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Estimation of the Reliability Function for a General Class of Probability Density Functions Using Transformation Method

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Abstract

In the present scripture, the problem for the estimation of the reliability function $R = Pr(X < Y)$for a general class of probability density functions due to Chaturvedi *et al*. (2009) has been taken into account. The expressions for the maximum likelihood estimator (MLE), uniformly minimum variance unbiased estimator (UMVUE) and interval estimation are derived for the considered probability density class. These estimators have been obtained using the method of transformation instead of the available conventional techniques.

Keywords: Stress-Strength model, Transformation method, MLE, UMVUE, Confidence interval.

1. INTRODUCTION

Life testing is the most important and obligatory area for engineering and survival analysis. The most frequently used function in the field of reliability engineering and life time data analysis is the reliability function, the function that gives the probability of an item operating for a certain amount of time without failure. Stress-Strength testing is a crucial topic in reliability engineering. In Stress-Strength testing, one statistical model is the probability$R = Pr(X < Y)$. This probability represents the performance of an item of Strength Y subject to a Stress X, where, X and Y both are non-negative and independently distributed continuous random variables. The term Stress-Strength was first introduced by Church and Harries (1970).

A lot of work has been done by various authors on the measure $R = Pr(X < Y)$, for a brief review one may refers to Downton (1973), Tong (1974), Kelly (1976), Sathe and Shah (1981), Chao (1982), Awad and Gharraf (1986) and Constantine, Karson and Tse (1986).

In the present paper, we consider a family of distributions suggested by Chaturvedi *et al*. (2009).This family of distributions covers many lifetime distributions as its particular cases. The UMVUE and MLE of R for the above family of distributions have been derived. In order to obtain these estimators, deviating from the conventional techniques the method of transformation is used.

The random variable X follows the distribution having the probability density function (pdf),

$$f(x; a, \lambda, \underline{\theta}) = \lambda G'(x; a, \underline{\theta}) \exp\{-\lambda G(x; a, \underline{\theta})\}; x \geq a > 0, \lambda > 0 \tag{1.1}$$

here, $G(x; a, \underline{\theta})$ is a function of $x$and may also depend on the parameter 'a' and '$\underline{\theta}$' may be vector valued. Moreover, $G(x; a, \underline{\theta})$ is monotonically increasing in $x$with$G(a; a, \underline{\theta}) = 0$, $G(\infty; a, \underline{\theta}) = \infty$ and $G'(x; a, \underline{\theta})$ denotes the derivative of $G(x; a, \underline{\theta})$ w.r.t. x.

We assume that λ is unknown but $a$and $\underline{\theta}$are known. If we consider the transformation

$$Y = G(x; a, \underline{\theta}) \tag{1.2}$$

then the probability density function of this new random variable comes

$$f_Y(y|\lambda) = \lambda \exp\{-\lambda y\} \tag{1.3}$$

which is the pdf of exponential variable with mean $1/_\lambda$.

So, we can draw the inferences by transformation method by considering the transformation (1.2). Here, we only need to deal with exponential distribution.

For example if we take $G(x; a, \underline{\theta}) = x^2$and $a = 0$, then $f(x; a, \underline{\theta})$ becomes Rayleigh distribution (Sinha (1986, p.200)) and by taking $Y = X^2$,we can transform the pdf of Rayleigh distribution into the pdf of exponential distribution.

2. MLE OF $R = Pr(X < Y)$ FOR A SUGGESTED CLASS OF LIFETIME DISTRIBUTIONS

Now, let X and Y are independent random variables with pdf of the form given in (1.1) with parameters $(\lambda_1, a_1, \underline{\theta_1})and(\lambda_2, a_2, \underline{\theta_2})$, respectively. For the given samples $\underline{X}$ and $\underline{Y}$ of size $n_1$and n_2, respectively, we are interested in estimation of reliability function$R = Pr(X < Y)$.

Theorem 1: The MLE of reliability function $R = Pr(X < Y)$ is given by

$$\tilde{R}_{XY} = \frac{T_y}{T_x + T_y} \tag{2.1}$$

where,

$$T_x = \frac{1}{n_1}\sum_{i=1}^{n_1} G\left(x_i; a_1, \underline{\theta_1}\right) \tag{2.2}$$

and

$$T_y = \frac{1}{n_2}\sum_{i=1}^{n_2} G\left(y_i; a_2, \underline{\theta_2}\right) \tag{2.3}$$

Proof: Let us consider the transformations $\epsilon = G\left(x; a_1, \underline{\theta_1}\right)$ and $\eta = G\left(y; a_2, \underline{\theta_2}\right)$, then we have two independent exponential variates ϵ and η with means '1/ λ_1' and '1/ λ_2', respectively. Now the reliability function $R_{XY} = Pr(X < Y) = Pr(\epsilon < \eta) = R_{\epsilon\eta}$.

