Generalized S-modules

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Abstract
Let $R$ be a non-necessarily commutative ring with unity $1 \neq 0$ and $M$ a unitary left $R$-module. An $R$-module $M$ is said to be generalized hopfian if every surjective $R$-endomorphism of $M$ is superfluous. It is well known that any noetherian module is generalized hopfian but converse is not always true. For instance the $\mathbb{Z}$-module $\mathbb{Q}$ of rational numbers is generalized hopfian but it is not noetherian. For a fixed ring $R$, we study $R$-modules for which every generalized hopfian module of $\sigma[M]$ is noetherian. Such modules are said to be generalized $S$-modules. In this paper, some properties and important characterizations of generalized $S$-modules are given.

Keyword: hopfian module, generalized hopfian module, generalized $S$-module

Introduction
Let $R$ be a non-necessarily commutative ring with unity $1 \neq 0$ and $M$ a left module over $R$. Let $M$ and $N$ be two objects of $R$-Mod. We say that $N$ is generated by $M$ if there are a set $\Lambda$ and a surjective homomorphism $\phi : M^{(\Lambda)} \rightarrow N$. A submodule of $N$ is said to be subgenerated by $M$. The set of all $R$-modules subgenerated by $M$ constitutes the category $\sigma[M]$. It’s a full subcategory of $R$-Mod.
A submodule $N$ of $M$ is called superfluous in $M$ if every submodule $L$ of $M$, the relation $N + L = M$ implies $L = M$. An epimorphism $f : M \to N$ is said to be superfluous if $\ker f$ is superfluous. A module $N$ is said to be generalized hopfian if every surjective $R$-endomorphism of $N$ has a superfluous kernel. We know that every noetherian module is generalized hopfian but the inverse is not true in general. For example the $\mathbb{Z}$-module $\mathbb{Q}$ of rational numbers is generalized hopfian but it is not noetherian. The goal of this work is, for a fixed ring $R$, to find the $R$-modules $M$ for which every generalized hopfian module in $\sigma[M]$ is noetherian. These modules are said to be generalized $S$-modules.

A projective, finitely generated and generator object of $\sigma[M]$ is said to be progenerator. An $R$-module $N$ is hopfian if every surjective $R$-endomorphism of $N$ is an automorphism. $M$ is called $S$-module if every hopfian object of $\sigma[M]$ is noetherian. A module $M$ is said to be locally noetherian (resp. locally of finite length) if every finitely generated submodule of $M$ is noetherian (resp. of finite length). An $R$-module $N$ is called uniserial if its submodules are linearly ordered by inclusion. A module is called serial if it is a direct sum of uniserial modules. A $M$ is of serial representation type if every object of $\sigma[M]$ is serial. A $M$ is serial type if every object of $\sigma[M]$ is direct sum of uniserial modules of finite length. An $R$-module $M$ is said to have dual Goldie dimension, if there exist nonzero submodules $N_1, \ldots, N_k$ and a surjection $M \to \prod_{i=1}^{k} N_i$. An $R$-module $M$ is said to be Quasi-noetherian module if for every ascending chain $N_1 \subseteq \ldots \subseteq N_n \subseteq \ldots$ of $R$-submodules of $M$, there exists $m \in \mathbb{Z}^+$ such that $R^m(\bigcup_{n} N_n) \subseteq N_m$.

**SOME PROPERTIES OF GENERALIZED $S$-MODULES**

**Proposition 1:** Let $M$ be a left $R$-module.

1. If $M$ is a generalized $S$-module, then so is every submodule of $M$.

2. If a module $M$ is a generalized $S$-module, then every homomorphic image of $M$ is a generalized $S$-module.

3. Let $M = \prod_{i \in I} M_i$ be a direct product of module $M_i$ with $\sigma[M_i] \cap \sigma[M_j] = 0$ for every $i \neq j$, then $M$ is a generalized $S$-module if and only if $I$ is finite and $M_i$ is a generalized $S$-module.
**Proof**

1) Let $N$ be a submodule of $M$. In particular, $M \in \sigma[M]$. Since $\sigma[M]$ is closed under submodule, then $N \in \sigma[M]$. The category $\sigma[N]$ becomes a full subcategory of $\sigma[M]$. Let $K$ be an object of $\sigma[N]$. $K$ is also an element of $\sigma[M]$. If $K$ is generalized hopfian, then $K$ is noetherian. Thus $N$ is a generalized $S$-module.

