Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 15, Number 6 (2019), pp. 1121-1128

http://www.ripublication.com/gjpam.htm

Local α-Closure Function in Ideal Topological Spaces

P Periyasamy

Department of Mathematics,(9555), Kamaraj College,Thoothukudi-3.
ManonmaniamSundaranar University, Abishekapatti,
Tirunelveli-12. Tamilnadu, India.
e-Mail:periyasamyvpp@gmail.com.

M Navaneethakrishnan

Department of Mathematics, Kamaraj College,Thoothukudi-3.
ManonmaniamSundaranar University, Abishekapatti,
Tirunelveli-12. Tamilnadu, India.
e-Mail:navaneethan65@yahoo.co.in.

Abstract:

In this paper $(X, \tau, \boldsymbol{I})$ denotes an ideal topological space. We define an operator $\zeta_\alpha(A)(\boldsymbol{I},\ \tau)$ called the local α-closure function of A with respect to ideal $\boldsymbol{I}$ and topology τ by: $\zeta_\alpha(A)(\boldsymbol{I},\ \tau) = \{x \in X : cl_\alpha(U) \cap A \notin \boldsymbol{I}$ for every $U \in \tau(x)\}$. We investigate the basic properties and characterizations of $\zeta_\alpha(A)(\boldsymbol{I},\ \tau)$. Also we investigate an operator Ψ_{ζ_α}: P(X) → τ satisfying $\Psi_{\zeta_\alpha}(A) = X - \zeta_\alpha(X - A)$ foreach A ∈ P(X) by using $\zeta_\alpha(A)(\boldsymbol{I},\ \tau)$.

Keywords: Ideal Topology, Local Function, Local α-closure function.

1. INTRODUCTION

Kuratowski [6], vaidhyanathaswamy [11] was studied the notion of ideal topological spaces Dontchev et al [5], Navaneethakrishnan et al [7], Jankovic et al[3], Mukherjee et al [8], Nasef et al [2] etc., were investigated applications to various fields of ideal topology. The purpose of this paper is to introduce an operator $\zeta_\alpha(A)(\boldsymbol{I},\ \tau)$ and investigate its basic properties and characterizations. Also introduce and study an operator $\Psi_{\zeta_\alpha}(A)$ using $\zeta_\alpha(A)(\boldsymbol{I},\ \tau)$.

2. PRELIMINARIES

In a topological space (X, τ) with no separation properties assumed, for a subset A, cl(A) and int(A) denotes the closure and interior of A in (X, τ) as smallest closed set containing A and the largest open set contained in A respectively. An ideal ***I*** on a set X is a nonempty collection of subsets of X which Satisfies: (i) A ∈ ***I*** and B ⊆ A implies B ∈ ***I*** (heredity), (ii) A ∈ ***I*** and B ∈ ***I*** implies A ∪ B ∈ ***I*** (finite additive). An ideal topological space is denoted by the triplet (*X, τ, **I***). Given a topological space (X, τ) with an ideal ***I*** on X and if P(X) is the collection of all subsets of X a set operator A* : P(X) → P(X) called a local function [6] of a subset A of X with respect to ***I*** and τ is defined as, A*(***I***, τ) = {x ∈ U ∩ $A \notin \boldsymbol{I}$ for every U ∈ $\tau(x)$ where $\tau(x)$ = {U ∈ $\tau(x)$ / x ∈ U}. A kuratowski closure operator $cl^*(A)$ for a topology $\tau^*(\boldsymbol{I}, \tau)$ called *-topology finer than τ is defined by $cl^*(A) = A$ ∪ A*(***I***, τ) [11]. In [10] O.Njasted investigated the notation of α-closed sets. A subset A of X is said to be α-open (resp. α-closed) set if A ⊆ int(cl(int(A))) (resp. cl(int(cl(A)))) ⊆ A. It is worth to mention that, the set of all α-open sets forms topology and it is denoted by τ^α. The closure operator onτ^α for a subset A of X is denoted by cl_α(A) or αcl(A). Velicko.N. [9] investigated the operator cl_θ(A) for a subset A of X, defined by cl_θ(A) = θcl(A) = {x ∈ X : cl(U) ∩ A ≠ ϕ}, for each U ∈ τ(x)}. A subset A of X is said to be a θ-closed[9] set if cl_θ(A) = A. Ahmad Al-Omari and Takashi Noiri[1] introduced and investigated the operator Γ(A)(**I**, τ) = {x ∈ X : cl(U) ∩ A ∉ ***I***, for every U ∈ τ(x)}. For convenience, denote A* for A*(***I***, τ).

In this paper we introduce and investigate an operator $\zeta_\alpha(A)(\boldsymbol{I}, \tau)$ called local α-closure function of A with respect to ***I*** and τ. Also, investigate an operator $\Psi_{\zeta_\alpha}: P(X) \to \tau$ using $\zeta_\alpha(A)$.