Recall that the MLE of $R_{\epsilon\eta}$ based on samples $\underline{\epsilon}$ and $\underline{\eta}$ of size n_1 and n_2 respectively will be given in the form

$$\tilde{R}_{\epsilon\eta} = \frac{\bar{\eta}}{\bar{\epsilon} + \bar{\eta}} \tag{2.4}$$

where,

$$\bar{\epsilon} = \frac{1}{n_1}\sum_{i=1}^{n_1} \epsilon_i \qquad \text{and} \qquad \bar{\eta} = \frac{1}{n_2}\sum_{i=1}^{n_2} \eta_i$$

replacing ϵ_i by $G\left(x_i; a_1, \underline{\theta_1}\right)$ and η_i by $G\left(y_i; a_2, \underline{\theta_2}\right)$ in the expression of $\tilde{R}_{\epsilon\eta}$, we get

$$\tilde{R}_{\epsilon\eta} = \frac{T_y}{T_x + T_y} = \tilde{R}_{XY}$$

where,

$$T_x = \frac{1}{n_1}\sum_{i=1}^{n_1} G\left(x_i; a_1, \underline{\theta_1}\right) = \frac{1}{n_1}\sum_{i=1}^{n_1} \epsilon_i = \bar{\epsilon}$$

and similarly

$$T_y = \frac{1}{n_2}\sum_{i=1}^{n_2} G\left(y_i; a_2, \underline{\theta_2}\right) = \frac{1}{n_2}\sum_{i=1}^{n_2} \eta_i = \bar{\eta}$$

Hence, the theorem proved.

Corollary 2.1: For $G\left(x; a, \underline{\theta}\right) = x^2$ and $a = 0$, pdf $f(x; a, \lambda, \underline{\theta})$ becomes Rayleigh distribution and by taking $Y = X^2$, we can transform the pdf of Rayleigh distribution into the pdf of exponential distribution with parameter λ, where, λ is the scale parameter. For the case MLE of R_{XY} based on samples $\underline{x}$ and $\underline{y}$ of size n_1 and n_2, respectively, will be given in the form $\tilde{R}_{XY}$, comes out to be

$$\tilde{R}_{XY} = \frac{T_y}{T_x + T_y}$$

where,

$$T_x = \frac{1}{n_1}\sum_{i=1}^{n_1} G(x_i; a, \underline{\theta}_1) = \frac{1}{n_1}\sum_{i=1}^{n_1} x_i^2 \tag{2.5}$$

and

$$T_y = \frac{1}{n_2}\sum_{i=1}^{n_2} G(y_i; a, \underline{\theta}_2) = \frac{1}{n_2}\sum_{i=1}^{n_2} y_i^2 \tag{2.6}$$

Corollary 2.2: If we consider the transformation,$G(x; a, \underline{\theta}) = x^p$, where $p > 0$ and $a = 0$, equation (1.1) turns into the pdf of Weibull distribution (Johnson and Kotz (1970, p.250)) with parameters p and λ, where, p is shape parameter and λ is scale parameter. Considering p is known, MLE of R_{XY} based on samples $\underline{x}$ and $\underline{y}$ of size $n_1$and n_2, respectively,$\tilde{R}_{XY}$ comes out

$$\tilde{R}_{XY} = \frac{T_y}{T_x + T_y}$$

where,

$$T_x = \frac{1}{n_1}\sum_{i=1}^{n_1} G(x_i; a, \underline{\theta}_1) = \frac{1}{n_1}\sum_{i=1}^{n_1} x_i^{p_1} \tag{2.7}$$

and

$$T_y = \frac{1}{n_2}\sum_{i=1}^{n_2} G(y_i; a, \underline{\theta}_2) = \frac{1}{n_2}\sum_{i=1}^{n_2} y_i^{p_2} \tag{2.8}$$

