Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 15, Number 6 (2019), pp. 875-888

http://www.ripublication.com/gjpam.htm

Characterizing Semigroups by Their Generalized $(\in_\gamma, \in_\gamma \vee q_\delta)$-Fuzzy Ideals

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Abstract

Generalizing the notions of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideals, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals of semigroups, the notions of Δ-fuzzy left ideals, Δ-fuzzy right ideals and Δ-fuzzy ideals are introduced. In particular, some related properties of the above algebraic structures are investigated. Moreover, characterizations of regular, intra-regular and semisimple semigroups are given in term of Generalized $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals

Keywords: regular semigroup; intra-regular semigroup; semisimple semigroup; Δ-fuzzy ideal; $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal

AMS Classification 2010: 20M12, 20M17, 03E72, 08A72

1. Introduction

Most of real world problems in almost all disciplines like in automata theory, artificial intelligence, robotics, engineering, physics, computer science, management sciences, and medical science are mostly full of complexities and consist of several types of uncertainties. To overcome these difficulties of uncertainties, Zadeh [1] introduced fuzzy set theory which has appropriate approach to deal with uncertainties. Since then, many notions of mathematics are extended to fuzzy sets, and various properties of these notions in the context of fuzzy sets are established. Rosenfeld [2] laid the foundations of fuzzy algebra in 1971. He introduced the notion of fuzzy subgroups (subgroupoids) of groups (groupoids). Kuroki [3, 4] initiated the study of fuzzy semigroups. The monograph by Mordeson et al. [5] concentrates on the theory of fuzzy semigroups and its applications in fuzzy coding theory, fuzzy finite state machines and fuzzy languages.

Bhakat and Das [6, 7] used the "belongs to" ($\in$) and "quasi-coincidence with" (q) relations, given in [8, 9], and defined $(\in, \in \vee q)$-fuzzy subgroups which are generalizations of Rosenfeld's fuzzy subgroups. Jun and Song in [10] proposed the study of (α, β)-fuzzy interior ideals in a semigroup. Shabir et al. [11] introduced concepts of $(\in, \in \vee q)$-fuzzy left (right) ideals, $(\in, \in \vee q)$-fuzzy ideals and $(\in, \in \vee q)$-fuzzy quasi-ideals of semigroups and characterized regular semigroups by using these fuzzy subsets.

Generalizing the idea of the concept of the quasi-coincidence" relation of a fuzzy point with a fuzzy set, Jun [12] defined $(\in, \in \vee q_k)$-fuzzy subalgebras in BCK/BCI-algebras. Shabir et al. [13] characterized regular and intra-regular semigroups by the properties of $(\in, \in \vee q_k)$-fuzzy left (right, quasi-, two-sided) ideals. Khan and Anis [14] characterized right weakly regular semigroups by the properties of their $(\in, \in \vee q_k)$-fuzzy ideals.

General concepts of fuzzy subsystems of semigroups are developed continuously, Shadir and Ali [15] introduced $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideal, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal of semigroups, and used properties of these fuzzy ideals to characterize regular, intra-regular and semisimple semigroups. It is natural to investigate similar type of generalizations of the existing fuzzy subsystems with other algebraic structures. Ma et al. [16] introduced $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals of BCI-algebras and characterized positive implicative, implicative and commutative BCI-algebras by these generalized fuzzy ideals. In [17], Zhan and Yin discussed relationship between prime $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals and strong prime $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals of near-rings. Rehman and Shabir [18, 19] introduced concepts of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideals, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideals of ternary semigroups and characterized ternary semigroups in terms of the properties of their $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals. The notions of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-ideals in hemirings was introduced in [20]. Huang et al. [21] initiated notions of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy hyperideals of semihyperrings, and characterized hyperregular and left duo semihyperrings by their $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy hyperideals.

In this paper, we discuss more general forms of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideals, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals of semigroups. We introduce the concepts of Δ-fuzzy left ideals, Δ-fuzzy right ideals and Δ-fuzzy ideals of semigroups and investigate some related properties. Regular, intra-regular and semisimple semigroups are characterized in terms of these fuzzy ideals. Taking $\Delta = (\gamma, \delta]$ in main results, we immediately obtain characterizations of such semigroups by their $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals.

