,(2I+B)_{n+k}}{k\,!(n-k)!}(2I)_k^{-1}(2I+B)_n^{-1}\left(\frac{z-\sqrt{w}}{2}\right)^{k-1}$$

$$= \frac{(2I+B+nI)}{4}\sum_{k=1}^{n}\ \frac{(2I)_n\,(2I+B)_{n+k+1}}{k\,!(n-(k+1))!}(2I)_{k+1}^{-1}(2I+B)_n^{-1}\left(\frac{z-\sqrt{w}}{2}\right)^k\,.$$

Write this relationship down now as follows:

$$\Lambda P_n^{(I,B)}(z,w)=\frac{(2I+B+nI)}{4}\sum_{k=1}^{n}\frac{(2I)_n(2I+B)_{n+k+1}}{k![(n-1)-k]!(2I)_{k+1}(2I+B)_n}\left(\frac{z-\sqrt{w}}{2}\right)^k$$

$$=\frac{(2I+B+nI)}{4}\sum_{k=1}^{n}\frac{(3I)_{n-1}[3I+(I+B)]_{(n-1)+k}}{k!(n-(k+1))!(3I)_k[3I+(I+B)]_{n-1}}\left(\frac{z-\sqrt{w}}{2}\right)^k$$

$$=\frac{(2I+B+nI)}{4}\sum_{k=1}^{n}\frac{(3I)_{n-1}[3I+(I+B)]_{(n-1)+k}(3I)_k^{-1}[3I+(I+B)]_{n-1}^{-1}}{k!((n-1)-k)!}\left(\frac{z-\sqrt{w}}{2}\right)^k$$

$$\Lambda P_n^{I,B}(z,w)=\frac{(2I+B+nI)}{4}P_{n-1}^{2I,B+I}(z,w).$$

Then the Jacobi matrix polynomials satisfies matrix differential equation:

$$\Lambda P_n^{I,B}(z,w)-\frac{(2I+B+nI)}{4}P_{n-1}^{2I,B+I}(z,w)=0. \tag{2.11}$$

Also, by putting A=I in the different images of the Jacobi matrix polynomials and with appropriate differential operators we get similar results as follows:

$$P_n^{(I,B)}(z,w)=\sum_{k=0}^{n}\frac{(2I)_n(2I+B)_{n+k}(2I)_k^{-1}(2I+B)_n^{-1}}{k!(n-k)!}\left(\frac{z-\sqrt{w}}{2\sqrt{w+1}}\right)^k$$

$$P_n^{(I,B)}(z,w)=\sum_{k=0}^{n}\frac{(2I)_n(I+B)_n(2I)_k^{-1}(I+B)_{n-k}^{-1}}{k!(n-k)!}\left(\frac{z-\sqrt{w+1}}{2\sqrt{w+1}}\right)^k\left(\frac{z+\sqrt{w+1}}{2\sqrt{w+1}}\right)^{n-k}$$

and

$$P_n^{(I,B)}(z,w)=\sum_{k=0}^{n}\frac{(-1)^{n-k}(I+B)_n(2I+B)_{n+k}(I+B)_k^{-1}(2I+B)_n^{-1}}{k!(n-k)!}\left(\frac{z+\sqrt{w+1}}{2\sqrt{w+1}}\right)^k.$$

4- Finely putting B=I in the Jacobi matrix polynomials and with differential operator (2.6) we get:

$$\left(\frac{\partial}{\partial z}+\sqrt{w}\frac{\partial}{\partial w}\right)P_n^{(A,I)}(z,w)$$

$$=\left(\frac{\partial}{\partial z}+\sqrt{w}\frac{\partial}{\partial w}\right)\sum_{k=0}^{n}\frac{(I+A)_n(2I+A)_{n+k}}{k!(n-k)!}(I+A)_k^{-1}(2I+A)_n^{-1}\left(\frac{z-\sqrt{w}}{2}\right)^k$$

$$=\frac{1}{4}\sum_{k=0}^{n}\frac{k(I+A)_n(2I+A)_{n+k}}{k!(n-k)!}(I+A)_k^{-1}(2I+A)_n^{-1}\left(\frac{z-\sqrt{w}}{2}\right)^{k-1}$$

$$=\frac{1}{4}\sum_{k=1}^{n}\frac{(I+A)_n(2I+A)_{n+k+1}}{k!(n-(k+1))!}(I+A)_{k+1}^{-1}(2I+A)_n^{-1}\left(\frac{z-\sqrt{w}}{2}\right)^k.$$

Write this relationship down now as follows:

$$\Lambda P_n^{A,I}(z,w) = -\frac{}{4}\sum_{k=1}^{n} \frac{(2I+A+nI)[I+(I+A)]_{n-1}[3I+(I+A)]_{(n-1)+k}}{k!(n-(k+1))![I+(I+A)]_k [3I+(I+A)]_{n-1}}\left(\frac{z-\sqrt{w}}{2}\right)^k$$

$$= \frac{(2I+A+nI)}{4}\sum_{k=1}^{n} \frac{[I+(I+A)]_{n-1}[3I+(I+A)]_{(n-1)+k}[I+(I+A)]_k^{-1}[3I+(I+A)]_{n-1}^{-1}}{k!((n-1)-k)!}\left(\frac{z-\sqrt{w}}{2}\right)^k$$

$$= \frac{(2I+A+nI)}{4}\sum_{k=1}^{n} \frac{[I+(I+A)]_{n-1}[3I+(I+A)]_{(n-1)+k}[I+(I+A)]_k^{-1}[3I+(I+A)]_{n-1}^{-1}}{k!((n-1)-k)!}\left(\frac{z-\sqrt{w}}{2}\right)^k$$

i.e.

$$\Lambda P_n^{A,I}(z,w) = \frac{(2I+A+nI)}{4} P_{n-1}^{A+I,2I}(z,w).$$

Then the Jacobi matrix polynomials satisfies matrix differential equation:

$$\Lambda P_n^{A,I}(z,w) - \frac{(2I+A+nI)}{4} P_{n-1}^{A+I,2I}(z,w) = 0. \quad (2.12)$$

We also get similar results for the A = I case by putting B = I in the various images of the Jacobi matrix polynomials and with suitable differential operators.

CONCLUSIONS

A new comparison for the study of some important features of some matrix special functions matrix recurrence relationships, matrix differential recurrence relationships, and matrix differential equation was introduced in this paper. Some other special matrix functions in mathematical physics that play a required role can also be studied using the method developed in this paper.

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