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Positivity of Sums for Higher Order $\nabla-$Convex Sequences and Functions

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Abstract

In present article, some general identities of Popoviciu type for discrete case for sums $\sum_{j=1}^{M}\sum_{l=1}^{N} r_{jl} f(y_j, z_l)$ and $\sum_{j=1}^{M}\sum_{l=1}^{N} r_{jl} a_{jl}$ are deduced for function and sequence involving higher order ∇ divided difference respectively, then by applying obtained identities, positivity of these expressions will be characterised for higher order $\nabla-$convex sequences and functions.

Keywords: Convex functions, $\nabla-$convex functions

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1. INTRODUCTION

In an introductory section, we start from some definitions and preliminaries then we go next section, in this we would like to obtain discrete identities of two variables function $f(y_j, z_l)$ and sequence a_{jl} involving higher order ∇ divided difference. Using obtained identities we derive various significant results. We will also discuss the characterisation of Popoviciu$-$type positivity of these discrete sums involving $\nabla-$convex sequences and functions.

Let us recall, few useful definitions and significant results regarding the convex functions extracted from [7] (see also [1, 2, 4, 5]). Throughout the chapter J is an interval in $\mathbb{R}$ and m, n, M, N are natural numbers.

Definition 1.1. The $m-$*divided difference* of a function $f : J \to \mathbb{R}$, at different points $y_j, y_{j+1}, \ldots, \; y_{j+m} \in J = [a, b] \subset \mathbb{R}$, where $i \in \mathbb{N}$ is stated as:

$$\begin{aligned} [y_l; f] &= f(y_l), \quad l \in \{j, j+1, \ldots, j+m\} \\ [y_j, \ldots, y_{j+m}; f] &= \frac{[y_{j+1}, \ldots, y_{j+m}; f] - [y_j, \ldots, y_{j+m-1}; f]}{y_{j+m} - y_j}. \end{aligned}$$

Remark 1.2. Let us denote $[y_j, y_{j+1}, \cdots, y_{j+m}; f]$ by $\Delta_{(m)} f(y_j)$. The value $[y_j, \ldots, y_{j+m}; f]$ is independent of order of points $y_j, y_{j+1}, \ldots, y_{j+m}$. This definition can extended by including the cases for more than one points coincide by applying the respective limits.

Definition 1.3. Let $E = \{y_1, y_2, \ldots, y_M\} \subset \mathbb{R}$. A function $f : E \to \mathbb{R}$ is called the discrete $m-$convex function if inequality $[y_j, \ldots, y_{j+m}; f] \geq 0$ holds $\forall (m+1)$ different points $y_j, \ldots, y_{j+m} \in E$.

Definition 1.4. A function $f : E \to \mathbb{R}$, is said to be discrete $m - \nabla-$convex or discrete $\nabla-$*convex of order* m, if $\forall (m+1)$ different points $y_j, y_{j+1}, \ldots, y_{j+m}$ we have $\nabla_{(m)} f(y_j) = (-1)^m \Delta_{(m)} f(y_j) \geq 0$.

We extend all the aforementioned definitions up to order (m, n). For that let us denote $J \times L \subset \mathbb{R}^2$.

Definition 1.5. Let $f : J \times J \to \mathbb{R}$, be a function, then $(m, n)-$*divided difference* or *divided difference of order* (m, n), of a function f at different points $y_j, \ldots, y_{j+m} \in J$, $z_l, \ldots, z_{l+n} \in L$ for some $j, l \in \mathbb{N}$, is stated as $\Delta_{(m,n)} f(y_j, z_l) = [y_j, \ldots, y_{j+m}; [z_l, \ldots, z_{l+n}; f]]$.

Definition 1.6. Let $E = \{y_1, y_2, \ldots, y_M\}, F = \{z_1, z_2, \ldots, z_N\} \subset \mathbb{R}$. A function $f : E \times F \to \mathbb{R}$ is called a discrete $(m, n)-$convex function if inequality $[y_j, \ldots, y_{j+m}; [z_l, \ldots, z_{l+n}; f]] \geq 0$ holds $\forall (m+1)$ different points $y_j, \ldots, y_{j+m} \in E$ and $(n+1)$ different points $z_l, \ldots, z_{l+n} \in F$.

Definition 1.7. *Finite difference of order (m,n)* of a function $f : J \times L \to \mathbb{R}$, where $h, k \in \mathbb{R}$ and $y \in J$, $z \in L$, is stated as

$$\begin{aligned} \Delta_{h,k}^{m,n} f(y, z) &= \Delta_h^m (\Delta_k^n f(y, z)) = \Delta_k^n (\Delta_h^m f(y, z)) \\ &= \sum_{j=0}^{m} \sum_{l=0}^{n} (-1)^{m+n-j-l} \binom{m}{j} \binom{n}{l} f(y + jh, z + lk). \end{aligned}$$

where $y + jh, z + lk \in J, L$ respectively and $j \in \{0, 1, 2, \ldots, m-1, m\}$; $l \in \{0, 1, 2, \ldots, n-1, n\}$. Moreover, a function $f : J \times L \to \mathbb{R}$ is called the $(m, n)-$convex, if the following conditions hold $\Delta_{h,k}^{m,n} f(y, z) \geq 0 \; \forall y \in J, \, z \in L$.