3. LOCAL α-CLOSURE FUNCTION

Definition 3.1. Let (*X, τ, **I***) be an ideal topological space. For a subset A of X. we define the following operator $\zeta_\alpha(A)(\boldsymbol{I},\tau) = \{x \in X: A \cap cl_\alpha(U) \notin \boldsymbol{I}, forevery\ U \in \tau(x)\}$ where $\tau(x)$ = {U ∈ $\tau(x)$ / x ∈ U}. For convenience $\zeta_\alpha(A)(\boldsymbol{I}, \tau)$ is briefly denoted by $\zeta_\alpha(A)$ and is called the local α-closure function with respect ***I*** and τ.

The following Example shows that the existence of $\zeta_\alpha(A)$ for a subset A of X.

Example 3.2. Let X = {a, b, c, d}, τ = {X, ϕ, {b},{c, d}, {b, c, d}}, and I = {ϕ, {b}, {c}, {b, c}}. Then $\zeta_\alpha(\{d\})$ = {a, c, d}.

Lemma 3.3. Let (X, τ, ***I***) be an ideal topological space and A ⊆ X. Then A* ⊆ $\zeta_\alpha(A)$

Proof. Let x ∈ A* then for every U ∈ $\tau(x)$, A ∩ $U \notin \boldsymbol{I}$ and hence A ∩ $cl_\alpha(U) \notin \boldsymbol{I}$, for every $U \in \tau(x)$. Therefore x ∈ $\zeta_\alpha(A)$.

Remark 3.4. The following Example shows that the reverse inclusion is not always hold.

Example 3.5. In Example 3.2, $\zeta_\alpha(\{a\}) = \zeta_\alpha(\{a,\ b\}) = \zeta_\alpha(\{a,\ c\})$ = X and {a}* = {a, b}* = {a, c}* = {a}.

Lemma 3.6.[1] Let (X, τ, ***I***) be an ideal topological space. For A ⊆ X, then

(i)If A is open then cl(A) = $cl_{\theta}(A)$.

(ii) If A is closed then int(A) = $int_{\theta}(A)$.

Theorem 3.7. Let (X,τ, ***I***) be an ideal topological space ***I*** and ***J*** be two ideals on X and let A and B be subsets of X. Then the following properties hold:

(i) If A ⊆ B then $\zeta_{\alpha}(A) \subseteq \zeta_{\alpha}(B)$.

(ii) If ***I*** ⊆ ***J*** then $\zeta_{\alpha}(A)(\boldsymbol{J}) \subseteq \zeta_{\alpha}(A)(\boldsymbol{I})$.

(iii)$\zeta_{\alpha}(A) = cl(\zeta_{\alpha}(A)) \subseteq cl_{\theta}(A)$ and $\zeta_{\alpha}(A)$ is closed.

(iv)If A ⊆ $\zeta_{\alpha}(A)$ and $\zeta_{\alpha}(A)$ is open, then $\zeta_{\alpha}(A) = cl_{\theta}(A)$.

(v) If A ∈ $\boldsymbol{I}$ *then* $\zeta_{\alpha}(A) = \phi$.

Proof. (i) If A ⊆ B. Suppose x ∉ $\zeta_{\alpha}(B)$, then there exists $U \in \tau(x)$ such that B ∩ $cl_{\alpha}(U) \in \boldsymbol{I}$. Since A ∩ $cl_{\alpha}(U)$ ⊆ B ∩ $cl_{\alpha}(U) \in \boldsymbol{I}$. Hence A∩ cl_{α}(U) ∈ ***I***, for every $U \in \tau(x)$ which implies x ∉ $\zeta_{\alpha}(A)$. Thus X \ $\zeta_{\alpha}(B)$ ⊆ X \ $\zeta_{\alpha}(A)$.

(ii) Suppose, x ∉ $\zeta_{\alpha}(A)(\boldsymbol{I})$, then there exists $U \in \tau(x)$ such that A ∩ $cl_{\alpha}(U) \in \boldsymbol{I}$. Since ***I*** ⊆ ***J***, A ∩ $cl_{\alpha}(U) \in \boldsymbol{J}$ and hence x ∉ $\zeta_{\alpha}(A)(\boldsymbol{J})$. Therefore, $\zeta_{\alpha}(A)(\boldsymbol{J}) \subseteq \zeta_{\alpha}(A)(\boldsymbol{I})$.

