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Generalized S-modules

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Abstract

Let R be a non-necessarily commutative ring with unity $1 \neq 0$ and M a unitary left R-module. An R-module M is said to be generalized hopfian if every surjective R-endomorphism of M is superfluous. It is well known that any noetherian module is generalized hopfian but converse is not always true. For instance the $\mathbb{Z}$-module $\mathbb{Q}$ of rational numbers is generalized hopfian but it is not noetherian. For a fixed ring R, we study R-modules for which every generalized hopfian module of $\sigma[M]$ is noetherian. Such modules are said to be generalized S-modules. In this paper, some properties and important characterizations of generalized S-modules are given.
Keyword: hopfian module, generalized hopfian module, generalized S-module

Introduction

Let R be a non-necessarily commutative ring with unity $1 \neq 0$ and M a left module over R. Let M and N be two objects of R-Mod. We say that N is generated by M if there are a set Λ and a surjective homomorphism $\phi : M^{(\Lambda)} \to N$. A submodule of N is said to be subgenerated by M. The set of all R-modules subgenerated by M constitutes the category $\sigma[M]$. It's a full subcategory of R-Mod.

A submodule N of M is called superfluous in M if every submodule L of M, the relation $N + L = M$ implies $L = M$. An epimorphism $f : M \longrightarrow N$ is said to be superfluous if $\ker f$ is superfluous. A module N is said to be generalized hopfian if every surjective R-endomorphism of N has a superfluous kernel. We know that every noetherian module is generalized hopfian but the inverse is not true in general. For example the $\mathbb{Z}$-module $\mathbb{Q}$ of rational numbers is generalized hopfian but it is not noetherian. The goal of this work is, for a fixed ring R, to find the R-modules M for which every generalized hopfian module in $\sigma[M]$ is noetherian. These modules are said to be generalized S-modules.

A projective, finitely generated and generator object of $\sigma[M]$ is said to be progenerator. An R-module N is hopfian if every surjective R-endomorphism of N is an automorphism. M is called S-module if every hopfian object of $\sigma[M]$ is noetherian. A module M is said to be locally noetherian (resp. locally of finite length) if every finitely generated submodule of M is noetherian (resp. of finite length). An R-module N is called uniserial if its submodules are linearly ordered by inclusion. A module is called serial if it is a direct sum of uniserial modules. A M is of serial representation type if every object of $\sigma[M]$ is serial. A M is serial type if every object of $\sigma[M]$ is direct sum of uniserial modules of finite length. An R-module M is said to have dual Goldie dimension, if there exist nonzero submodules $N_1, ..., N_k$ and a surjection $M \to \prod_{i=1}^{k} N_i$. An R-module M is said to be Quasi-noetherian module if for every ascending chain $N_1 \subseteq ..... \subseteq N_n \subseteq ...$ of R-submodules of M, there exists $m \in \mathbb{Z}^+$ such that $R^m(\cup_n N_n) \subseteq N_m$.

SOME PROPERTIES OF GENERALIZED S-MODULES

Proposition 1: Let M be a left R-module.

1. If M is a generalized S-module, then so is every submodule of M.
2. If a module M is a generalized S-module, then every homomorphic image of M is a generalized S-module.
3. Let $M = \prod_{i \in I} M_i$ be a direct product of module M_i with $\sigma[M_i] \cap \sigma[M_j] = 0$ for every $i \neq j$, then M is a generalized S-module if and only if I is finite and M_i is a generalized S-module.

Proof

1) Let N be a submodule of M. In particular, $M \in \sigma[M]$. Since $\sigma[M]$ is closed under submodule, then $N \in \sigma[M]$. The category $\sigma[N]$ becomes a full subcategory of $\sigma[M]$. Let K be an object of $\sigma[N]$. K is also an element of $\sigma[M]$. If K is generalized hopfian, then K is noetherian. Thus N is a generalized S-module.

2) Let f:$M \to M'$ be a homomorphic image of M. That implies M' is generated by M. Hence, $M' \in \sigma[M]$ by referring to 1) M' is a generalized S-module.

3)Assume $M = \prod_{i \in I} M_i$ be a generalized S-module. Let $\pi_j : M \to M_j$ for all $j \in I$ be a surjective homomorphism, then by referring to 2) M_j is a generalized S-module for all $j \in I$. Now, let's suppose that M_i is a generalized S-module and I is finite. Let N be a generalized hopfian object of $\sigma[M]$. Let g: $\prod_{i \in I} M_i \longrightarrow \bigoplus_{i \in I} M_i$ be a homomorphism

Since I is finite, then :

$$\prod_{i \in I} M_i \cong \bigoplus_{i \in I} M_i$$

Then $N \in \sigma[\bigoplus_{i \in I} M_i]$. As $\sigma[M_i] \cap \sigma[M_j] = 0$ for any $i \neq j$, then by referring to [6], $N = \oplus_{i \in I} N_i$ with $N_i \in \sigma[M_i]$. Since N is generalized hopfian, then N_i is generalized hopfian for each $i \in I$. As M_i is generalized S-module, then N_i is noetherian. Therefore N is noetherian. Hence, M is generalized S-module.