2. PRELIMINARIES

In the following discussions, S always stands for a semigroup, γ and δ are two constant numbers such that $0 \le \gamma < \delta \le 1$, and Δ is a nonempty subset of the closed interval $[0,1]$ such that $|\Delta| > 1$, unless otherwise specified. A nonempty subset I of S is called an **idempotent** subset if $I^2 = I$. A nonempty subset I of S is called a **left (**resp., **right) ideal** of S if $SI \subseteq I$ (resp., $IS \subseteq I$). A nonempty subset of S is called a **two-sided ideal** (or simply an **ideal**) of S if it is both a left ideal and a right ideal of S. A semigroup S is called **regular** if for every $a \in S$ there exists $x \in S$ such that $a = axa$ and S is called **intra-regular** if for every $a \in S$ there exist $x, y \in S$ such that $a = xa^2y$. In general, neither a regular semigroup is intra-regular nor an intra-regular semigroup is regular [15]. A semigroup S is called **semisimple** if every ideal of S is idempotent. The following characterization theorems are well known in semigroups and used for this paper.

Theorem 2.1. *For a semigroup* S *the following conditions are equivalent.*

(i) S *is regular.*

(ii) $R \cap L = RL$ *for every right ideal* R *and every left ideal* L *of* S.

Theorem 2.2. *For a semigroup* S *the following conditions are equivalent.*

(i) S *is intra-regular.*

(ii) $R \cap L \subseteq LR$ *for every right ideal* R *and every left ideal* L *of* S.

Theorem 2.3. *A semigroup* S *is semisimple if and only if* $a \in (SaS)(SaS)$ *for every* $a \in S$.

A **fuzzy subset** [1] of S is described as a function: $f : S \to [0,1]$. Let f and g be fuzzy subsets of S. Then their products $f \wedge g$ and $f \circ g$ are fuzzy subsets of S defined by

$$(f \wedge g)(x) = \min\{f(x), g(x)\}$$

and

$$(f \circ g)(x) = \begin{cases} \sup\limits_{x=ab}\{\min\{f(a), g(b)\}\} & \text{if } x \text{ is expressible as } x = ab \text{ for some } a, b \in S \\ 0 & \text{otherwise,} \end{cases}$$

for all $x \in S$. We define the order relation "$\le$" on the set $F(S)$ of all fuzzy subsets of S as follows: $f \le g$ if and only if $f(x) \le g(x)$ for all $x \in S$. For subset A of S, the characteristic function C_A of A is the function $C_A : S \to \{0,1\}$ defined by for all

$x \in S$,

$$C_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 2.1. [5] *If A and B are subsets of a semigroup S, then $C_A \wedge C_B = C_{A \cap B}$ and $C_A \circ C_B = C_{AB}$.*

Definition 2.1. [15] Let f and g be fuzzy subsets of a semigroup S. The fuzzy subsets f^*, $f \wedge^* g$ and $f * g$ of S defined by for all $x \in S$,

$$f^*(x) = \max\{\min\{f(x), \delta\}, \gamma\},$$

$$(f \wedge^* g)(x) = \max\{\min\{(f \wedge g)(x), \delta\}, \gamma\},$$

and

$$(f * g)(x) = \max\{\min\{(f \circ g)(x), \delta\}, \gamma\}.$$

From Definition 2.1, we get $f \wedge^* g = f^* \wedge g^*$ and $f^* \circ g^* \le f * g$ [15]. If $S = S^2$, then

$$f^* \circ g^* = f * g.$$

Note that for every element x of S, we see that $f^*(x) \le g^*(x)$ if and only if $\min\{f(x), \delta\} \le \max\{g(x), \gamma\}$.

Next, we will give important tools to use for this paper.

Definition 2.2. Let f and g be fuzzy subsets of a semigroup S. We define the subset $S[f;\Delta]$ and the fuzzy subset f_Δ of S, and the relations "$\le_\Delta$" and "$=_\Delta$" on $F(S)$ as follows:

(i) $S[f;\Delta] = \{x \in S \mid f(x) \le \inf(\Delta), \text{or } f(x) \in \Delta, \text{or } f(x) \ge \sup(\Delta)\}$

(ii) $$f_\Delta(x) = \begin{cases} \max\{\min\{f(x), \sup(\Delta)\}, \inf(\Delta)\} & \text{if } x \in S[f;\Delta] \\ \inf(\Delta) & \text{otherwise,} \end{cases}$$

for all $x \in S$.

(iii) $f \le_\Delta g$ if and only if for all $x \in S$ if $x \in S[g;\Delta]$, then

$$\min\{f(x), \sup(\Delta)\} \le \max\{g(x), \inf(\Delta)\}$$

(iv) $f =_\Delta g$ if and only if for all $x \in S$ if $x \in S[g;\Delta]$, then $f_\Delta(x) = g_\Delta(x)$.

From Definition 2.2, we have the following proposition.

Proposition 2.2. *Let* f, g *and* h *be fuzzy subsets of a semigroup* S*, the following are true.*

(i) *If* $f \leq g$ *and* $g \leq_\Delta h$*, then* $f \leq_\Delta h$.

(ii) *If* $f = g$*, then* $f =_\Delta g$.