(iii) We have $\zeta_{\alpha}(A)$ ⊆ cl($\zeta_{\alpha}(A)$). Let x ∈ $cl(\zeta_{\alpha}(A))$. Then $\zeta_{\alpha}(A) \cap U_x \neq \phi$ for every $U_x \in \tau(x)$. Therefore, there exists some y ∈ $\zeta_{\alpha}(A) \cap U_x$ and $U_x \in \tau(y)$. Since y ∈ $\zeta_{\alpha}(A)$, A ∩ $cl_{\alpha}(U_x) \notin I$ for every $U_x \in \tau(y)$ and hence x ∈ $\zeta_{\alpha}(A)$. Now, let x ∈ $\zeta_{\alpha}(A)$, then A ∩ $cl_{\alpha}(U_x) \neq \phi$ for every $U_x \in \tau(x)$.This implies A ∩ $cl(U_x) \neq \phi$ for every $U_x \in \tau(x)$. Therefore, x ∈ $cl_{\theta}(A)$. This shows that $\zeta_{\alpha}(A) = cl(\zeta_{\alpha}(A)) \subseteq cl_{\theta}(A)$.

(iv) For any subset A of X. We have $\zeta_{\alpha}(A) = cl(\zeta_{\alpha}(A)) \subseteq cl_{\theta}(A)$. Since A ⊆ $\zeta_{\alpha}(A)$ and $\zeta_{\alpha}(A)$ is open then $cl_{\theta}(A) \subseteq cl_{\theta}(\zeta_{\alpha}(A)) = cl(\zeta_{\alpha}(A)) = \zeta_{\alpha}(A) \subseteq cl_{\theta}(A)$by Lemma 3.6. Hence $\zeta_{\alpha}(A)$ = $cl_{\theta}(A)$.

(v) Suppose x ∈ $\zeta_{\alpha}(A)$. Then for any $U \in \tau(x)$, A ∩ cl_{α}(U) ∉ ***I***. But, since A∈ ***I***, A∩ $cl_{\alpha}(U) \in \boldsymbol{I}$, for every $U \in \tau(x)$. There is a contradiction. Hence $\zeta_{\alpha}(A) = \phi$.

Theorem 3.8. Let (X, τ, ***I***) be an ideal topological space, A and B any two subsets of X. Then the following hold:

(i) $\zeta_{\alpha}(\phi) = \phi$.

(ii) $\zeta_{\alpha}(A \cup B) = \zeta_{\alpha}(A) \cup \zeta_{\alpha}(B)$.

Proof. (i) Proof is obvious.

(ii) By Theorem 3.7(i), $\zeta_{\alpha}(A \cup B) \supseteq \zeta_{\alpha}(A) \cup \zeta_{\alpha}(B)$. To prove the reverse inclusion, let x ∉ $\zeta_{\alpha}(A) \cup \zeta_{\alpha}(B)$. Then $x \notin \zeta_{\alpha}(A)$ and $x \notin \zeta_{\alpha}(B)$. Therefore there exist $U_x, V_x \in \tau(x)$ such that $cl_{\alpha}(U_x)$ ∩ A ∈ ***I*** and $cl_{\alpha}(V_x)$ ∩ B ∈ ***I***. Then ($cl_{\alpha}(U_x)$ ∩ A) ∪ ($cl_{\alpha}(V_x)$ ∩ B) ∈ ***I*** since ***I*** is additive. Furthermore, since ***I*** is heredity, $cl_{\alpha}(U_x \cap V_x) \cap (A \cup B) \subseteq [(cl_{\alpha}(U_x) \cap (cl_{\alpha}(V_x)] \cap (A \cup B)] = [(cl_{\alpha}(U_x) \cap cl_{\alpha}(V_x)) \cap A] \cup [(cl_{\alpha}(U_x) \cap$

$cl_\alpha(V_x)) \cap B] = ([cl_\alpha(U_x) \cap A] \cap [cl_\alpha(V_x) \cap A]) \cup ([cl_\alpha(U_x) \cap B] \cap [cl_\alpha(V_x) \cap B]) \subseteq [cl_\alpha(U_x) \cap A] \cup [(cl_\alpha(V_x) \cap B] \in \boldsymbol{I}$. Also, $U_x \cap V_x \in \tau(x)$. Therefore, x $\notin \zeta_\alpha(A \cup B)$.

Theorem 3.9. Let (X, τ, ***I***) be an idealtopological space and A, B any two subsets of X. Then $\zeta_\alpha(A)\backslash\zeta_\alpha(B) = \zeta_\alpha(A\backslash B)\backslash\zeta_\alpha(B) \subseteq \zeta_\alpha(A\backslash B)$.

Proof. Let A = (A\B) ∪ (A ∩ B), then by Theorem 3.8(ii), $\zeta_\alpha(A) = \zeta_\alpha(A\backslash B) \cup \zeta_\alpha(A \cap B)$. $\zeta_\alpha(A)\backslash\zeta_\alpha(B) = \zeta_\alpha(A) \cap \left(X\backslash\zeta_\alpha(B)\right) = \left(\zeta_\alpha(A\backslash B) \cup \zeta_\alpha(A \cap B)\right) \cap \left(X\backslash \zeta_\alpha(B)\right) = [\zeta_\alpha(A\backslash B) \cap \left(X\backslash\zeta_\alpha(B)\right)] \cup [\zeta_\alpha(A \cap B) \cap (X\backslash\zeta_\alpha(B))] = [\zeta_\alpha(A\backslash B)\backslash \zeta_\alpha(B)] \subseteq \zeta_\alpha(A\backslash B)$.