Definition 1.8. *Finite difference and Divided difference of* (m, n) *order, of a sequence* (a_{jl}) *are stated as* $\Delta^{m,n}a_{jl} = \Delta^{m,n}_{1,1}f(y_j, z_l)$ *and* $\Delta_{(m,n)}a_{jl} = \Delta_{(m,n)}f(y_j, z_l)$ *respectively, where* $j \in \{1, 2, 3, \ldots, m-1, m\}$, $l \in \{1, 2, 3, \ldots, n-1, n\}$. *If* $y_j = j$, $z_l = l$, *then* $f : \{1, \ldots, m\} \times \{1, \ldots, n\} \to \mathbb{R}$ *is the function which is* $f(j, l) = a_{jl}$. *Moreover, a sequence* (a_{jl}) *is called a* $(m, n)-$*convex*, if following conditions hold $\Delta^{m,n}a_{jl} \geq 0$ for $m, n \geq 0$ and $j, l \in \{1, 2, 3, \ldots\}$.

Further, in this paper we would use the following notations: $J \times L \subset \mathbb{R} \times \mathbb{R}$. For some real sequence (a_m), $m \in \mathbb{N}$ and $n \in \{2, 3, \ldots\}$:

$$\nabla^{(1)}a_m = \nabla a_m = a_m - a_{m+1}, \quad \nabla^{(n)}a_m = \nabla(\nabla^{(n-1)}a_m).$$

Also for m distinct real numbers y_j, $j \in \{1, \ldots, m\}$ and $n \geq 0$:

$$(y_k - y_j)^{\{n+1\}} = (y_k - y_j)(y_{k-1} - y_j) \ldots (y_{k-n} - y_j), \quad (y_k - y_j)^{\{0\}} = 1.$$

Definition 1.9. A function $f : E \times F \to \mathbb{R}$ is called discrete $(m, n) - \nabla-$convex if inequality $\nabla_{(m,n)}f(y_j, z_l) = (-1)^{m+n}\Delta_{(m,n)}f(y_j, z_l) \geq 0$, holds $\forall$ different points $y_j, \ldots, y_{j+m} \in J$, $z_l, \ldots, z_{l+n} \in L$.

Let us give brief explanation the formate of our article as follows, after introduction and preliminaries in the next section, we will get identities for the sums $\sum_{j=1}^{M}\sum_{l=1}^{N} r_{jl}f(y_j, z_l)$ and $\sum_{j=1}^{M}\sum_{l=1}^{N} r_{jl}a_{jl}$ for two dimension involving higher order ∇ divided difference and investigate the inequality $\sum_{j=1}^{M}\sum_{l=1}^{N} r_{jl}f(y_j, z_l) \geq 0$ for $\nabla-$convex functions and $\nabla-$convex sequences of order (m, n) in two dimension.

2. DISCRETE CASE FOR SEQUENCE OF TWO DIMENSION

Under the given heading, we would consider a discrete sequence of two dimension. Firstly, we will get identities for sequence $\sum_{j=1}^{M}\sum_{l=1}^{N} r_{jl}a_{jl}$ in which involves higher order ∇ divided difference for $j \in \{1, \ldots, M\}$ and $l \in \{1, \cdots, N\}$ as well as $j, l \in \{1, \ldots, M\}$. Further that we can split this sequence into two sequences as an especial case by using $a_{jl} = a_j b_l$.

In the paper [6] the following result for a real sequence (a_M) was proved:

Proposition 2.1. *Let $r_j \in \mathbb{R}$ for $j \in \{1,\dots,M\}$, then the following identity for any real sequence (a_M) holds:*

$$\begin{aligned}\sum_{j=1}^{M} r_j a_j &= \sum_{k=0}^{m-1} \frac{1}{k!}\nabla^{(k)} a_{M-k} \sum_{j=1}^{M-k} (M-j)^{\{k\}} r_j \\ &+ \frac{1}{(m-1)!}\sum_{k=1}^{M-m}\left(\sum_{j=1}^{k}(m+k-1-j)^{\{m-1\}} r_j\right)\nabla^{(m)} a_k. \quad (2.1)\end{aligned}$$

Now we would like to obtain the following main theorem for a real sequence (a_{MN}).

Theorem 2.2. *Let $r_{jl} \in \mathbb{R}$ and a_{jl} be a sequence, where $j \in \{1,\dots,M\}$ and $l \in \{1,\cdots,N\}$, then the following identity*

