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On Commutative GS-rings

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Abstract

Let R be a commutative ring and M a left R-module. An R-module M is said to be generalized Hopfian if any endomorphism surjective of M is superfluous. It is well-know that any noetherian module M is generalized Hopfian but the converse is not always true. For example the $\mathbb{Z}$-module, $M = \bigoplus \mathbb{Z}/p\mathbb{Z}$ where p is a prime number, is generalized Hopfian but it is not noetherian. A ring R is said to be GS-ring if every generalized Hopfian module is noetherian. In this note, we give some characterizations of such rings.

Keywords: Generalized Hopfian module, Hopfian module, GS-ring.

INTRODUCTION AND DEFINITIONS

In this paper, R is commutative ring and M a left R-module. A module M is said to be generalized Hopfian if every endomorphism surjective of M is superfluous. Any noetherian module is generalized Hopfian but the reciprocal is not true in general. For instance, let $M = \bigoplus \mathbb{Z}/p\mathbb{Z}$ be a left $\mathbb{Z}$-module where p is a prime number, it is obvious to see that M is generalized Hopfian but it is not noetherian. On this fact, we introduce the notion of GS-ring, it is a ring for which every generalized Hopfian module is noetherian.

A module M is said to be Hopfian when every endomorphism surjective of M is injective. A ring R is a S-ring if every Hopfian module is noetherian. A module is uniserial if its submodules are linearly ordered by inclusion. A module M is faithful if its annihilator $Ann(M) = 0$. A module M is said to be a prime module if $Ann(M) = Ann(N)$ for every submodule N of M. A ring R is said to be of finite representation type, if R as a left R-module is of finite length and there are only a finite number of non-isomorphic finitely generated indecomposable modules. A ring is said to be semilocal if it has a finite number of maximal ideals. A module M is said to be hollow if every proper submodule of M is superfluous.

Proposition 1:
Let R be a commutative ring, if R is a GS-ring then R is a S-ring.
Proof:
Let M be a Hopfian module. It results from corollary 1.4 of [3] that M is generalized Hopfian. As R is a GS-ring, then M is noetherian. Thus R is a S-ring.

Corollary 1:
Let R be a commutative semisimple ring, then R is a GS-ring if and only if R is a S-ring.
Proof:
It has been shown in proposition 1 that a GS-ring is a S-ring. For the reverse, let M be a generalised Hopfian R-module. Since R is semisimple, then M is a semisimple R-module. It is well know that any semisimple module is quasi-projective. It results from corollary 1.4 of [3] that M is Hopfian. Hence, M is noetherian. Thus R is a GS-ring.

Proposition 2:
Let R be a commutative ring and M a finitely generated, faithful and prime module. Then R is a GS-ring if and only if R is a field.
Proof:
Let's assume R to be a GS-ring, hence R is a S-ring, by proposition 1. It follows from [5], that R is a artinian principal ideal ring. Since any quotient of an artinian ring is artinian then $R/Ann(M)$ is artinian. Let $f : R \rightarrow R/Ann(M)$ be a canonical projection. Let $h : R \rightarrow M$ an epimorphism.

$$\begin{array}{c} f : R \longrightarrow M \\ \downarrow \ \swarrow \\ R/Ann(M) \end{array}$$

Hence, by the first isomorphism theorem $R/Ann(M)$ is isomorphic to M. Hence M is artinian. It is well-know that any artinian module has a simple submodule, let's say N_0. Let $g : R \rightarrow N_0$ be another epimorphism.

$$\begin{array}{c} g : R \longrightarrow N_0 \\ \downarrow \ \swarrow \\ R/Ann(N_0) \end{array}$$

Then, $R/Ann(N_0)$ is isomorphic to N_0. As M is prime, $Ann(N_0) = Ann(M)$. Thus $R/Ann(M) = R/Ann(N_0)$ is simple. Since M is faithful therefore R is a field.
Now let's suppose that R is a field and M a module over R. Let $f : M \to R$ and $g : R \to M$ be homomorphisms with $f, g \neq 0$. It obvious to see that f is surjective and g is injective. Assume $g \circ f : M \to M$ to be surjective hence, g is surjective. We deduce that g in an isomorphism. Thus M is simple hence noetherian.

Proposition 3:
Let R be a commutative GS-ring.
(1) Every prime ideal of R is maximal;
(2) There exists a finite number of prime ideals.
Proof:
(1) It has been shown in the proof of proposition 2 that R is artinian. Let I be a prime ideal. It is well known that the quotient of any artinian ring is artinian. Hence R/I is artinian which is also an integral domain ring. By the lemma 12.2.5 of [2], R/I is a field. Therefore I is maximal.
(2) Let $P = \{I_\lambda, \lambda \in \Lambda\}$ the set of all prime ideals of R. Assume $Q = \bigoplus_{\lambda\in\Lambda} R/I_\lambda$ and let's consider for each $\lambda \in \Lambda$, R/I_λ as a left R-module. Therefore R/I_λ is simple for any $\lambda \in \Lambda$. Hence R/I_λ is Hopfian for any $\lambda \in \Lambda$. Since any simple module is completely invariant then, $Q = \bigoplus_{\lambda\in\Lambda} R/I_\lambda$ is Hopfian. By corallary 1.4 of [3], Q is generalized Hopfian. As R is a GS-ring, therefore Q is noetherian. It results from 10.16 of [2] that Q is of finie length. Thus Λ is finite.