Corollary 3.10. Let (X, τ, ***I***) be an ideal topological space and A and B any two subsets of X with B ∈ ***I***. Then $\zeta_\alpha(A \cup B) = \zeta_\alpha(A) = \zeta_\alpha(A - B)$.

Proof. Since B ∈ ***I***,by Theorem 3.7 (v) $\zeta_\alpha(B) = \phi$. Hence by Theorem 3.9, $\zeta_\alpha(A) = \zeta_\alpha(A - B)$. Also, by Theorem 3.8(ii), we have $\zeta_\alpha(A \cup B) = \zeta_\alpha(A) \cup \zeta_\alpha(B) = \zeta_\alpha(A)$.

Theorem 3.11. Let (X, τ, ***I***) be an ideal topological space, A and B any two subsets of X. Then $(A \cap B) \subseteq \zeta_\alpha(A) \cap \zeta_\alpha(B)$. Moreover, $\zeta_\alpha(A \cap B) \subseteq \zeta_\alpha(A) \cup \zeta_\alpha(B)$.

Proof. Let x $\in \zeta_\alpha(A \cap B)$. Then $cl_\alpha(U) \cap (A \cap B) \notin \boldsymbol{I}$, for every U $\in \tau(x)$ which implies $(cl_\alpha(U) \cap A) \cap (cl_\alpha(U) \cap B) \notin \boldsymbol{I}$ and hence $cl_\alpha(U) \cap A \notin \boldsymbol{I}$ and $cl_\alpha(U) \cap B \notin \boldsymbol{I}$. That is, x $\in \zeta_\alpha(A)$ and x $\in \zeta_\alpha(B)$ and hence x $\in \zeta_\alpha(A) \cap \zeta_\alpha(B)$ and so x ∈ $\zeta_\alpha(A) \cup \zeta_\alpha(B)$.

Theorem 3.12. Let (X, τ, ***I***) be an ideal topological space. Then $\zeta_\alpha(A) \supseteq$ A\∪ $\{U \subseteq$ X : U ∈ ***I*** } for all A ⊆ X.

Proof. Take B = ∪ $\{U \subseteq$ X : X ∈ ***I***} and let x ∈ A – B. Then x ∉ B implies that x ∉ U for all U ∈ ***I***,so that {x} = {x} ∩ $A \notin \boldsymbol{I}$, because x ∈ A. For every V ∈ $\tau(x)$, we have {x}∩ A ⊆ cl_α(V) ∩ $A \notin \boldsymbol{I}$, by heredity of ideal. Hence x ∈ $\zeta_\alpha(A)$.

Theorem 3.13. Let (X, τ, ***I***) be an ideal topological space with $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ be any two ideals then $\zeta_\alpha(\boldsymbol{I}_1 \cap \boldsymbol{I}_2) \subseteq \zeta_\alpha(\boldsymbol{I}_1) \cap \zeta_\alpha(\boldsymbol{I}_2)$. Moreover, $\zeta_\alpha(\boldsymbol{I}_1 \cap \boldsymbol{I}_2) \subseteq \zeta_\alpha(\boldsymbol{I}_1) \cup \zeta_\alpha(\boldsymbol{I}_2)$

Proof. Let x $\in \zeta_\alpha(\boldsymbol{I}_1 \cap \boldsymbol{I}_2)$ which implies for every U ∈ $\tau(x)$, $cl_\alpha(U) \cap A \notin \boldsymbol{I}_1$ and $cl_\alpha(U) \cap A \notin \boldsymbol{I}_2$, for every U ∈ $\tau(x)$. Therefore, x ∈ $\zeta_\alpha(\boldsymbol{I}_1)$ and x ∈ $\zeta_\alpha(\boldsymbol{I}_2)$and hence x ∈ $\zeta_\alpha(\boldsymbol{I}_1) \cap \zeta_\alpha(\boldsymbol{I}_2)$ and so x ∈ $\zeta_\alpha(\boldsymbol{I}_1) \cup \zeta_\alpha(\boldsymbol{I}_2)$.

4. α-CLOSURE COMPATIBILITY OF TOPOLOGICAL SPACES

Definition 4.1. Let (X, τ, ***I***) be an ideal topological space. We say that τ is α-closure compatible with the ideal ***I*** denoted by τ $\sim\zeta_\alpha\boldsymbol{I}$, if the following conditions holds for every A ⊆ X, if for every x ∈ A there exists U ∈ $\tau(x)$ such that A ∩ $cl_\alpha(U) \in \boldsymbol{I}$, then A ∈ ***I***.

Theorem 4.2. Let (X, τ, ***I***) be an ideal topological space, then the following properties are equivalent:

(i) τ ~ $\zeta_\alpha\boldsymbol{I}$.