$$\begin{aligned}&\sum_{j=1}^{M}\sum_{l=1}^{N} r_{jl} a_{jl} \\ &= \sum_{k=0}^{n-1}\sum_{t=0}^{m-1}\sum_{s=1}^{M-t}\sum_{p=1}^{N-k} r_{sp}\frac{(M-s)^{\{t\}}}{t!}\frac{(N-p)^{\{k\}}}{k!}\nabla_{(t,k)} a_{(M-t,N-k)} \\ &+ \sum_{k=0}^{n-1}\sum_{t=1}^{M-m}\sum_{s=1}^{t}\sum_{p=1}^{N-k} r_{sp}\frac{(m+t-1-s)^{\{m-1\}}}{(m-1)!}\frac{(N-p)^{\{k\}}}{k!}\nabla_{(m,k)} a_{(t,N-k)} \\ &+ \sum_{k=1}^{N-n}\sum_{t=0}^{m-1}\sum_{s=1}^{M-t}\sum_{p=1}^{k} r_{sp}\frac{(M-s)^{\{t\}}}{t!}\frac{(n+k-1-p)^{\{n-1\}}}{(n-1)!}\nabla_{(t,n)} a_{(M-t,k)} \\ &+ \sum_{k=1}^{N-n}\sum_{t=1}^{M-m}\sum_{s=1}^{t}\sum_{p=1}^{k} r_{sp}\frac{(m+t-1-s)^{\{m-1\}}}{(m-1)!}\frac{(n+k-1-p)^{\{n-1\}}}{(n-1)!}\nabla_{(m,n)} a_{(t,k)}\end{aligned} \quad (2.2)$$

holds.

Proof. We have

$$\sum_{j=1}^{M}\sum_{l=1}^{N} r_{jl} a_{jl} = \sum_{j=1}^{M}\left(\sum_{l=1}^{N} q_l A_j\right),$$

where $r_{jl} = q_l$ and $A_j : l \mapsto a_{(j,l)}$. Using (2.1) in the inner sum we get

$$\begin{aligned}
\sum_{j=1}^{M}\sum_{l=1}^{N} r_{jl}a_{jl} &= \sum_{j=1}^{M}\sum_{k=0}^{n-1}\frac{1}{k!}\nabla_{(k)}A_{j(N-k)}\left(\sum_{l=1}^{N-k} q_l(N-l)^{\{k\}}\right)\\
&+ \sum_{j=1}^{M}\sum_{k=1}^{N-n}\frac{1}{(n-1)!}\nabla_{(n)}A_{j(k)}\left(\sum_{l=1}^{k} q_l(n+k-1-l)^{\{n-1\}}\right)\\
&= \sum_{k=0}^{n-1}\left(\sum_{j=1}^{M}\frac{1}{k!}\nabla_{(k)}A_{j(N-k)}\left(\sum_{l=1}^{N-k} q_l(N-l)^{\{k\}}\right)\right)\\
&+ \sum_{k=1}^{N-n}\left(\sum_{j=1}^{M}\frac{1}{(n-1)!}\nabla_{(n)}A_{j(k)}\left(\sum_{l=1}^{k} q_l(n+k-1-l)^{\{n-1\}}\right)\right)\\
&= \sum_{k=0}^{n-1}\left(\sum_{j=1}^{M} w_jB_j\right) + \sum_{k=1}^{N-n}\left(\sum_{j=1}^{M} v_jC_j\right),
\end{aligned}$$

where $w_j = \sum_{l=1}^{N-k} q_l(N-l)^{\{k\}} = \sum_{l=1}^{N-k} r_{jl}(N-l)^{\{k\}}$, $\quad v_j = \sum_{l=1}^{k} q_l(n+k-1-l)^{\{n-1\}}$, $B_j = \frac{1}{k!}\nabla_{(k)}A_{j(N-k)}$, and $C_j = \frac{1}{(n-1)!}\nabla_{(n)}A_{j(k)}$.

Using again (2.1) in inner sums, then we have

$$\begin{aligned}
\sum_{j=1}^{M}\sum_{l=1}^{N} r_{jl}a_{jl} &= \sum_{k=0}^{n-1}\sum_{p=0}^{m-1}\frac{1}{p!}\nabla_{(p)}B_{(M-p)}\left(\sum_{j=1}^{M-p} w_j(M-j)^{\{p\}}\right)\\
&+ \sum_{k=0}^{n-1}\sum_{p=1}^{M-m}\frac{1}{(m-1)!}\nabla_{(m)}B_{(p)}\left(\sum_{j=1}^{p} w_j(m+p-1-j)^{\{m-1\}}\right)\\
&+ \sum_{k=1}^{N-n}\sum_{t=0}^{m-1}\frac{1}{t!}\nabla_{(t)}C_{(M-t)}\left(\sum_{j=1}^{M-t} v_j(M-j)^{\{t\}}\right)\\
&+ \sum_{k=1}^{N-n}\sum_{t=1}^{M-m}\frac{1}{(m-1)!}\nabla_{(m)}C_{(t)}\left(\sum_{j=1}^{t} v_j(m+t-1-j)^{\{m-1\}}\right)\\
&= \sum_{k=0}^{n-1}\sum_{p=0}^{m-1}\sum_{j=1}^{M-p}\sum_{l=1}^{N-k} r_{jl}\frac{(M-j)^{\{p\}}}{p!}\frac{(N-l)^{\{k\}}}{k!}\nabla_{(p,k)}a_{(M-p,N-k)}\\
&+ \sum_{k=0}^{n-1}\sum_{p=1}^{M-m}\sum_{j=1}^{p}\sum_{l=1}^{N-k} r_{jl}\frac{(m+p-1-j)^{\{m-1\}}}{(m-1)!}\frac{(N-l)^{\{k\}}}{k!}\nabla_{(m,k)}a_{(p,N-k)}\\
&+ \sum_{k=1}^{N-n}\sum_{t=0}^{m-1}\sum_{j=1}^{M-t}\sum_{l=1}^{k} r_{jl}\frac{(M-j)^{\{t\}}}{t!}\frac{(n+k-1-l)^{\{n-1\}}}{(n-1)!}\nabla_{(t,n)}a_{(M-t,k)}\\
&+ \sum_{k=1}^{N-n}\sum_{t=1}^{M-m}\sum_{j=1}^{t}\sum_{l=1}^{k} r_{jl}\frac{(m+t-1-j)^{\{m-1\}}}{(m-1)!}\frac{(n+k-1-l)^{\{n-1\}}}{(n-1)!}\nabla_{(m,n)}a_{(t,k)}.
\end{aligned}$$