Corollary 2: Let R be a commutative GS-ring then R is semilocal ring.
Proof
It results from the definition of semilocal ring.

Lemma 1: let R be a commutative ring such that $R = \prod_{i\in I} R_i$, where R_i are rings. If M is R-module then M is a R_i-module.

Proposition 4:
Let R be a commutative ring.
(1) An homomorph image of a GS-ring is a GS-ring.

(2) Let $R = \prod_{i \in I} R_i$ a direct product of rings and I a finite set, R is a GS-ring if and only if R_i is a GS-ring for each $i \in I$.

Proof:

(1) Let R be a GS-ring and $f : R \longrightarrow f(R) = A$ a homomorphism surjective. Assume M a generalized Hopfian module over A. M is also a module over R by the following map:

$$\begin{aligned} R \times M &\longrightarrow M \\ (r, n) &\longmapsto f(r)n \end{aligned}$$

As R is a GS-ring, then M is noetherian. Hence A is a GS-ring.

(2) Let's suppose that R is a GS-ring. Let $f_i : R \longrightarrow R_i$ an homomorphism surjective for all $i \in I$. It results from (1) that R_i is a GS-ring for any $i \in I$.

Now assume that R_i is a GS-ring for all $i \in I$ and let M be a generalized Hopfian R-modules. It follows from lemma 2 that M is module over R_i which is a GS-ring for all $i \in I$. Therefore M is noetherian. Hence, R is a GS-ring.

Theorem 1:

Let R be commutative ring and M a module, then the following conditions are equivalent:

(1) R is a GS-ring;

(2) R is a S-ring;

(3) R is artinian principal ideal rings.

Proof:

(1) $\Rightarrow$ (2) It results from proposition 1.

(2) $\Leftrightarrow$ (3) It follows from [5].

(3) $\Rightarrow$ (1) Since R is an artinian principal ideal ring, it results from corollary 10 of [5] that any left R-module is a direct sum of cyclic submodules. So M can be written as the form $M = \bigoplus M_i$ where M_i are cyclic submodules for any $i \in \mathbb{N}$.

Let $M = \bigoplus M_i$ be a generalized Hopfian module that is not noetherian then there exists in M only a family $(L_i)_{i \in I}$ of finite number of indecomposable non-isomorphic cyclic submodules and a direct summand of infinite countable family $(N_j)_{j \in \mathbb{N}}$ of cyclic submodules such that any two of them are isomorphic. So M can be written as the form $M = L \bigoplus N$ where $L = \bigoplus_{i \in I} L_i$ and $N = \bigoplus_{i \in \mathbb{N} - I} N_i$. It results from corollary 1.3 of [3] that N is generalized Hopfian and from proposition 2.3 of [7] that N is a not generalized. Therefore M is noetherian. Thus R is a GS-ring.

Theorem 2:
Let R be a commutative ring. The following conditions are equivalent:
(1) R is a GS-ring;
(2) R is a S-ring;
(3) R is SCS-ring;
(4) R is an artinian principal ideal ring.

Proof:
$(1) \Leftrightarrow (4)$ theorem 1, $(2) \Leftrightarrow (4)$ theorem 9 of [5], and $(3) \Leftrightarrow (4)$ theorem 2.5 of [6].

Theorem 3:
Let R be a commutative ring, M a left R-module and $(I_\lambda)_{\lambda\in\Lambda}$ an ascending chain conditions of prime ideals of R. Then following conditions are equivalent:
(1) R is a GS-ring;
(2) M is of finite length.
Proof
$(1) \Rightarrow (2)$ Let M be a left R-module. Here we are going to show first that the following ascending chain is stationary and of finite length.

$$0 = M_0 \subset M_1 \subset ... \subset M_i \subset ... \subset M$$

It is well know that any module $M \neq 0$ has an associated prime ideal which is the annihilator of an element $m \in M$(i.e there exists $m \in M$ such that $I_1 = Ann(m)$). Let $M_1 = Rm$, then we have the following diagram

$$\begin{array}{c} f : R \longrightarrow Rm \\ \downarrow \ \nearrow \\ R/I_1 \end{array}$$

since f is an epimorphism therefore $R/I_1 \simeq Rm = M_1$. If we do the same diagram for M/M_1 we will have

$$\begin{array}{c} f_1 : R \longrightarrow M/M_1 \\ \downarrow \ \nearrow \\ R/I_2 \end{array}$$

where $R/I_2 \subset M/M_1$. Therefore there is $M_2 \subset M$ containing M_1 such that $R/I_2 \simeq M_2/M_1$. By continuing the process, we will have the following sequence:

$$M_1 \simeq R/I_1, M_2/M_1 \simeq R/I_2, ..., M_i/M_{i-1} \simeq R/I_i, ... \quad (0)$$