(ii) If a subset A of X has a cover of open sets each of whose α closure intersection with A is in $\boldsymbol{I}$, then A ∈ $\boldsymbol{I}$.

(iii) For every A ⊆ X, A ∩ $\zeta_\alpha(A) = \phi$ implies that A ∈ $\boldsymbol{I}$.

(iv) For every A ⊆ X , A $-\ \zeta_\alpha(A) \in \boldsymbol{I}$.

(v) For every A ⊆ X , if A contains no nonempty subset B with B ⊆ $\zeta_\alpha(B)$, then A ∈ $\boldsymbol{I}$.

Proof. (i) ⇒ (ii). The proof is obvious by Definition.

(ii) ⇒ (iii). Let A ⊆ X and x ∈ A. Then x ∉ $\zeta_\alpha(A)$ and there exist G ∈ $\tau(x)$ such that $cl_\alpha(G) \cap A \in \boldsymbol{I}$. Therefore, we have A ⊆ ∪ $\{G : x \in A\}$ and G ∈ $\tau(x)$ and by (ii), A ∈ $\boldsymbol{I}$.

(iii)⇒ (iv). For any A ⊆ X , A $-\ \zeta_\alpha(A)$ ⊆ A and (A $-\ \zeta_\alpha(A)$) ∩ ζ_α(A $-\ \zeta_\alpha$(A)) ⊆ (A $-\ \zeta_\alpha(A)$) ∩ $\zeta_\alpha(A) = \phi$. Therefore by (iii), A $-\ \zeta_\alpha(A) \in \boldsymbol{I}$.

(iv) ⇒ (v). By (iv), for every A ⊆ X, A $-\ \zeta_\alpha(A) \in \boldsymbol{I}$. Let A $-\ \zeta_\alpha(A)$ = J ∈ $\boldsymbol{I}$, Then A = J ∪ (A ∩ $\zeta_\alpha(A)$). By Theorem 3.8 (ii) and by Theorem 3.7(v), $\zeta_\alpha(A) = \zeta_\alpha(J$ ∪ (A ∩ $\zeta_\alpha(A)$) = $\zeta_\alpha(J) \cup \zeta_\alpha$ (A ∩ $\zeta_\alpha(A)$) = ζ_α(A ∩ $\zeta_\alpha(A)$). Therefore, A ∩ $\zeta_\alpha(A)$ = A ∩ $\zeta_\alpha(A \cap \zeta_\alpha(A))$ and $A \cap \zeta_\alpha$(A) ⊆ A. By assumption, $A \cap \zeta_\alpha(A) = \phi$ and hence A = A $-\ \zeta_\alpha(A) \in \boldsymbol{I}$.

(v) ⇒ (i). Let A ⊆ X and assume that for every x ∈ A , there exist U ∈ $\tau(x)$ such that $cl_\alpha(U) \cap A \in \boldsymbol{I}$. Then $A \cap \zeta_\alpha(A)$ ⊆ A. Therefore by (v), $A \cap \zeta_\alpha(A) = \phi$. Suppose that A contains B such that $B \subseteq \zeta_\alpha(B)$. Then B = B ∩ $\zeta_\alpha(B)$ ⊆ $A \cap \zeta_\alpha(A) = \phi$. Therefore, A contains no non empty subset B with $B \subseteq \zeta_\alpha(B)$. Hence A ∈ $\boldsymbol{I}$.

Theorem 4.3. Let (X, τ, $\boldsymbol{I}$) be an ideal topological space. If τ is α-closure compatible with $\boldsymbol{I}$, then the following equivalent properties hold:

(i) For every A ⊆ X, $A \cap \zeta_\alpha(A) = \phi$ implies that $\zeta_\alpha(A) = \phi$.

(ii) For every A ⊆ X, $\zeta_\alpha\big(A - \zeta_\alpha(A)\big) = \phi$.

(iii) For every A ⊆ X , $\zeta_\alpha(A \cap \zeta_\alpha(A)) = \zeta_\alpha(A)$.

Proof. First we show that (i) holds if τ is α-closure compatible with $\boldsymbol{I}$. Let A ⊆ X and $A \cap \zeta_\alpha(A) = \phi$, by Theorem 4.2(iii), A ∈ $\boldsymbol{I}$ and by Theorem 3.7(v) $\zeta_\alpha(A) = \phi$.

(i) ⇒ (ii). Assume that for every A ⊆ X, $A \cap \zeta_\alpha(A) = \phi$ implies that $\zeta_\alpha(A) = \phi$. Let B = A – $\zeta_\alpha(A)$, then B ∩ $\zeta_\alpha(B)$ = $[A - \zeta_\alpha(A)] \cap \zeta_\alpha[A - \zeta_\alpha(A)]$ = (A ∩ $(X - \zeta_\alpha(A))$) ∩ $\zeta_\alpha(A \cap (X - \zeta_\alpha(A))) \subseteq \Big(\mathrm{A} \cap \big(X - \zeta_\alpha(A)\big)\Big) \cap (\zeta_\alpha(A) \cap \zeta_\alpha\big(X - \zeta_\alpha(A)\big)) = \phi$. Therefore by (i), we have $\zeta_\alpha(B) = \phi$. Hence $\zeta_\alpha(A - \zeta_\alpha(A)) = \phi$.