Corollary 2.3: If we take $G(x; a, \underline{\theta}) = log\left(\frac{x}{a}\right)$, pdf $f(x; a, \lambda, \underline{\theta})$ in equation (1.1) becomes the pdf of Pareto distribution (Johnson and Kotz (1970, p.233)) with parameters a and λ. Considering a is known, MLE of λ based on samples $\underline{x}$ size n is given by

$$\tilde{\lambda} = \frac{1}{n}\sum_{i=1}^{n} log\left(\frac{x_i}{a}\right) \tag{2.9}$$

Hence, the MLE of reliability function R_{XY} for the case, based on samples $\underline{x}$ and $\underline{y}$ of size $n_1$and n_2, respectively,$\tilde{R}_{XY}$ will be given by

$$\tilde{R}_{XY} = \frac{T_y}{T_x + T_y}$$

where,

$$T_x = \frac{1}{n_1}\sum_{i=1}^{n_1} log\left(\frac{x_i}{a_1}\right) \tag{2.10}$$

and

$$T_y = \frac{1}{n_2}\sum_{i=1}^{n_2} log\left(\frac{y_i}{a_2}\right) \tag{2.11}$$

Corollary 2.4: Taking $G(x; a, \underline{\theta}) = log(1 + x^p)$, where $p > 0$ and $a = 0$, pdf (1.1) becomes the pdf of Burr distribution with parameters p and λ (Burr (1942) and Cislak and Burr (1968)). Considering p is known, MLE of λ, based on samples $\underline{x}$ size n, is given by

$$\tilde{\lambda} = \frac{1}{n}\sum_{i=1}^{n} log\left(1 + x_i^p\right) \quad (2.9)$$

and by invariant property of maximum likelihood estimator the MLE of reliability function R_{XY} for this case, based on the samples $\underline{x}$ and $\underline{y}$ of size n_1 and n_2, respectively, $\tilde{R}_{XY}$ will be given by

$$\tilde{R}_{XY} = \frac{T_y}{T_x + T_y}$$

where,

$$T_x = \frac{1}{n_1}\sum_{i=1}^{n_1} log\left(1 + x_i^{p_1}\right) \quad (2.10)$$

and

$$T_y = \frac{1}{n_2}\sum_{i=1}^{n_2} log\left(1 + y_i^{p_2}\right) \quad (2.11)$$

3. UMVUE of $R = Pr(X < Y)$ for the suggested class of lifetime distributions

Theorem 2: The UMVUE of reliability function $R = Pr(X < Y)$ is given by

$$\hat{R}_{YX} = \begin{cases} Q_1\left(n_1, n_2, n_1T_x, n_2T_y\right); & if\ n_1T_x \geq n_2T_y \\ Q_2\left(n_1, n_2, n_1T_x, n_2T_y\right); & if\ n_1T_x < n_2T_y \end{cases} \quad (3.1)$$

where, Q_1 and Q_2 are defined as

$$Q_1(a, b, u, v) = \sum_{i=0}^{a-2}(-1)^i \frac{\Gamma a \Gamma b}{\Gamma(a-i-1)\Gamma(b+i+1)}\left(\frac{v}{u}\right)^{i+1} \quad (3.2)$$

$$Q_2(a, b, u, v) = \sum_{i=0}^{b-1} \frac{\Gamma a \Gamma b}{\Gamma(a+i)\Gamma(b-i)}\left(\frac{u}{v}\right)^{i} \quad (3.3)$$

and T_x and T_y are

$$T_x = \frac{1}{n_1}\sum_{i=1}^{n_1} G\left(x_i; a_1, \underline{\theta_1}\right),\ T_y = \frac{1}{n_2}\sum_{i=1}^{n_2} G\left(y_i; a_2, \underline{\theta_2}\right) \quad (3.4)$$

Proof: To obtain the UMVUE of reliability function $R = Pr(X < Y)$ based on samples $\underline{x}$ and $\underline{y}$ of size n_1 and n_2, respectively. Here, X and Y are two independent random variables following the distribution with pdf of the form given in (1.1) with parameters $\left(\lambda_1, a_1, \underline{\theta_1}\right)$ and $\left(\lambda_2, a_2, \underline{\theta_2}\right)$, respectively. Considering the transformations $\epsilon = G\left(x; a_1, \underline{\theta_1}\right)$ and $\eta = G\left(y; a_2, \underline{\theta_2}\right)$, we get two independent exponential variates 'ϵ' and 'η' with means '$1/\lambda_1$' and '$1/\lambda_2$', respectively and the reliability function