It results from the proposition 3 that there is a finite number of prime ideals. So the sequence (0) is stationary meaning there is $n \in \mathbb{N}$ such that

$$M_1 \simeq R/I_1, M_2/M_1 \simeq R/I_2, ..., M_n/M_{n-1} \simeq R/I_n \quad (1)$$

Since $(I_\lambda)_{\lambda\in\Lambda}$ is an ascending chain conditions of prime ideals, then we have

$$0 = I_1 \subset I_2 \subset ... \subset I_n = I_{n+1} \text{ which implies } R/I_1 \supset R/I_2 \supset ... \supset R/I_n = R/I_{n+1}$$

It results from (1) that

$$M_1 \supset M_2/M_1 \supset ... \supset M_n/M_{n-1} \text{ hence, } 0 = M_0 \subset M_1 \subset ... \subset M_n = M$$

Therefore the sequence $0 = M_0 \subset M_1 \subset ... \subset M_n = M$ is stationary. It follows from proposition 3 that $M_{i-1}/M_i \simeq R/I_i$ is simple for any $0 \leq i \leq n$. Thus M is of finite length.
$(2) \Rightarrow (1)$ It is obvious: let M be an generalized Hopfian module. Since M is of finite length, hence noetherian.

Theorem 4:
Let R be a commutative ring and M a left R-module such that any module has small kernel, then the following statements are equivalent:
(1) R is a GS-ring;
(2) R is a of finite representation type.

Proof.
$(1) \Rightarrow (2)$ Let's prove first that R is of finite length as a left R-module. Indeed, it follows from theorem 1 that R is artinian. Let M be a finitely generated module. Since R is commutative then M is Hopfian, hence generalized Hopfian. As R is a GS-ring, therefore M is noetherian. It results from 10.19 of [3] that R is noetherian. Let $(M_i)_{i\in\mathbb{N}}$ be a complete system of non-isomorphic class of simple modules. Let's suppose $M = \bigoplus_{i\in\mathbb{N}} M_i$. For each $i \in \mathbb{N}$, M_i is simple hence Hopfian. Since simple modules are fully invariants therefore $M = \bigoplus_{i\in\mathbb{N}} M_i$ is Hopfian. Hence M is generalized Hopfian. As R is a GS-ring, then M is noetherian. It results from 10.16 of [2] that M is of finite length. Hence there is a finite number of non-isomorphic class of simple(i.e finitely generated and indecomposable) modules. Thus R is a of finite

representation type as a left R-module.
$(2) \Rightarrow (1)$ Let's consider R as a left R-module of finite length. Let f and g be the following homomorphisms:

$$f : R \longrightarrow M \text{ and } g : M \longrightarrow R$$

It is well know that the endomorphism of finite length module is an automorphism. As $g \circ f$ is an automorphism therefore f is a monomorphism. Let m be an element of M, since $g \circ f$ is an epimorphism, then there is $r \in R$ such that $g(f(r)) = g(m)$. That means $m = f(r) + (m - f(r)) \in$ Im f + ker g. Therefore M = Im f + ker g. Since M has small kernel then M = Im f. Hence f is surjective, so R is isomorphic to M. Thus M is noetherian.

Corollary 3:
Let R be a commutative ring such that every left R-module M is hollow, then the following statements are equivalent:
(1) R is a GS-ring;
(2) R is a of finite representation type.

Proof.
$(1) \Longrightarrow (2)$ It results from Theorem 4.
$(2) \Longrightarrow (1)$ It follows from theorem 4 that if M = Im f + ker g. Since M is hollow, then M = Im f or M = ker g. If M = Im f, then M is noetherian by theorem 4. If not M = ker g meaning g is injective. As $g \circ f$ is surjective so is g. Hence $M \simeq R$.

REFERENCES

[1] Chambert-Loir: *Algèbre commutative*, Centre de Mathématique, Ecole polytechnique, 91128 Palaiseau Cedex, Paris 6,2001.

[2] Anderson F.W and Fuller K. (1974) *Rings and categories of modules*, Springer-Verlag

[3] A. Ghorbani and A. Haghany: *Generalized Hopfian modules*, Journal of Algebra 255, (2002) 324-341

[4] C. Faith, *On Köthe Ring* , Math. Annalen 164, 207-212(1966)

[5] Kaidi, A.M et Sangharé, M: *Une caractérisation des anneaux artiniens à idéaux principaux.* L.Notes in Math.Springer Verlag (1988) pp.245 - 254

[6] Mbaye A., Sangharé M., Touré S. D., *On Commutative SCI-Rings and Commutative SCS-Rings*, International Journal of Algebra, Vol. 4, 2010, no. 12, 585 - 590.

[7] S. D Touré, M. Sangharé and I. Labou, *On Weakly Commutative SCI-Rings and Generalized Commutative SCS-Rings*.