If change $j \to s$, $l \to p$ in all sums and put $p \to t$ in first and second sums, then obtain the required identity (2.3). □

Corollary 2.3. *Let* $r_{jl} \in \mathbb{R}$ *and* a_{jl} *be a sequence, where* $j, l \in \{1, \ldots, M\}$*, then*

$$\begin{aligned}
&\sum_{j=1}^{M}\sum_{l=1}^{M} r_{jl}a_{jl} \\
&= \sum_{k=0}^{n-1}\sum_{t=0}^{m-1}\sum_{s=1}^{M-t}\sum_{p=1}^{M-k} r_{sp}\frac{(M-s)^{\{t\}}}{t!}\frac{(M-p)^{\{k\}}}{k!}\nabla_{(t,k)}a_{(M-t,M-k)} \\
&+ \sum_{k=0}^{n-1}\sum_{t=1}^{M-m}\sum_{s=1}^{t}\sum_{p=1}^{M-k} r_{sp}\frac{(m+t-1-s)^{\{m-1\}}}{(m-1)!}\frac{(M-p)^{\{k\}}}{k!}\nabla_{(m,k)}a_{(t,M-k)} \\
&+ \sum_{k=1}^{M-n}\sum_{t=0}^{m-1}\sum_{s=1}^{M-t}\sum_{p=1}^{k} r_{sp}\frac{(M-s)^{\{t\}}}{t!}\frac{(n+k-1-p)^{\{n-1\}}}{(n-1)!}\nabla_{(t,n)}a_{(M-t,k)} \\
&+ \sum_{k=1}^{M-n}\sum_{t=1}^{M-m}\sum_{s=1}^{t}\sum_{p=1}^{k} r_{sp}\frac{(m+t-1-s)^{\{m-1\}}}{(m-1)!}\frac{(n+k-1-p)^{\{n-1\}}}{(n-1)!}\nabla_{(m,n)}a_{(t,k)}
\end{aligned} \tag{2.3}$$

holds.

Remark 2.4. If we simply put $a_{jl} = a_j b_l$ in Theorem 2.2, then we obtain similar result for two a_j and b_l sequences as below.

Corollary 2.5. *Let* $r_{jl} \in \mathbb{R}$*,* $b : l \mapsto b_l$ *and* $a : j \mapsto a_j$ *be two sequences, where* $l \in \{1, \cdots, N\}$ *and* $j \in \{1, \ldots, M\}$*, then*

$$\begin{aligned}
&\sum_{j=1}^{M}\sum_{l=1}^{N} r_{jl}a_j b_l \\
&= \sum_{k=0}^{n-1}\sum_{t=0}^{m-1}\sum_{s=1}^{M-t}\sum_{p=1}^{N-k} r_{sp}\frac{(M-s)^{\{t\}}}{t!}\nabla_{(t)}a_{(M-t)}\frac{(N-p)^{\{k\}}}{k!}\nabla_{(k)}b_{(N-k)} \\
&+ \sum_{k=0}^{n-1}\sum_{t=1}^{M-m}\sum_{s=1}^{t}\sum_{p=1}^{N-k} r_{sp}\frac{(m+t-1-s)^{\{m-1\}}}{(m-1)!}\nabla_{(m)}a_{(t)}\frac{(N-p)^{\{k\}}}{k!}\nabla_{(k)}b_{(N-k)} \\
&+ \sum_{k=1}^{N-n}\sum_{t=0}^{m-1}\sum_{s=1}^{M-t}\sum_{p=1}^{k} r_{sp}\frac{(M-s)^{\{t\}}}{t!}\nabla_{(t)}a_{(M-t)}\frac{(n+k-1-p)^{\{n-1\}}}{(n-1)!}\nabla_{(n)}b_{(k)} \\
&+ \sum_{k=1}^{N-n}\sum_{t=1}^{M-m}\sum_{s=1}^{t}\sum_{p=1}^{k} r_{sp}\frac{(m+t-1-s)^{\{m-1\}}}{(m-1)!}\nabla_{(m)}a_{(t)}\frac{(n+k-1-p)^{\{n-1\}}}{(n-1)!}\nabla_{(n)}b_{(k)}.
\end{aligned}$$

We would obtain necessary and sufficient conditions of Theorem 2.2 for Popoviciu type characterisation of positivity of sums for sequence in two dimension $\sum_{j=1}^{M}\sum_{l=1}^{N} r_{jl}a_{jl} \geq 0$ holds, for every $(m,n) - \nabla-$convex sequences in last as a remark.