Lemma 1

Let M be a left R-module. If M is hopfian module, then M is generalized hopfian module.

Proof

Let f be a surjective endomorphism of M.

Since M is hopfian, then $Kerf = 0$.

$$Kerf + N = 0 + N = M \Rightarrow N = M.$$

Thus M is generalized hopfian module.

Proposition 2: Let M be a left R-module. If M is a generalized S-module, then M is a S-module.

Proof

Let K be a hopfian object of $\sigma[M]$, then K is a generalized hopfian object of $\sigma[M]$. Since M is a generalized S-module, then K is noetherian.

Thus M is a S-module.

Proposition 3:
Let M be a left R-module. If M is a generalized S-module, then every indecomposable projective object of $\sigma[M]$ is noetherian.
Proof
Let N be a projective object of $\sigma[M]$ and f a surjective endomorphism of N. We assume the following exact sequence :

$$0 \to kerf \to N \to N \to 0$$

We have $N = kerf \bigoplus N$. As N is indecomposable, then $kerf = 0$.
Therefore N is generalized hopfian. That implies N is noetherian.

Proposition 4:
Let M be a left R-module. If M is a generalized S-module, then the projective cover of every simple object of $\sigma[M]$, if it exists, is noetherian.
Proof
Let P be a simple object of $\sigma[M]$ with projective cover $\bar{P}$. To show that the projective cover is noetherian, we have to prove that $\bar{P}$ is indecomposable. Let P_1 and P_2 be two submodules of $\bar{P}$. We suppose $\bar{P} = P_1 \bigoplus P_2$ and a surjective homomorphism $f : \bar{P} \to P$ such that kerf is superfluous in $\bar{P}$. Let $f_1 \neq 0$ be the restriction of f on P_1. As P is simple, then f_1 is surjective. Then $P_2 \subseteq kerf$ that implies $P_2 = 0$. Therefore $\bar{P}$ is indecomposable. It results from proposition 3 that $\bar{P}$ is noetherian.

Proposition 5:
Let M be a R-module. If M is a generalized S-module, then there exists a finite number of non isomorphic simple modules in $\sigma[M]$.
Proof Let $(N_j)_{j \in J}$ be a complete system of non-isomorphic class of simple objects of $\sigma[M]$. Assume $N = \oplus_{j \in J} N_j$. Since N is hopfian, then N is generalized hopfian, therefore N is noetherian. Hence J is finite.

Lemma 2
If M is a direct sum of an infinite countable family $(M_n)_{n \in N}$ of submodules of M such that any two of them are isomorphic, then M is not generalized hopfian module.

Proof

It results from proposition 2.3 of [4].

Lemma 3

A direct summand of a generalized hopfian module is a generalized hopfian module.

Proof

Let M be a module and N a direct summand of M. We can write $M = N \bigoplus K$ where K is a submodule of M. If M is a generalized hopfian module and f a surjective endomorphism of N, then $\phi : M = N \bigoplus K \rightarrow M = N \bigoplus K$ such that $\phi(n+k) = f(n)+k$. ϕ is a surjective endomorphism of M. Therefore, $Ker\phi = Kerf$ is superfluous in M. If L is a submodule of N such that $kerf + L = N$, then $M = kerf+L+K$ and consequently $M = L+K$. Then $N = N \cap M = N \cap (L+K) = L + N \cap K = L + 0 = L$. Thus N is a generalized hopfian module.

Proposition 6:

Let R be a commutative ring and M a module over R.

If M is a generalized S-module, then M is locally noetherian.

Proof

Let L be a finitely generated object of $\sigma[M]$. Then every finitely generated module over a commutative ring is Hopfian module. Then L is Hopfian module. By referring to Lemma 1, L is generalized Hopfian module. Thus L is noetherian. Then M is locally noetherian.

Proposition 7:

Let R be a commutative ring and M a semisimple module.

If M is a generalized S-module then, every object of $\sigma[M]$ is noetherian.

Proof

Let N be a object of $\sigma[M]$. Since M is semisimple module, therefore N is semisimple module and $N = \oplus_{i \in I} N_i$ with N_i simple module. As N_i is Hopfian module and fully invariant, then N is Hopfian module. Then N is generalized Hopfian module. Thus N is noetherian.

Proposition 8:

Let R be a ring and M a generalized S-module. Then every object N of finite corank in $\sigma[M]$ is noetherian.

Proof

Let f:$N \rightarrow N$ be an epimorphism, and suppose that $coranckN = k$. There exist nonzero modules N_i and a surjection $\phi : N \rightarrow \prod_{i=1}^{k} N_i$ such that $Ker(\phi) \ll N$. Since $g = \phi f$ is an epimorphism, then we have $Ker(g) \ll N$.

Let $n \in Ker(f)$, $g(n) = (\phi)(f(n)) = \phi(0) = 0$, $n \in Ker(g)$ and $Ker(f) \subseteq Ker(g) \ll N$. Hence $Ker(f) \ll N$. Then N is noetherian.