Proof. It is straightforward. □

For every subset A of a semigroup S, we see that for all $x \in S$,

$$(C_A)_\Delta(x) = \begin{cases} \sup(\Delta) & \text{if } x \in A \\ \inf(\Delta) & \text{otherwise,} \end{cases}$$

and so we obtain the following proposition.

Proposition 2.3. *If* A *and* B *are subsets of a semigroup* S*, then the following are equivalent.*

(i) $A = B$.

(ii) $C_A =_\Delta C_B$.

(iii) $C_B =_\Delta C_A$.

Proof. It is straightforward. □

If we take $\Delta = (\gamma, \delta]$ in Definition 2.2, we have the following proposition.

Proposition 2.4. Let f and g be fuzzy subsets of a semigroup S. Then the following are true.

(i) $f_{(\gamma,\delta]} = f^*$.

(ii) $f \leq_{(\gamma,\delta]} g$ if and only if $f^* \leq g^*$.

(iii) $f =_{(\gamma,\delta]} g$ if and only if $f^* = g^*$.

Proof. It is straightforward. □

3. Characterizing Semigroups by Their Generalized $(\in_\gamma, \in_\gamma \vee q_\delta)$-Fuzzy Ideals

In this section, we introduce a Δ-fuzzy left ideal, a Δ-fuzzy right ideal and a Δ-fuzzy ideal of a semigroup which are generalizations of an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal, an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal and an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal, respectively. It is shown that the definitions of a Δ-fuzzy left (resp., right, two-sided) ideal and a $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy (resp., right, two-sided) ideal of any semigroup coincide when $\Delta=(\gamma,\delta]$. Finally, we characterize regular, intra-regular and semisimple semigroups characterized by such fuzzy subsets

Definition 3.1. Let f be a fuzzy subset of a semigroup S, then

(i) f is called a Δ**-fuzzy left ideal** of S if it satisfies the following condition:

for all $x,y\in S$, if $y\in S[f;\Delta]$ or $xy\in S[f;\Delta]$, the $\max\{f(xy),\inf(\Delta)\}\geq\min\{f(y),\sup(\Delta)\}$.

(ii) f is called a Δ**-fuzzy right ideal** of S if it satisfies the following condition:

for all $x,y\in S$, if $x\in S[f;\Delta]$ or $xy\in S[f;\Delta]$, then

$$\max\{f(xy),\inf(\Delta)\}\geq\min\{f(x),\sup(\Delta)\}.$$

(iii) f is called a Δ**-fuzzy two-sided ideal** (or simply a Δ**-fuzzy ideal**) of S if it is both a Δ-fuzzy left ideal and a Δ-fuzzy right ideal of S.

Theorem 3.1. *If f is a Δ-fuzzy left (right) ideal of a semigroup S and $\Omega\subseteq\Delta$ such that $|\Omega|>1$ and $S[f;\Omega]\subseteq S[f;\Delta]$, then f is a Ω-fuzzy left (right) ideal of S.*

Proof. It is straightforward. □

If we take $\Delta=(\gamma,\delta]$ in Definition 3.1, then it is reduced to the following definition.

Definition 3.2. [15] Let f be a fuzzy subset of a semigroup S, then

(i) f is called an $(\in_\gamma,\in_\gamma\vee q_\delta)$**-fuzzy left ideal** of S if

$$\max\{f(xy),\gamma\}\geq\min\{f(y),\delta\} \text{ for all } x,y\in S.$$

(ii) f is called an $(\in_\gamma,\in_\gamma\vee q_\delta)$**-fuzzy right ideal** of S if

$$\max\{f(xy),\gamma\}\geq\min\{f(x),\delta\} \text{ for all } x,y\in S.$$

(iii) f is called an $(\in_\gamma,\in_\gamma\vee q_\delta)$**-fuzzy two-sided ideal** (or simply an $(\in_\gamma,\in_\gamma\vee q_\delta)$**-fuzzy ideal**) of S if it is both an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy left ideal and an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy right ideal of S.

In case: $\Delta\subseteq(\gamma,\delta]$ and using Proposition 3.1, we have that every $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy left (right) ideal of a semigroup S is a Δ-fuzzy left (right) ideal of S. However, the following example demonstrates that the converse is not true.

Example 3.1. Let $\mathbb{N}$ be the set of all positive integers. Then $\mathbb{N}$ is a semigroup under the usual multiplication. Define the fuzzy subset f of $\mathbb{N}$ as follows:

$$f(x) = \begin{cases} 0.2 \text{ if } x \in \{2n+1 \mid n \in \mathbb{N} \cup \{0\}\}, \\ 0.4 \text{ if } x \in \{2(2n+1) \mid n \in \mathbb{N} \cup \{0\}\}, \\ 0.3 \text{ if } x \in \{4(2n+1) \mid n \in \mathbb{N} \cup \{0\}\}, \\ 0.5 \text{ if } x \in \{8(2n+1) \mid n \in \mathbb{N} \cup \{0\}\}, \\ 0.9 \text{ if } x \in \{16(2n+1) \mid n \in \mathbb{N} \cup \{0\}\}, \\ 0.8 \text{ if } x \in \{32n \mid n \in \mathbb{N}\}, \end{cases}$$

for all $x \in \mathbb{N}$.