3. DISCRETE CASE OF TWO DIMENSION FUNCTION

Under present heading, we would consider a discrete function of two variables that is defined in the interval $J_1 \times L_1 \subset \mathbb{R} \times \mathbb{R}$. Firstly, we will get identities for function $\sum_{j=1}^{M}\sum_{l=1}^{N} r_{jl}f(y_j, z_l)$ in which involves higher order ∇ divided difference. Moreover, we can split this function into two functions as an especial case by $f(y_j, z_l) = f(y_j)g(z_l)$ and also consider necessary and sufficient conditions of Theorem 3.2 for Popoviciu type characterisation of positivity of sums for discrete function of two-variables $\sum_{j=1}^{M}\sum_{l=1}^{N} r_{jl}f(y_j, z_l) \geq 0$ holds, for every $(m,n) - \nabla-$convex function.

The following result was proved in [1] for the real function involving ∇ divided difference and it is a generalisation of (2.1) which may be stated as:

Proposition 3.1. *Let r_j be real numbers for $j \in \{1, 2, \cdots, M\}$, where $M \geq m$. Let f be discrete function and y_j non mutual elements in the interval J for $j \in \{1, 2, \dots, M\}$, then following identity holds:*

$$\begin{aligned}\sum_{j=1}^{M} r_j f(y_j) &= \sum_{k=0}^{m-1} \nabla_{(k)} f(y_{M-k}) \left(\sum_{l=1}^{M-k} r_l (y_M - y_l)^{\{k\}} \right) \\ &+ \sum_{k=1}^{M-m} \nabla_{(m)} f(y_k)(y_{m+k} - y_k) \left(\sum_{l=1}^{k} r_l (y_{m+k-1} - y_l)^{\{m-1\}} \right). \quad (3.1)\end{aligned}$$

Now we are able to give our main general theorem for discrete function in two dimension.

Theorem 3.2. *Let $r_{jl} \in \mathbb{R}$ and $f : J_1 \times L_1 \to \mathbb{R}$ be discrete function, where $l \in \{1, \cdots, N\}$ and $j \in \{1, \ldots, M\}$, then*

$$\begin{aligned}
&\sum_{j=1}^{M}\sum_{l=1}^{N} r_{jl} f(y_j, z_l) \\
&= \sum_{k=0}^{n-1}\sum_{t=0}^{m-1}\sum_{s=1}^{M-t}\sum_{p=1}^{N-k} r_{sp}(z_N - z_p)^{\{k\}}(y_M - y_s)^{\{t\}}\nabla_{(t,k)} f(y_{M-t}, z_{N-k}) \\
&+ \sum_{k=0}^{n-1}\sum_{t=1}^{M-m}\sum_{s=1}^{t}\sum_{p=1}^{N-k} r_{sp}(z_N - z_p)^{\{k\}}(y_{m+t-1} - y_s)^{\{m-1\}}\nabla_{(m,k)} f(y_t, z_{N-k})(y_{m+t} - y_t) \\
&+ \sum_{k=1}^{N-n}\sum_{t=0}^{m-1}\sum_{s=1}^{M-t}\sum_{p=1}^{k} r_{sp}(z_{n+k-1} - z_p)^{\{n-1\}}(y_M - y_s)^{\{t\}}\nabla_{(t,n)} f(y_{M-t}, z_k)(z_{n+k} - z_k) \\
&+ \sum_{k=1}^{N-n}\sum_{t=1}^{M-m}\sum_{s=1}^{t}\sum_{p=1}^{k} r_{sp}(z_{n+k-1} - z_p)^{\{n-1\}}(y_{m+t-1} - y_s)^{\{m-1\}}\nabla_{(m,n)} f(y_t, z_k)\times \\
&\times (y_{m+t} - y_t)(z_{n+k} - z_k). \qquad (3.2)
\end{aligned}$$

holds, where $(y_j, z_l) \in J_1 \times L_1$ are distinct points.

Proof. We have

$$\sum_{j=1}^{M}\sum_{l=1}^{N} r_{jl} f(y_j, z_l) = \sum_{j=1}^{M}\left(\sum_{l=1}^{N} q_l G_j(z_l)\right),$$

where $r_{jl} = q_l$ and $G_j : z \mapsto f(y_j, z)$. Using (3.1) in the inner sum we get

$$\begin{aligned}
\sum_{j=1}^{M}\sum_{l=1}^{N} r_{jl} f(y_j, z_l) &= \sum_{j=1}^{M}\sum_{k=0}^{n-1} \nabla_{(k)} G_j(z_{N-k})\left(\sum_{l=1}^{N-k} q_l(z_N - z_l)^{\{k\}}\right) \\
&+ \sum_{j=1}^{M}\sum_{k=1}^{N-n} \nabla_{(n)} G_j(z_k)(z_{n+k} - z_k)\left(\sum_{l=1}^{k} q_l(z_{n+k-1} - z_l)^{\{n-1\}}\right) \\
&= \sum_{k=0}^{n-1}\left(\sum_{j=1}^{M} \nabla_{(k)} G_j(z_{N-k})\left(\sum_{l=1}^{N-k} q_l(z_N - z_l)^{\{k\}}\right)\right) \\
&+ \sum_{k=1}^{N-n}\left(\sum_{j=1}^{M} \nabla_{(n)} G_j(z_k)(z_{n+k} - z_k)\left(\sum_{l=1}^{k} q_l(z_{n+k-1} - z_l)^{\{n-1\}}\right)\right) \\
&= \sum_{k=0}^{n-1}\left(\sum_{j=1}^{M} w_j F(y_j)\right) + \sum_{k=1}^{N-n}\left(\sum_{j=1}^{M} v_j H(y_j)\right),
\end{aligned}$$