2) Let $f: M \to M'$ be a homomorphic image of $M$. That implies $M'$ is generated by $M$. Hence, $M' \in \sigma[M]$ by referring to 1) $M'$ is a generalized $S$-module.

3) Assume $M = \prod_{i \in I} M_i$ be a generalized $S$-module. Let $\pi_j : M \to M_j$ for all $j \in I$ be a surjective homomorphism, then by referring to 2) $M_j$ is a generalized $S$-module for all $j \in I$. Now, let’s suppose that $M_i$ is a generalized $S$-module and $I$ is finite. Let $N$ be a generalized hopfian object of $\sigma[M]$. Let $g: \prod_{i \in I} M_i \to \bigoplus_{i \in I} M_i$ be a homomorphism

Since $I$ is finite, then :

$$\prod_{i \in I} M_i \cong \bigoplus_{i \in I} M_i$$

Then $N \in \sigma[\bigoplus_{i \in I} M_i]$. As $\sigma[M_i] \cap \sigma[M_j] = 0$ for any $i \neq j$, then by referring to [6], $N = \bigoplus_{i \in I} N_i$ with $N_i \in \sigma[M_i]$. Since $N$ is generalized hopfian, then $N_i$ is generalized hopfian for each $i \in I$. As $M_i$ is generalized $S$-module, then $N_i$ is noetherian. Therefore $N$ is noetherian. Hence, $M$ is generalized $S$-module.

**Lemma 1**

Let $M$ be a left $R$-module. If $M$ is hopfian module, then $M$ is generalized hopfian module.

**Proof**

Let $f$ be a surjective endomorphism of $M$. Since $M$ is hopfian, then $Ker f = 0$.

$$Ker f + N = 0 + N = M \Rightarrow N = M.$$

Thus $M$ is generalized hopfian module.

**Proposition 2:** Let $M$ be a left $R$-module. If $M$ is a generalized $S$-module, then $M$ is a $S$-module.

**Proof**

Let $K$ be a hopfian object of $\sigma[M]$, then $K$ is a generalized hopfian object of $\sigma[M]$. Since $M$ is a generalized $S$-module, then $K$ is noetherian.
Thus $M$ is a $S$-module.

**Proposition 3:**
Let $M$ be a left $R$-module. If $M$ is a generalized $S$-module, then every indecomposable projective object of $\sigma[M]$ is noetherian.

**Proof**
Let $N$ be a projective object of $\sigma[M]$ and $f$ a surjective endomorphism of $N$. We assume the following exact sequence:

$$0 \rightarrow \ker f \rightarrow N \rightarrow N \rightarrow 0$$

We have $N = \ker f \bigoplus N$. As $N$ is indecomposable, then $\ker f = 0$. Therefore $N$ is generalized hopfian. That implies $N$ is noetherian.

**Proposition 4:**
Let $M$ be a left $R$-module. If $M$ is a generalized $S$-module, then the projective cover of every simple object of $\sigma[M]$, if it exists, is noetherian.

**Proof**
Let $P$ be a simple object of $\sigma[M]$ with projective cover $\bar{P}$. To show that the projective cover is noetherian, we have to prove that $\bar{P}$ is indecomposable. Let $P_1$ and $P_2$ be two submodules of $\bar{P}$. We suppose $\bar{P} = P_1 \bigoplus P_2$ and a surjective homomorphism $f : \bar{P} \rightarrow P$ such that $\ker f$ is superfluous in $\bar{P}$. Let $f_1 \neq 0$ be the restriction of $f$ on $P_1$. As $P$ is simple, then $f_1$ is surjective. Then $P_2 \subseteq \ker f$ that implies $P_2 = 0$. Therefore $\bar{P}$ is indecomposable. It results from proposition 3 that $\bar{P}$ is noetherian.

**Proposition 5:**
Let $M$ be a $R$-module. If $M$ is a generalized $S$-module, then there exists a finite number of non isomorphic simple modules in $\sigma[M]$.

**Proof** Let $(N_j)_{j \in J}$ be a complete system of non-isomorphic class of simple objects of $\sigma[M]$. Assume $N = \oplus_{j \in J} N_j$. Since $N$ is hopfian, then $N$ is generalized hopfian, therefore $N$ is noetherian. Hence $J$ is finite.

**Lemma 2**
If $M$ is a direct sum of an infinite countable family $(M_n)_{n \in \mathbb{N}}$ of submodules of $M$ such that any two of them are isomorphic, then $M$ is not generalized hopfian module.
Proof
It results from proposition 2.3 of [4].