(ii) ⇒ (iii). Assume that, for every A ⊆ X , $\zeta_\alpha\big(A - \zeta_\alpha(A)\big) = \phi$. Since A = $(A - \zeta_\alpha(A))$ ∪ ($A \cap \zeta_\alpha(A)$). $\zeta_\alpha(A) = \zeta_\alpha\big(A - \zeta_\alpha(A)\big) \cup \zeta_\alpha\big(A \cap \zeta_\alpha(A)\big) = \zeta_\alpha(A \cap \zeta_\alpha(A))$.

(iii) ⇒ (i). Assume for every A ⊆ X, $\zeta_\alpha(A \cap \zeta_\alpha(A)) = \zeta_\alpha(A)$ if $\zeta_\alpha(A \cap \zeta_\alpha(A)) = \phi$ then $\zeta_\alpha(A) = \phi$.

Theorem 4.4. Let (X, τ, $\boldsymbol{I}$) be an ideal topological space, then the following properties are equivalent:

(i) $cl_{\alpha}(\tau) \cap \boldsymbol{I} = \phi$ where $cl_{\alpha}(\tau)$ = {cl_{α}(V) : V ∈ τ }.

(ii) If I ∈ $\boldsymbol{I}$, then $int_{\theta}(I) = \phi$.

(iii) For every clopen V, V ⊆ $\zeta_{\alpha}(V)$.

(iv) X = $\zeta_{\alpha}(X)$.

Proof. (i) ⇒ (ii). Let I ∈ $\boldsymbol{I}$. Suppose that x ∈ $int_{\theta}(I)$ then there exist U ∈ τ such that x $\in U \subseteq cl_{\alpha}(U) \subseteq$ cl(U) ⊆ I. Since I ∈ $\boldsymbol{I}$ and hence ϕ ≠ {x} $\subseteq cl_{\alpha}(U) \in cl_{\alpha}(\tau) \cap \boldsymbol{I}$. This is contrary to $cl_{\alpha}(\tau) \cap \boldsymbol{I} = \phi$. Therefore, $int_{\theta}(I) = \phi$

(ii) ⇒ (iii). Let x ∈ V. Assume that, x $\notin \zeta_{\alpha}(V)$, then there exist U ∈ $\tau(x)$ such that, V ∩ cl_{α}(U) ∈ $\boldsymbol{I}$ and hence V ∩ U ∈ $\boldsymbol{I}$.Since V is clopen, by (ii) and Lemma 3.6, x $\in V \cap$ U $= int(V \cap \mathrm{U}) \subseteq \mathrm{int}\big(V \cap cl_{\alpha}(\mathrm{U})\big) = int_{\theta}\big(V \cap cl_{\alpha}(\mathrm{U})\big) = \phi$. There is a contradiction. Hence x ∈ $\zeta_{\alpha}(V)$ and so V ⊆ $\zeta_{\alpha}(V)$.

(iii) ⇒ (iv). Obvious as X is clopen.

(iv) ⇒ (i). X = $\zeta_{\alpha}(X) =$ {x $\in X\!: cl_{\alpha}(U) \cap X = cl_{\alpha}(U) \notin \boldsymbol{I}$ for every open set U containing x}. Hence $cl_{\alpha}(U) \cap \boldsymbol{I} = \phi$.

5. $\boldsymbol{\Psi_{\zeta_{\alpha}}}$– OPERATOR

Definition 5.1. Let (X, τ, $\boldsymbol{I}$) be an ideal topological space. An operator $\Psi_{\zeta_{\alpha}}\!: P(X) \to \tau$ is defined as follows: for every A ⊆ X,$\Psi_{\zeta_{\alpha}}$ (A) = { x ∈ X : there exist U ∈ $\tau(x)$such that $cl_{\alpha}(U) - A \in \boldsymbol{I}$}. Note that, $\Psi_{\zeta_{\alpha}}$ (A) = $X - \zeta_{\alpha}(X - A)$.

Theorem 5.2. Let (X, τ, $\boldsymbol{I}$) be an ideal topological space. Then the following properties hold:

(i) If A ⊆ X, then $\Psi_{\zeta_{\alpha}}$ (A) is open.

(ii) If A ⊆ B, then $\Psi_{\zeta_{\alpha}}$ (A) ⊆ $\Psi_{\zeta_{\alpha}}$(B).