where $w_j = \sum_{l=1}^{N-k} q_l(z_N - z_l)^{\{k\}} = \sum_{l=1}^{N-k} r_{jl}(z_N - z_l)^{\{k\}}$, $\quad v_j = \sum_{l=1}^{k} q_l(z_{n+k-1} - z_l)^{\{n-1\}}$, $F(y_j) = \nabla_{(k)} G_j(z_{N-k})$, and $H(y_j) = \nabla_{(n)} G_j(z_k)(z_{n+k} - z_k)$.

Using again (3.1) in the inner sums, then we have

$$\begin{aligned}\sum_{j=1}^{M}\sum_{l=1}^{N} r_{jl} f(y_j, z_l) \;&=\; \sum_{k=0}^{n-1}\sum_{p=0}^{m-1} \nabla_{(p)} F(y_{M-p}) \left(\sum_{j=1}^{M-p} w_j (y_M - y_j)^{\{p\}}\right)\\ &+\; \sum_{k=0}^{n-1}\sum_{p=1}^{M-m} \nabla_{(m)} F(y_p)(y_{m+p} - y_p) \left(\sum_{j=1}^{p} w_j (y_{m+p-1} - y_j)^{\{m-1\}}\right)\\ &+\; \sum_{k=1}^{N-n}\sum_{t=0}^{m-1} \nabla_{(t)} H(y_{M-t}) \left(\sum_{j=1}^{M-t} v_j (y_M - y_j)^{\{t\}}\right)\\ &+\; \sum_{k=1}^{N-n}\sum_{t=1}^{M-m} \nabla_{(m)} H(y_t)(y_{m+t} - y_t) \left(\sum_{j=1}^{t} v_j (y_{m+t-1} - y_j)^{\{m-1\}}\right)\end{aligned}$$

$$\sum_{j=1}^{M}\sum_{l=1}^{N} r_{jl} f(y_j, z_l) = \sum_{k=0}^{n-1}\sum_{p=0}^{m-1}\sum_{j=1}^{M-p}\sum_{l=1}^{N-k} r_{jl}(z_N - z_l)^{\{k\}}(y_M - y_j)^{\{p\}} \nabla_{(p,k)} f(y_{M-p}, z_{N-k})$$

$$+\sum_{k=0}^{n-1}\sum_{p=1}^{M-m}\sum_{j=1}^{p}\sum_{l=1}^{N-k} r_{jl}(z_N - z_l)^{\{k\}}(y_{m+p-1} - y_j)^{\{m-1\}} \nabla_{(m,k)} f(y_p, z_{N-k})(y_{m+p} - y_p)$$

$$+\sum_{k=1}^{N-n}\sum_{t=0}^{m-1}\sum_{j=1}^{M-t}\sum_{l=1}^{k} r_{jl}(z_{n+k-1} - z_l)^{\{n-1\}}(y_M - y_j)^{\{t\}} \nabla_{(t,n)} f(y_{M-t}, z_k)(z_{n+k} - z_k)$$

$$+\sum_{k=1}^{N-n}\sum_{t=1}^{M-m}\sum_{j=1}^{t}\sum_{l=1}^{k} r_{jl}(z_{n+k-1} - z_l)^{\{n-1\}}(y_{m+t-1} - y_j)^{\{m-1\}} \nabla_{(m,n)} f(y_t, z_k)\times$$

$$\times\,(y_{m+t} - y_t)(z_{n+k} - z_k).$$

If change $j \to s$, $l \to p$ in all sums and put $p \to t$ in first and second sums, then obtain the required identity (3.2). □

Corollary 3.3. *Let $r_{jl} \in \mathbb{R}$ and function $f : J_1^2 \to \mathbb{R}$ be discrete, where $j, l \in \{1, 2, 3, \ldots, M-1, M\}$, then following identity holds:*

$$\sum_{j=1}^{M}\sum_{l=1}^{M} r_{jl} f(y_j, z_l)$$
$$= \sum_{k=0}^{n-1}\sum_{t=0}^{m-1}\sum_{s=1}^{M-t}\sum_{p=1}^{M-k} r_{sp}(z_M - z_p)^{\{k\}}(y_M - y_s)^{\{t\}}\nabla_{(t,k)} f(y_{M-t}, z_{M-k})$$
$$+ \sum_{k=0}^{n-1}\sum_{t=1}^{M-m}\sum_{s=1}^{t}\sum_{p=1}^{M-k} r_{sp}(z_M - z_p)^{\{k\}}(y_{m+t-1} - y_s)^{\{m-1\}}\nabla_{(m,k)} f(y_t, z_{M-k})(y_{m+t} - y_t)$$
$$+ \sum_{k=1}^{M-n}\sum_{t=0}^{m-1}\sum_{s=1}^{M-t}\sum_{p=1}^{k} r_{sp}(z_{n+k-1} - z_p)^{\{n-1\}}(y_M - y_s)^{\{t\}}\nabla_{(t,n)} f(y_{M-t}, z_k)(z_{n+k} - z_k)$$
$$+ \sum_{k=1}^{M-n}\sum_{t=1}^{M-m}\sum_{s=1}^{t}\sum_{p=1}^{k} r_{sp}(z_{n+k-1} - z_p)^{\{n-1\}}(y_{m+t-1} - y_s)^{\{m-1\}}\nabla_{(m,n)} f(y_t, z_k)\times$$
$$\times (y_{m+t} - y_t)(z_{n+k} - z_k).$$