Lemma 3
A direct summand of a generalized hopfian module is a generalized hopfian module.

Proof
Let $M$ be a module and $N$ a direct summand of $M$. We can write $M = N \oplus K$ where $K$ is a submodule of $M$. If $M$ is a generalized hopfian module and $f$ a surjective endomorphism of $N$, then $\phi : M = N \oplus K \rightarrow M = N \oplus K$ such that $\phi(n + k) = f(n) + k$. $\phi$ is a surjective endomorphism of $M$. Therefore, $\text{Ker} \phi = \text{Ker} f$ is superfluous in $M$. If $L$ is a submodule of $N$ such that $\text{ker} f + L = N$, then $M = \text{ker} f + L + K$ and consequently $M = L + K$. Then $N = N \cap M = N \cap (L + K) = L + N \cap K = L + 0 = L$. Thus $N$ is a generalized hopfian module.

Proposition 6:
Let $R$ be a commutative ring and $M$ a module over $R$. If $M$ is a generalized $S$-module, then $M$ is locally noetherian.

Proof
Let $L$ be a finitely generated object of $\sigma[M]$. Then every finitely generated module over a commutative ring is Hopfian module. Then $L$ is Hopfian module. By referring to Lemma 1, $L$ is generalized Hopfian module. Thus $L$ is noetherian. Then $M$ is locally noetherian.

Proposition 7:
Let $R$ be a commutative ring and $M$ a semisimple module. If $M$ is a generalized $S$-module then, every object of $\sigma[M]$ is noetherian.

Proof
Let $N$ be a object of $\sigma[M]$. Since $M$ is semisimple module, therefore $N$ is semisimple module and $N = \oplus_{i \in I} N_i$ with $N_i$ simple module. As $N_i$ is Hopfian module and fully invariant, then $N$ is Hopfian module. Then $N$ is generalized Hopfian module. Thus $N$ is noetherian.

Proposition 8:
Let $R$ be a ring and $M$ a generalized $S$-module. Then every object $N$ of finite corank in $\sigma[M]$ is noetherian.
Proof
Let \( f: \mathbb{N} \to \mathbb{N} \) be an epimorphism, and suppose that \( \operatorname{corank}N = k \). There exist nonzero modules \( N_i \) and a surjection \( \phi: N \to \prod_{i=1}^{k} N_i \) such that \( \ker(\phi) \ll N \). Since \( g = \phi f \) is an epimorphism, then we have \( \ker(g) \ll N \).

Let \( n \in \ker(f) \), then \( g(n) = (\phi)(f(n)) = \phi(0) = 0 \), \( n \in \ker(g) \) and \( \ker(f) \subseteq \ker(g) \ll N \). Hence \( \ker(f) \ll N \). Then \( N \) is noetherian.

**Proposition 9:**
Let \( R \) be a ring and \( M \) a generalized \( S \)-module. Then every generalized hopfian object of \( \sigma[M] \) is Quasi-noetherian module.

**Proof**
Let \( N \) be a generalised Hopfian object of \( \sigma[M] \). Since \( M \) is a generalized \( S \)-module, then \( N \) is Noetherian. By referring to [8] that \( N \) is quasi-noetherian.

**Proposition 10:**
Let \( R \) be a ring and \( M \) a semisimple module.
If \( M \) is a generalized \( S \)-module, then the following conditions are verified:

1. every object of \( \sigma[M] \) is finite length;
2. \( M \) is serial type.

**Proof**
1)Let \( N \) be an object of \( \sigma[M] \). As \( M \) is semisimple module, then \( N = \bigoplus_{i \in I} N_i \) where \( N_i \) is a simple module. For every \( i \in I \), \( N_i \) is hopfian module and fully invariant. Therefore \( N \) is hopfian module, then \( N \) is generalized hopfian module. Since \( M \) is a generalized \( S \)-module, then \( N \) is noetherian. Since \( N \) is semisimple module, then \( N \) is finite length.
2)Let \( N \) be a direct sum of simple module \( N_i \). Then \( N_i \) is uniserial and finite length for every \( i \in I \). Thus \( M \) is serial type.

**MAIN RESULTS**

**Theorem 1:** Let \( R \) be a ring whose left ideal and the right ideal are two-sided. Let \( M \) be a module over \( R \). We suppose that \( \sigma[M] \) has a progenerator. Then the following conditions are equivalent:

1. \( M \) is a generalized \( S \)-module;
2. $M$ is a $S$-module;

3. every object of $\sigma[M]$ is direct sum of cyclic submodules.