$$R_{XY} = Pr(X < Y) = Pr(\epsilon < \eta) = R_{\epsilon\eta}.$$

Recall that the UMVUE of reliability function $R_{\epsilon\eta} = Pr(\epsilon < \eta)$ for exponential variates, which is given by

$$\hat{R}_{\epsilon\eta} = \begin{cases} Q_1(n_1, n_2, n_1\bar{\epsilon}, n_2\bar{\eta}); & if\ n_1\bar{\epsilon} \geq n_2\bar{\eta} \\ Q_2(n_1, n_2, n_1\bar{\epsilon}, n_2\bar{\eta}); & if\ n_1\bar{\epsilon} < n_2\bar{\eta} \end{cases} \tag{3.5}$$

where,Q_1 and Q_2 are defined as

$$Q_1(a,b,u,v) = \sum_{i=0}^{a-2} (-1)^i \frac{\Gamma a \Gamma b}{\Gamma(a-i-1)\Gamma(b+i+1)} \left(\frac{v}{u}\right)^{i+1}$$

$$Q_2(a,b,u,v) = \sum_{i=0}^{b-1} \frac{\Gamma a \Gamma b}{\Gamma(a+i)\Gamma(b-i)} \left(\frac{u}{v}\right)^{i}$$

here,

$$\bar{\epsilon} = \frac{1}{n_1}\sum_{i=1}^{n_1} \epsilon_i = \frac{1}{n_1}\sum_{i=1}^{n_1} G(x_i; a_1, \underline{\theta_1}) = T_x(say)$$

and

$$\bar{\eta} = \frac{1}{n_2}\sum_{i=1}^{n_2} \eta_i = \frac{1}{n_2}\sum_{i=1}^{n_2} G(y_i; a_2, \underline{\theta_2}) = T_y(say)$$

replacing ϵ_i and η_i by $G(x_i; a_1, \underline{\theta_1})$ and $G(y_i; a_2, \underline{\theta_2})$ respectively in the expression of $\hat{R}_{\epsilon\eta}$ we get

$$\hat{R}_{\epsilon\eta} = \hat{R}_{XY} = \begin{cases} Q_1(n_1, n_2, n_1T_x, n_2T_y); & if\ n_1T_x \geq n_2T_y \\ Q_2(n_1, n_2, n_1T_x, n_2T_y); & if\ n_1T_x < n_2T_y \end{cases}$$

Hence the theorem proved.

Corollary 3.1: For $G(x; a, \underline{\theta}) = x^2$and $a = 0$, $f(x; a, \lambda, \underline{\theta})$ becomes Rayleigh distribution and by taking $Y = X^2$,we can transform the pdf of Rayleigh distribution into the pdf of exponential distribution with parameterλ. In this case UMVUE of reliability function $R_{XY} = Pr(X < Y)$, based on samples $\underline{x}$ and $\underline{y}$ of size n_1 and n_2, respectively, will be given by equation (3.1) with T_x and T_y defined as

$$T_x = \frac{1}{n_1}\sum_{i=1}^{n_1} G(x_i; a, \underline{\theta_1}) = \frac{1}{n_1}\sum_{i=1}^{n_1} x_i^2 \tag{3.6}$$

and

$$T_y = \frac{1}{n_2}\sum_{i=1}^{n_2} G(y_i; a, \underline{\theta_2}) = \frac{1}{n_1}\sum_{i=1}^{n_1} y_i^2 \tag{3.7}$$

Corollary 3.2: Consider the transformation, $G(x;a,\underline{\theta}) = x^p$, where $p > 0$ and $a = 0$, equation (1.1) turns into the pdf of Weibull distribution with parameters p and λ. p is shape parameter and λ is scale parameter. Considering p is known, the UMVUE of reliability function, $\hat{R}_{XY}$, based on samples $\underline{x}$ and $\underline{y}$ of size n_1 and n_2, respectively, will be given by equation (3.1). In this case T_x and T_y defined as

$$T_x = \frac{1}{n_1}\sum_{i=1}^{n_1} G(x_i;a,\underline{\theta}_1) = \frac{1}{n_1}\sum_{i=1}^{n_1} x_i^{p_1} \tag{3.8}$$

and

$$T_y = \frac{1}{n_2}\sum_{i=1}^{n_2} G(y_i;a,\underline{\theta}_2) = \frac{1}{n_1}\sum_{i=1}^{n_1} y_i^{p_2} \tag{3.9}$$