(iii) A, B ∈ P(X), then $\Psi_{\zeta_{\alpha}}$ (A ∩ B) = $\Psi_{\zeta_{\alpha}}$(A) ∩ $\Psi_{\zeta_{\alpha}}$(A).

(iv) If A ⊆ X, then $\Psi_{\zeta_{\alpha}}$(A) = $\Psi_{\zeta_{\alpha}}(\Psi_{\zeta_{\alpha}}$(A)) iff $\zeta_{\alpha}(X - A) = \zeta_{\alpha}(\zeta_{\alpha}(X - A))$.

(v) If A ∈ $\boldsymbol{I}$, then $\Psi_{\zeta_{\alpha}}$ (A) = X − $\zeta_{\alpha}(X)$.

(vi) If A ⊆ X, I ∈ $\boldsymbol{I}$, then $\Psi_{\zeta_{\alpha}}$ (A − I) = $\Psi_{\zeta_{\alpha}}$(A).

(vii) If A ⊆ X, I ∈ $\boldsymbol{I}$, then $\Psi_{\zeta_{\alpha}}$ (A ∪ I) = $\Psi_{\zeta_{\alpha}}$(A).

(viii)If (A − B) ∪ (B − A) ∈ $\boldsymbol{I}$, then $\Psi_{\zeta_{\alpha}}$ (A) = $\Psi_{\zeta_{\alpha}}$(B).

Proof. (i) Proof is obvious by Definition.

(ii) Let A ⊆ B and x ∈ $\Psi_{\zeta_{\alpha}}(A)$ then there exists U ∈ $\tau(x)$ such that cl(U) − A ∈ $\boldsymbol{I}$. Since A ⊆ B which implies $cl_{\alpha}(U) - B \in \boldsymbol{I}$, for $U \in \tau(x)$. Therefore, $x \in \Psi_{\zeta_{\alpha}}(B)$.

(iii) Let A, B ∈ P(X) then $\Psi_{\zeta_\alpha}(A \cap B)$ = X $- \zeta_\alpha(X-(A\cap B))$ = X – $\zeta_\alpha((X-A) \cup (X-B))$ = [X – ζ_α $(X-A)$] ∩ [X– $\zeta_\alpha(X-B)$] = Ψ_{ζ_α} (A) ∩ Ψ_{ζ_α}(B).

(iv)If A⊆ X and assume ζ_α(X− A) $=\zeta_\alpha(\zeta_\alpha(X-A))$. We have Ψ_{ζ_α}(A) = X $-\zeta_\alpha(X-A)$. *Now,* $\Psi_{\zeta_\alpha}\left(\Psi_{\zeta_\alpha}(A)\right) = \Psi_{\zeta_\alpha}(X-\zeta_\alpha(X-A)) = X-\zeta_\alpha(X-(X-\zeta_\alpha(X-A))) = X-\zeta_\alpha(\zeta_\alpha(X-A)) = X-\zeta_\alpha(X-A) = \Psi_{\zeta_\alpha}$(A). Conversely, assume that, Ψ_{ζ_α}(A) = $\Psi_{\zeta_\alpha}(\Psi_{\zeta_\alpha}$(A)). Then, X $-\zeta_\alpha(X-A) = X-\zeta_\alpha\left(X-(X-\zeta_\alpha(X-A))\right)$. Therefore, $\zeta_\alpha(X-A)=\zeta_\alpha(\zeta_\alpha(X-A))$.

(v) If A ∈ I than Ψ_{ζ_α} (A) = X $-(X-A)$ = X $-\zeta_\alpha(X)$ by corallory 3.10.

(vi) If A ⊆ X and I ∈ ***I*** then, Ψ_{ζ_α} (A − ***I***) = X – $\zeta_\alpha(X-(A-I))$ = X $-\zeta_\alpha((X-A) \cup$ ***I***)) = X – $[\zeta_\alpha(X-A)\cup\zeta_\alpha(I))]$ = X $-\zeta_\alpha(X-A)\cup\ \phi$ = X $-\zeta_\alpha(X-A)=\Psi_{\zeta_\alpha}(A)$ by Theorem 3.7(v).

(vii) If A ⊆ X, I ∈ ***I***, then Ψ_{ζ_α}(A ∪ ***I***) = X – $\zeta_\alpha(X-(A\cup$ ***I***)) = X– $\zeta_\alpha((X-A)-I)$ = X – $\zeta_\alpha(X-A)$ = Ψ_{ζ_α}(A), by Corollary 3.10.

(viii)Assume that, (A − B) ∪ $(B-A)$ ∈ ***I***. Let A − B = I and B − A = J. Observe that I, J ∈ ***I*** by heredity. Also observe that, B = (A − I) ∪ J . Thus Ψ_{ζ_α}(A) = Ψ_{ζ_α}(A − I) = Ψ_{ζ_α}((A − I) ∪ J) = Ψ_{ζ_α} (B), by Corollary 3.10.