**Proof**

1 $\Rightarrow$ 2 Assume $M$ is a generalized $S$-module. By referring to proposition 2, $M$ is a $S$-module.

2 $\Rightarrow$ 3 Let $P$ be a progenerator of $\sigma[M]$ and $x_i \in P$ for every $i \in I$, $\sigma[Rx_i]$ is a subcategory of $\sigma[M]$. Let $K$ be a hopfian object of $\sigma[Rx_i]$, then $K$ is a hopfian object of $\sigma[M]$. Since $M$ is a $S$-module, then $K$ is noetherian. Therefore $Rx_i$ is a $S$-module. By referring to [8], $\sigma[Rx_i] = R/\text{Ann}(x_i)$-Mod, then $R/\text{Ann}(x_i)$ is a $S$-ring. Hence $P = \bigoplus_{i=1}^{n} Rx_i$ with $Rx_i$ simple and for all $i \neq k$ $Rx_i$ and $Rx_k$ non isomorphic. Let $N \in \sigma[P]$, $N = \bigoplus_{i=1}^{n} N_i$ with $N_i \in R/\text{Ann}(x_i)$-Mod (following[7]). Since $\sigma[P] \subseteq \sigma[M]$ then $N \in \sigma[M]$. Since $Rx_i$ is a $S$-module for all $i$, then $P$ is a $S$-module. Therefore the quotient ring $R/\text{Ann}(x_i)$ is a $S$-ring. Following [3] theorem 2 $N_i$ is direct sum of cyclic modules.

3 $\Rightarrow$ 1 Let $L = \bigoplus_{i \in I} L_i$ be a generalized hopfian object of $\sigma[M]$ which is direct sum of cyclic submodules $L_i$. If $L$ is not noetherian, then all the class of isomorphic of the cyclic modules of $\sigma[M]$ is finite, there exists an infinite countable family $L_n$ of the family $L_i$, constituted of cyclic submodules of $L$ such that any two of them are isomorphic. Then

$$L = Q \oplus N \text{ where } N = \bigoplus_{n \in \mathbb{N}} L_n.$$ 

By referring to Lemma 2, $N$ is not generalized hopfian module. Since $N$ is a summand direct by referring to Lemma 3, $L$ is generalized hopfian module which is a contradiction.

**Theorem 2:**

Let $R$ be a ring and $M$ a semisimple $R$-module.

We assume that $\sigma[M]$ has a finite number of uniserial modules. Then the following conditions are equivalent:

1. $M$ is a generalized $S$-module;

2. $M$ is of serial representation type and of finite length.

**Proof**

1 $\Rightarrow$ 2 Let $N$ be an object of $\sigma[M]$. Since $M$ is semisimple module, then $N$ is
semisimple module and $N = \bigoplus N_i$ where $N_i$ is simple. As $N_i$ uniserial, then $N$ is serial. Thus $M$ is of serial representation type. $M$ is semisimple module, then $M$ is hopfian module. Thus $M$ is generalized hopfian module. As $M$ is a generalized S-module, then $M$ is noetherian. By referring to [4] corollary 10.16 that $M$ is finite length.

2 $\Rightarrow$ 1 Let $N$ be a generalized hopfian object of $\sigma[M]$. Since $M$ is of serial representation type, then $N$ is serial. $N = \bigoplus_{i \in I} N_i$ where $N_i$ are uniserial. As $\sigma[M]$ has a finite number of uniserial module, then $N$ is finite length. Therefore $N$ is noetherian.

Theorem 3:
Let $R$ be a commutative ring and $M$ a regular $R$-module. We suppose that $\sigma[M]$ has a finite number of simple $R$-module. Then the following conditions are equivalent:

1. $M$ is a generalized $S$-module;

2. $M$ is locally noetherian module.

Proof
1 $\Rightarrow$ 2 It results from proposition 6 that $M$ is locally noetherian.

2 $\Rightarrow$ 1. Let $N$ be a generalized hopfian object $\sigma[M]$. $M$ is regular module and By the condition 2 $M$ is locally noetherian module. By referring [8] 37 that $M$ is semisimple module. Since $M$ is semisimple module, then $N$ is semisimple module. $N = \bigoplus_{i \in I} N_i$ where $N_i$ are simple module. Since $\sigma[M]$ has a finite number of simples module, then $N$ is of finite length. Then $N$ is noetherian, therefore $M$ is a generalized S-module.

REFERENCES