Corollary 3.3: If we consider the transformation,$G(x;a,\underline{\theta}) = log\left(\frac{x}{a}\right)$, pdf $f(x;a,\lambda,\underline{\theta})$ in equation (1.1) becomes the pdf of Pareto distribution with parameters a and λ. Here we take parameter a as known constant. Let X and Y are two independent random variables following Pareto distribution with parameters (λ_1, a_1) and (λ_2, a_2) respectively. Based on samples $\underline{X}$ and $\underline{Y}$ of size n_1 and n_2 respectively. The UMVUE of reliability function, $R_{XY} = Pr(X < Y)$, will be given by equation (3.1) with T_x and T_y defined as

$$T_x = \frac{1}{n_1}\sum_{i=1}^{n_1} log\left(\frac{x_i}{a_1}\right) \tag{3.10}$$

and

$$T_y = \frac{1}{n_2}\sum_{i=1}^{n_2} log\left(\frac{y_i}{a_2}\right) \tag{3.11}$$

Corollary 3.4: Considering the transformation,$G(x;a,\underline{\theta}) = log(1 + x^p)$, where $p > 0$ and $a = 0$, pdf $f(x;a,\lambda,\underline{\theta})$ in equation (1.1) becomes the pdf of Burr distribution with parameters p and λ. Considering p is known. Let us consider two independent random variables $X$and Y, following Burr distribution with parameters (λ_1, p_1) and (λ_2, p_2) respectively. UMVUE of reliability function R_{XY} for this case, based on the samples $\underline{x}$ and $\underline{y}$ of size $n_1$and n_2, respectively, will be given by equation (3.1) with T_x and T_y defined as

$$T_x = \frac{1}{n_1}\sum_{i=1}^{n_1} log(1 + x_i^{p_1}) \tag{3.12}$$

and

$$T_y = \frac{1}{n_2}\sum_{i=1}^{n_2} log(1 + y_i^{p_2}) \tag{3.13}$$

Corollary 3.5: If we consider the transformation,$G(x;a,\underline{\theta}) = log\left(1 + \frac{x}{v}\right)$, pdf $f(x;a,\lambda,\underline{\theta})$ in equation (1.1) becomes the pdf of Lomax (Lomax (1954)) distribution with parameters v and λ. Here, we take parameter v as known constant. Let X and Y are two independent random variables following Lomax distribution with parameters (λ_1, v_1) and (λ_2, v_2) respectively. Assume that v_1 and $v_2$are known. Based on samples $\underline{X}$ and $\underline{Y}$ of size n_1 and n_2 respectively. The UMVUE of reliability function,$R_{XY} = Pr(X < Y)$, will be given by equation (3.1) with T_x and T_y defined as

$$T_x = \frac{1}{n_1}\sum_{i=1}^{n_1} log\left(1 + \frac{x_i}{v_1}\right) \tag{3.14}$$

and

$$T_y = \frac{1}{n_2}\sum_{i=1}^{n_2} log\left(1 + \frac{y_i}{v_2}\right) \tag{3.15}$$

4. Construction of confidence interval for $R = Pr(X < Y)$for the suggested class of lifetime distributions

To construct the confidence interval for reliability function $R_{XY} = Pr(X < Y)$, we need to obtain the UMVUE for variance of UMVUE of reliability function, $\hat{R}_{XY}$.

Theorem 3: The UMVUE for variance of UMVUE of reliability function, $R_{XY} = Pr(X < Y)$, $\widehat{var}(\hat{R}_{XY})$ or ${\hat{\sigma}_{XY}}^2$is given by

$${\hat{\sigma}_{XY}}^2 = \left(\hat{R}_{XY}\right)^2 - \frac{(n_1-1)(n_1-2)(n_2-1)(n_2-2)}{{n_1}^2{n_2}^2(T_x)^{n_1-1}(T_y)^{n_2-1}} H\left(n_1, n_2, T_x, T_y\right) \tag{4.1}$$

here, $\hat{R}_{XY}$ is the UMVUE of $R_{XY}$and $H(n_1, n_2, \bar{\epsilon}, \bar{\eta})$ is defined as