Proposition 9:

Let R be a ring and M a generalized S-module . Then every generalized hopfian object of $\sigma[M]$ is Quasi-noetherian module.

Proof

Let N be a generalised Hopfian object of $\sigma[M]$.

Since M is a generalized S-module, then N is Noetherian. By referring to [8] that N is quasi-noetherian.

Proposition 10:

Let R be a ring and M a semisimple module.

If M is a generalized S-module, then the following conditions are verified :

1. every object of $\sigma[M]$ is finite length;
2. M is serial type.

Proof

1)Let N be a object of $\sigma[M]$. As M is semisimple module, then $N = \oplus_{i \in I} N_i$ where N_i is a simple module. For every $i \in I$, N_i is hopfian module and fully invariant. Therefore N is hopfian module, then N is generalized hopfian module. Since M is a generalized S-module, then N is neotherian. Since N is semisimple module, then N is finite length.

2)Let N be a direct sum of simple module N_i. Then N_i is uniserial and finite length for every $i \in I$. Thus M is serial type.

MAIN RESULTS

Theorem 1: Let R be a ring whose left ideal and the right ideal are two-sided. Let M be a module over R. We suppose that $\sigma[M]$ has a progenerator. Then the following conditions are equivalent :

1. M is a generalized S-module;

2. M is a S-module;

3. every object of $\sigma[M]$ is direct sum of cyclic submodules.

Proof

$1 \Rightarrow 2$ Assume M is a generalized S-module. By referring to proposition 2, M is a S-module.

$2 \Rightarrow 3$ Let P be a progenerator of $\sigma[M]$ and $x_i \in P$ for every $i \in I$, $\sigma[Rx_i]$ is a subcategory of $\sigma[M]$. Let K be a hopfian object of $\sigma[Rx_i]$, then K is a hopfian object of $\sigma[M]$. Since M is a S-module, then K is noetherian. Therefore Rx_i is a S-module. By referring to [8], $\sigma[Rx_i] = R/Ann(x_i)$-Mod, then $R/Ann(x_i)$ is a S-ring. Hence $P = \oplus_{i=1}^{n} Rx_i$ with Rx_i simple and for all $i \neq k$ Rx_i and Rx_k non isomorphic. Let $N \in \sigma[P]$, $N = \oplus_{i=1}^{n} N_i$ with $N_i \in R/Ann(x_i)$-Mod (following[7]). Since $\sigma[P] \subset \sigma[M]$ then $N \in \sigma[M]$. Since Rx_i is a S-module for all i, then P is a S-module. Therefore the quotient ring $R/Ann(x_i)$ is a S-ring. Following [3] theorem 2 N_i is direct sum of cyclic modules.

$3 \Rightarrow 1$ Let $L = \bigoplus_{i \in I} L_i$ be a generalized hopfian object of $\sigma[M]$ which is direct sum of cyclic submodules L_i. If L is not noetherian, then all the class of isomorphic of the cyclic modules of $\sigma[M]$ is finite, there exists an infinite countable family L_n of the family L_i, constitued of cyclic submodules of L such that any two of them are isomorphic. Then

$L = Q \oplus N$ where $N = \oplus_{n \in \mathbf{N}} L_n$.

By referring to Lemma 2, N is not generalized hopfian module. Since N is a summand direct by referring to Lemma 3, L is generalized hopfian module which is a contradiction.

Theorem 2:

Let R be a ring and M a semisimple R-module.

We assume that $\sigma[M]$ has a finite number of uniserial modules. Then the following conditions are equivalent:

1. M is a generalized S-module;

2. M is of serial representation type and of finite length.

Proof

$1 \Rightarrow 2$ Let N be an object of $\sigma[M]$. Since M is semisimple module, then N is

semisimple module and $N = \oplus N_i$ where N_i is simple. As N_i uniserial, then N is serial. Thus M is of serial representation type. M is semisimple module, then M is hopfian module. Thus M is generalized hopfian module. As M is a generalized S-module, then M is noetherian. By referring to [4] corollary 10.16 that M is finite length.
$2 \Rightarrow 1$ Let N be a generalized hopfian object of $\sigma[M]$. Since M is of serial representation type, then N is seial. $N = \oplus_{i \in I} N_i$ where N_i are uniserial. As $\sigma[M]$ has a finite number of uniserial module, then N is finite length. Therefore N is noetherian.

Theorem 3:
Let R be a commutative ring and M a regular R-module.We suppose that $\sigma[M]$ has a finite number of simple R-module. Then the following conditions are equivalent :

1. M is a generalized S-module;
2. M is locally noetherian module.

Proof
$1 \Rightarrow 2$ It results from proposition 6 that M is locally noetherian.
$2 \Rightarrow 1$. Let N be a generalized hopfian object $\sigma[M]$. M is regular module and By the condition 2 M is locally noetherian module. By referring [8] 37 that M is semisimple module. Since M is semisimple module, then N is semisimple module. $N = \oplus_{i \in I} N_i$ where N_i are simple module. Since $\sigma[M]$ has a finite number of simples module, then N is of finite length. Then N is noetherian, therefore M is a generalized S-module.

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