Corollary 5.3. Let (X, τ, ***I***) be an ideal topological space. Then U ⊆ Ψ_{ζ_α}(U) for every θ-open set U ⊆ X.

Proof. We know that, Ψ_{ζ_α} (U) = X $-(X-U)$. Now ζ_α(X − U) ⊆ cl_θ(X − U) = X − U, since X − U is θ-closed. Therefore, U = X − (X − U) ⊆ X $-\zeta_\alpha(X-U)=\Psi_{\zeta_\alpha}$(U).

Theorem 5.4. Let (X,τ,***I***) be an ideal topological space and A⊆ X, then the following holds:

(i) Ψ_{ζ_α}(A) = ∪{U ∈ τ: $cl(U)-A\in$ ***I***}.

(ii) Ψ_{ζ_α}(A) ⊇ ∪{U ∈ τ: $(cl_\alpha(U)-A)\cup(A-cl_\alpha(U))\in$ ***I***}.

Proof. (i)This follows from the definition of Ψ_{ζ_α} −operator.

(ii)Since ***I*** is heredity, it is obvious that ∪{U ∈ τ: $(cl_\alpha(U)-A)\cup(A-cl_\alpha(U))\in$ ***I***} ⊆ {U ∈ τ: $(cl_\alpha(U)-A)\in$ ***I***} = $\Psi_{\zeta_\alpha}(A)$, for every A ⊆ X.

Theorem 5.5. Let (X, τ, ***I***) be an ideal topological space. If $\alpha_0=\{A\subseteq X : A\subseteq \Psi_{\zeta_\alpha}(A)\}$ then $\alpha_0$is a topology for X.

Proof. Let $\alpha_0=\{A\subseteq X : A\subseteq\Psi_{\zeta_\alpha}(A)\}$. Since ϕ ∈ ***I*** by Theorem 3.7(v), $\zeta_\alpha(\phi)=\phi$and $\Psi_{\zeta_\alpha}(X)=X-\zeta_\alpha(X-X)=X-\zeta_\alpha(\phi)=X$. Also, $\Psi_{\zeta_\alpha}(\phi)=X-\zeta_\alpha(X-\phi)=X-X=\phi$. Therefore, we obtained ϕ ⊆ $\Psi_{\zeta_\alpha}(\phi)$ and X ⊆ $\Psi_{\zeta_\alpha}(X)$. Thus ϕ, X ∈ α_0. Now, if A, B ∈ α_0 then by Theorem 5.2(iii), $A\cap B\subseteq\Psi_{\zeta_\alpha}(A)\cap\Psi_{\zeta_\alpha}(B)=\Psi_{\zeta_\alpha}(A\cap B)$ which implies $A\cap B\in\alpha_0$. If $\{A_\alpha:\alpha\in\Delta\}\subseteq\alpha_0$ then $A_\alpha\subseteq\Psi_{\zeta_\alpha}(A_\alpha)\subseteq$

$\Psi_{\zeta_\alpha}(\cup A_\alpha)$ for all α, and hence $\cup A_\alpha \subseteq \Psi_{\zeta_\alpha}(\cup A_\alpha)$. Therefore, α_0 is a topology.

Theorem 5.6. Let (X, τ, ***I***) be an ideal topological space then $\tau \sim \zeta_\alpha \boldsymbol{I}$ iffΨ_{ζ_α}(A) – A ∈ ***I*** for every A ⊆ X.

Proof. Necessity. Assume $\tau \sim \zeta_\alpha \boldsymbol{I}$ and let A ⊆ X observe that x ∈ Ψ_{ζ_α}(A) − A if and only if x ∉ A and x $\notin \zeta_\alpha(X - A)$ if and only if x ∉ A and there exists $U \in \tau(x)$ such that $cl_\alpha(U) - A \in \boldsymbol{I}$ If and only if $U \in \tau(x)$ such that $x \in cl_\alpha(U) - A \in \boldsymbol{I}$. Now, for each $x \in \Psi_{\zeta_\alpha}$(A)−A and $U \in \tau(x)$, $cl_\alpha(U) \cap \Psi_{\zeta_\alpha}(A) - A \in \boldsymbol{I}$ by heredity and hence $\Psi_{\zeta_\alpha}(A) - A \in \boldsymbol{I}$, by assumption that $\tau \sim \zeta_\alpha \boldsymbol{I}$.

Sufficiency. Let A ⊆ X and assume that x ∈ A there exists $U \in \tau(x)$ such that $cl_\alpha(U) \cap A \in \boldsymbol{I}$. Observe that $\Psi_{\zeta_\alpha}(X - A) - (X - A) = A - \zeta_\alpha(A)$ = {x: there exists $U \in \tau(x)$ such that $x \in cl_\alpha(U) \cap A \in \boldsymbol{I}$}. Therefore we have, A ⊆ Ψ_{ζ_α}(X − A) ∈ ***I***and hence $A \in \boldsymbol{I}$, since ***I*** is heredity.

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