$$H(n_1, n_2, \bar{\epsilon}, \bar{\eta}) = \iint (\bar{\epsilon} - \frac{\epsilon_1+\epsilon_2}{n_1})^{n_1-3}(\bar{\eta} - \frac{\eta_1+\eta_2}{n_2})^{n_2-3}\, d\epsilon_1\, d\epsilon_2\, d\eta_1\, d\eta_2 \tag{4.2}$$

integral is over the region W^* defined as

$$W^* = \{(\epsilon_1, \epsilon_2, \eta_1, \eta_2): \epsilon_1 + \epsilon_2 < n_1\bar{\epsilon}, \eta_1 + \eta_2 < n_2\bar{\eta}, 0 < \eta_1 < \epsilon_1, 0 < \eta_2 < \epsilon_2\} \tag{4.3}$$

Proof: Let X and Y are twoindependent random variables following the distribution with pdf of the form given in (1.1) with parameters $(\lambda_1, a_1, \underline{\theta_1})$and $(\lambda_2, a_2, \underline{\theta_2})$, respectively. Let us consider the transformations $\epsilon = G(x; a_1, \underline{\theta_1})$and $\eta = G(y; a_2, \underline{\theta_2})$. For known $(a_1, \underline{\theta_1})and(a_2, \underline{\theta_2})$,$\epsilon$ and η will come out independent exponential variates with means '1/ λ_1' and '1/ λ_2', respectively and the reliability function will be

$$R_{XY} = Pr(X < Y) = Pr(\epsilon < \eta) = R_{\epsilon\eta}$$

Now, as we know that for exponential variates$\epsilon$and η, the UMVUE of variance of $\hat{R}_{\epsilon\eta}$ based on samples $\underline{\epsilon}$ and $\underline{\eta}$ of size n_1 and n_2 respectively is defined as

$${\hat{\sigma}_{\epsilon\eta}}^2 = (\hat{R}_{\epsilon\eta})^2 - \frac{(n_1-1)(n_1-2)(n_2-1)(n_2-2)}{{n_1}^2{n_2}^2(\bar{\epsilon})^{n_1-1}(\bar{\eta})^{n_2-1}} H(n_1, n_2, \bar{\epsilon}, \bar{\eta}) \tag{4.4}$$

where, $\hat{R}_{\epsilon\eta}$ is the UMVUE of $R_{\epsilon\eta}$ and $H(n_1, n_2, \bar{\epsilon}, \bar{\eta})$ is defined as

$$H(n_1, n_2, \bar{\epsilon}, \bar{\eta}) = \iint (\bar{\epsilon} - \frac{\epsilon_1 + \epsilon_2}{n_1})^{n_1-3}(\bar{\eta} - \frac{\eta_1 + \eta_2}{n_2})^{n_2-3}\, d\epsilon_1\, d\epsilon_2\, d\eta_1\, d\eta_2$$

integral is over the region W^* defined as

$$W^* = \{(\epsilon_1, \epsilon_2, \eta_1, \eta_2): \epsilon_1 + \epsilon_2 < n_1\bar{\epsilon}, \eta_1 + \eta_2 < n_2\bar{\eta}, 0 < \epsilon_1 < \eta_1, 0 < \epsilon_2 < \eta_2\}$$

where,

$$\bar{\epsilon} = \frac{1}{n_1}\sum_{i=1}^{n_1} \epsilon_i \quad \text{and} \quad \bar{\eta} = \frac{1}{n_2}\sum_{i=1}^{n_2} \eta_i$$

replacing ϵ_i and η_i by $G(x_i; a_1, \underline{\theta_1})$ and $G(y_i; a_2, \underline{\theta_2})$ respectively in the expression of $\hat{\sigma}_{\epsilon\eta}{}^2$ we will get the UMVUE for variance of $\hat{R}_{XY}$, i.e., $\hat{\sigma}_{XY}{}^2$ as follows

$$\hat{\sigma}_{\epsilon\eta}{}^2 = \hat{\sigma}_{XY}{}^2 = \widehat{var}(\hat{R}_{XY})$$
$$= (\hat{R}_{XY})^2 - \frac{(n_1-1)(n_1-2)(n_2-1)(n_2-2)}{n_1{}^2 n_2{}^2 (T_x)^{n_1-1}(T_y)^{n_2-1}} H(n_1, n_2, T_x, T_y)$$

where,

$$T_x = \frac{1}{n_1}\sum_{i=1}^{n_1} G(x_i; a_1, \underline{\theta_1}) = \frac{1}{n_1}\sum_{i=1}^{n_1} \epsilon_i = \bar{\epsilon}$$

and similarly

$$T_y = \frac{1}{n_2}\sum_{i=1}^{n_2} G(y_i; a_2, \underline{\theta_2}) = \frac{1}{n_2}\sum_{i=1}^{n_2} \eta_i = \bar{\eta}$$

Hence, the theorem proved.