The above corollary uses to proof the famous Čebyšev's identity (see [3]).

Remark 3.4. If we simply put $f(y_j, z_l) = f(y_j)g(z_l)$ in Theorem 3.2, then we obtain similar result for both f and g functions as below.

Corollary 3.5. *Let $r_{jl} \in \mathbb{R}$ and functions $f : J_1 \to \mathbb{R}$ and $g : L_1 \to \mathbb{R}$ be both discrete, where $l \in \{1, \cdots, N\}$ and $j \in \{1, \ldots, M\}$, then*

$$\sum_{j=1}^{M}\sum_{l=1}^{N} r_{jl} f(y_j)g(z_l)$$
$$= \sum_{k=0}^{n-1}\sum_{t=0}^{m-1}\sum_{s=1}^{M-t}\sum_{p=1}^{N-k} r_{sp}(y_M - y_s)^{\{t\}}\nabla_{(t)} f(y_{M-t})(z_N - z_p)^{\{k\}}\nabla_{(k)} g(z_{N-k})$$

$$+ \sum_{k=0}^{n-1}\sum_{t=1}^{M-m}\sum_{s=1}^{t}\sum_{p=1}^{N-k} r_{sp}(z_N - z_p)^{\{k\}}\nabla_{(k)} g(z_{N-k})(y_{m+t-1} - y_s)^{\{m-1\}}\nabla_{(m)} f(y_t)(y_{m+t} - y_t)$$
$$+ \sum_{k=1}^{N-n}\sum_{t=0}^{m-1}\sum_{s=1}^{M-t}\sum_{p=1}^{k} r_{sp}(z_{n+k-1} - z_p)^{\{n-1\}}\nabla_{(n)} g(z_k)(z_{n+k} - z_k)(y_M - y_s)^{\{t\}}\nabla_{(t)} f(y_{M-t})$$
$$+ \sum_{k=1}^{N-n}\sum_{t=1}^{M-m}\sum_{s=1}^{t}\sum_{p=1}^{k} r_{sp}(z_{n+k-1} - z_p)^{\{n-1\}}\nabla_{(n)} g(z_k)(z_{n+k} - z_k)\times$$
$$\times (y_{m+t-1} - y_s)^{\{m-1\}}\nabla_{(m)} f(y_t)(y_{m+t} - y_t).$$

holds, where $(y_j, z_l) \in J_1 \times L_1$ are distinct points.

Now its time to present necessary and sufficient conditions of Theorem 3.2 for Popoviciu type characterisation of positivity of sums for discrete function of two-variables involving $(m,n)-\nabla-$convex functions as well as in last we also present these necessary and sufficient conditions for Theorem 2.2 involving $(m,n)-\nabla-$convex sequences as a Remark 3.8.

Theorem 3.6. *Let $r_{jl} \in \mathbb{R}$ and $f : J_1 \times L_1 \to \mathbb{R}$ be discrete function, where $j \in \{1,\dots,M\}$, $l \in \{1,\cdots,N\}$ and $J_1 = \{y_{M-p}, y_{M-p+1},\dots,y_M\}$, $L_1 = \{z_{N-k}, z_{N-k+1},\dots,z_N\}$ and $y_{M-p} < \cdots < y_M$, $z_{N-k} < \cdots < z_N$, then the below inequality holds for all $(m,n)-\nabla-$convex function f.*

$$\sum_{j=1}^{M}\sum_{l=1}^{N} r_{jl} f(y_j, z_l) \geq 0 \tag{3.3}$$

iff

$$\sum_{s=1}^{M-t}\sum_{p=1}^{N-k} r_{sp}(z_N - z_p)^{\{k\}}(y_M - y_s)^{\{t\}} = 0, \quad \begin{array}{l} k \in \{0,1,2,\cdots,n-1\} \\ t \in \{0,1,2,\cdots,m-1\} \end{array} \tag{3.4}$$

$$\sum_{s=1}^{t}\sum_{p=1}^{N-k} r_{sp}(z_N - z_p)^{\{k\}}(y_{m+t-1} - y_s)^{\{m-1\}} = 0, \quad \begin{array}{l} k \in \{0,1,2,\cdots,n-1\} \\ t \in \{1,2,3,\cdots,M-m\} \end{array} \tag{3.5}$$

$$\sum_{s=1}^{M-t}\sum_{p=1}^{k} r_{sp}(z_{n+k-1} - z_p)^{\{n-1\}}(y_M - y_s)^{\{t\}} = 0, \quad \begin{array}{l} k \in \{1,2,3,\cdots,N-n\} \\ t \in \{0,1,2,\cdots,m-1\} \end{array} \tag{3.6}$$

$$\sum_{s=1}^{t}\sum_{p=1}^{k} r_{sp}(y_{m+t-1} - y_s)^{\{m-1\}}(z_{n+k-1} - z_p)^{\{n-1\}} \geq 0, \quad \begin{array}{l} k \in \{1,2,3,\cdots,N-n\} \\ t \in \{1,2,3,\cdots,M-m\}. \end{array} \tag{3.7}$$