Now, the asymptotic confidence interval for reliability function $R_{XY} = Pr(X < Y)$ with confidence coefficient $(1-\gamma)$ is

$$(\hat{R}_{XY} - z_{\gamma/2}\hat{\sigma}_{XY}, \hat{R}_{XY} - z_{\gamma/2}\hat{\sigma}_{XY}) \quad (4.5)$$

where,$\hat{\sigma}_{XY}{}^2$ is defined above in equation (4.1) and $z_{\gamma/2}$ is the $(1-\frac{\gamma}{2})$ percentile of standard normal distribution.

Corollary 4.1: For $G(x; a, \underline{\theta}) = x^2$and $a = 0$, pdf $f(x; a, \lambda, \underline{\theta})$ becomes the pdf of Rayleigh distribution and by taking $Y = X^2$, we can transform the pdf of Rayleigh distribution into the pdf of exponential distribution with parameter λ. In this case UMVUE of variance of estimate of reliability function, $R_{XY} = Pr(X < Y)$, $\widehat{var}(\hat{R}_{XY})$, based on samples $\underline{x}$ and $\underline{y}$ of size n_1 and n_2, respectively, will be given by equation (4.1) with T_x and T_y defined as

$$T_x = \frac{1}{n_1}\sum_{i=1}^{n_1} G(x_i; a, \underline{\theta_1}) = \frac{1}{n_1}\sum_{i=1}^{n_1} x_i^2 \quad (4.6)$$

and

$$T_y = \frac{1}{n_2}\sum_{i=1}^{n_2} G(y_i; a, \underline{\theta_2}) = \frac{1}{n_1}\sum_{i=1}^{n_1} y_i^2 \quad (4.7)$$

Now, the asymptotic confidence interval for reliability function R_{XY} can be defined as in equation (4.5).

Corollary 4.2: Consider the transformation, $G(x; a, \underline{\theta}) = x^p$, where $p > 0$ and $a =$

0, equation (1.1) turns into the pdf of Weibull distribution with parameters p and λ. p is shape parameter and λ is scale parameter. Considering the known shape parameter, the UMVUE of estimate of reliability function, based on samples $\underline{x}$ and $\underline{y}$ of size $n_1$and n_2, respectively, will be given as in equation (4.1). In this case T_x and T_y defined as

$$T_x = \frac{1}{n_1}\sum_{i=1}^{n_1} G\left(x_i; a, \underline{\theta}_1\right) = \frac{1}{n_1}\sum_{i=1}^{n_1} x_i^{p_1} \tag{4.7}$$

and

$$T_y = \frac{1}{n_2}\sum_{i=1}^{n_2} G\left(y_i; a, \underline{\theta}_2\right) = \frac{1}{n_2}\sum_{i=1}^{n_1} y_i^{p_2} \tag{4.8}$$

Similarly, as in Corollary 2.2, the asymptotic confidence interval for reliability function R_{XY} can be defined as in equation (4.5).

Corollary 4.3: If we consider the transformation,$G\left(x; a, \underline{\theta}\right) = log\left(\frac{x}{a}\right)$,$f(x; a, \lambda, \underline{\theta})$ in equation (1.1) becomes the pdf of Pareto distribution with parameters a and λ. Here we take parameter a as known constant. Let X and Y are two independent random variables following Pareto distribution with parameters (λ_1, a_1) and (λ_2, a_2) respectively. Based on samples $\underline{X}$ and $\underline{Y}$ of size n_1 and n_2 respectively. The UMVUE of estimate of reliability function, will be given by equation (4.1) with T_x and T_y defined as

$$T_x = \frac{1}{n_1}\sum_{i=1}^{n_1} log\left(\frac{x_i}{a_1}\right) \tag{4.9}$$

and

$$T_y = \frac{1}{n_2}\sum_{i=1}^{n_2} log\left(\frac{y_i}{a_2}\right) \tag{4.10}$$

and the asymptotic confidence interval for reliability function R_{XY} can be defined as in equation (4.5).

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