Proof. If (3.4), (3.5) and (3.6) hold, then 1st, 2nd and 3rd terms are zero in (3.2), then by using (3.7) we obtain the required inequality (3.3).

Conversely, if substitute the following functions in (3.3). Then we obtain the required equality (3.4)

$$f_1(y_s, z_p) = (z_N - z_p)^{\{k\}}(y_M - y_s)^{\{t\}} \quad \text{and} \quad f_2 = -f_1$$

for $0 \leq t \leq m-1$ and $0 \leq k \leq n-1$ such that $\nabla_{(m,n)} f_l \geq 0, \quad l \in \{1,2\}$

$$\sum_{s=1}^{M-t}\sum_{p=1}^{N-k} r_{sp}(z_N - z_p)^{\{k\}}(y_M - y_s)^{\{t\}} = 0, \quad 0 \leq k \leq n-1; \quad 0 \leq t \leq m-1.$$

In the similar manner, if take the following functions in (3.3) for $0 \le k \le n-1$ and $1 \le t \le M-m$

$$f_3(y_s, z_p) = \begin{cases} (z_N - z_p)^{\{k\}}(y_{m+t-1} - y_s)^{\{m-1\}}, & s < t \\ 0, & s \ge t \end{cases}$$

$$f_4 = -f_3$$

such that $\nabla_{(m,n)} f_l \ge 0, \quad l \in \{3, 4\}$, we obtain the equality (3.5) i.e.

$$\sum_{s=1}^{t}\sum_{p=1}^{k} r_{sp}(z_N - z_p)^{\{k\}}(y_{m+t-1} - y_s)^{\{m-1\}} = 0, \quad 0 \le k \le n-1; \quad 1 \le t \le M-m.$$

Similarly, if take the following functions in (3.3) for $0 \le t \le m-1$ and $1 \le k \le N-n$

$$f_5(y_s, z_p) = \begin{cases} (z_{n+k-1} - z_p)^{\{n-1\}}(y_M - y_s)^{\{t\}}, & p < k \\ 0, & p \ge k \end{cases}$$

$$f_6 = -f_5$$

such that $\nabla_{(m,n)} f_l \ge 0, \quad l \in \{5, 6\}$, we obtain the equality (3.6) as above, i.e.

$$\sum_{s=1}^{M-t}\sum_{p=1}^{k} r_{sp}(z_{n+k-1} - z_p)^{\{n-1\}}(y_M - y_s)^{\{t\}} = 0, \quad 0 \le t \le m-1; \quad 1 \le k \le N-n.$$

We get the last inequality (3.7) by considering the following function in (3.3) for $1 \le t \le M-m$ and $1 \le k \le N-n$

$$f_7(y_s, z_p) = \begin{cases} (y_{m+t-1} - y_s)^{\{m-1\}}(z_{n+k-1} - z_p)^{\{n-1\}}, & s < t, \quad p < k \\ 0, & s \ge t \ \text{ or } \ p \ge k. \end{cases}$$

□

Remark 3.7. Similar remark as given in Remark 3.4 for above result.

Remark 3.8. The above result also holds for a_{jl} sequence and $a_{jl} = a_j b_l$ sequences.

4. CONCLUSION

At the end of this article we would like to give conclusion that we have obtained discrete identity for sequence $\sum_{j=1}^{M}\sum_{l=1}^{N} r_{jl}a_{jl}$ for two dimension in which involves higher order ∇ divided difference. We have also obtained the similar result as Theorem 2.2 for two a_j and b_l sequences. Similarly, we have obtained discrete identity for function $\sum_{j=1}^{M}\sum_{l=1}^{N} r_{jl}f(y_j, z_l)$ for two independent variables in the interval $J_1 \times L_1 \subset \mathbb{R} \times \mathbb{R}$

in which involves higher order ∇ divided difference and also given result for function $\sum_{j=1}^{M}\sum_{l=1}^{M} r_{jl} f(y_j, z_l)$ for two independent variables in the interval $J_1^2 \subset \mathbb{R} \times \mathbb{R}$. We have further obtained the similar result as Theorem 3.2 for two f and g functions and also given the result about the necessary and sufficient conditions of Theorem 3.2 for Popoviciu type characterisation of positivity of sums for discrete function of two variables $\sum_{j=1}^{M}\sum_{l=1}^{N} r_{jl} f(y_j, z_l) \geq 0$ using $(m,n) - \nabla-$convex functions and we also presented these necessary and sufficient conditions for Theorem 2.2 involving $(m,n) - \nabla-$convex sequences as a Remark 3.8.

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