If $\Delta = (0.2, 0.8] \setminus \{0.3, 0.4\}$, then routine calculation show that f is a Δ-fuzzy ideal of $\mathbb{N}$ but not an $(\in_{0.2}, \in_{0.2} \vee q_{0.8})$-fuzzy ideal of $\mathbb{N}$ because

$$\max\{f(4), 0.2\} = 0.3 < 0.4 = \min\{f(2), 0.8\}.$$

Theorem 3.2. A nonempty subset A of a semigroup S is a left (right) ideal of S if and only if C_A is a Δ-fuzzy left (right) ideal of S.

Proof. It is straightforward. □

Theorem 3.3. *Let f be a fuzzy subset of a semigroup S. Then the following are true.*

(i) *If f is a Δ-fuzzy left ideal of S, then $C_S \circ f \leq_\Delta f$.*

(ii) *If f is a Δ-fuzzy right ideal of S, then $f \circ C_S \leq_\Delta f$.*

Proof. (i) Suppose that $\min\{(C_S \circ f)(x), \sup(\Delta)\} > \max\{f(x), \inf(\Delta)\}$ for some $x \in S[f; \Delta]$. Then there exist $a, b \in S$ such that $x = ab$ and

$$\min\{f(b), \sup(\Delta)\} > \max\{f(x), \inf(\Delta)\}.$$

Since f is a Δ-fuzzy left ideal of S, we have

$$\begin{aligned} \max\{f(x), \inf(\Delta)\} &= \max\{f(ab), \inf(\Delta)\} \\ &\geq \min\{f(b), \sup(\Delta)\} \\ &> \max\{f(x), \inf(\Delta)\}. \end{aligned}$$

This is a contradiction. Therefore $C_S \circ f \leq_\Delta f$. □

We give the characterization of regular semigroups by Δ-fuzzy left ideals and Δ-fuzzy right ideals.

Theorem 3.4. *Let* S *be a semigroup. The following conditions are equivalent:*

(i) S *is regular.*

(ii) $f \circ g =_{\Delta} f \wedge g$ *for every* Δ*-fuzzy right ideal* f *and every* Δ*-fuzzy left ideal* g *of* S.

Proof. First assume that (i) holds. Let f and g be a Δ-fuzzy right ideal and a Δ-fuzzy left ideal of S, respectively and let $x \in S[f \wedge g; \Delta]$. Suppose that $(f \wedge g)(x) = f(x)$. By assumption (i), we get $x = xax$ for some $a \in S$. Since f *is a* Δ-fuzzy right ideal of S and Theorem 3.3, we have $f \circ C_S \leq_{\Delta} f$ and then

$$\begin{aligned} \max\{(f \wedge g)(x), \inf(\Delta)\} &= \max\{f(x), \inf(\Delta)\} \\ &\geq \min\{(f \circ C_S)(x), \sup(\Delta)\} \\ &\geq \min\{(f \circ g)(x), \sup(\Delta)\} \end{aligned}$$

and

$$\begin{aligned} \max\{(f \circ g)(x), \inf(\Delta)\} &\geq \max\{\min\{f(xa), g(x)\}, \inf(\Delta)\} \\ &= \min\{\max\{f(xa), \inf(\Delta)\}, \max\{g(x), \inf(\Delta)\}\} \\ &\geq \min\{\min\{f(x), \sup(\Delta)\}, g(x)\} \\ &\geq \min\{(f \wedge g)(x), \sup(\Delta)\}. \end{aligned}$$

Next, we consider three cases as follows.

Case 1, let $f(x) \leq \inf(\Delta)$. Then

$$\inf(\Delta) = \max\{(f \wedge g)(x), \inf(\Delta)\} \geq \min\{(f \circ g)(x), \sup(\Delta)\}$$

and so $(f \circ g)(x) \leq \inf(\Delta)$. Thus $(f \circ g)_{\Delta}(x) = (f \wedge g)_{\Delta}(x)$.

Case 2, let $f(x) \geq \sup(\Delta)$. Then

$$\max\{(f \circ g)(x), \inf(\Delta)\} \geq \min\{(f \wedge g)(x), \sup(\Delta)\} = \sup(\Delta)$$

and so $(f \circ g)(x) \geq \sup(\Delta)$. Thus $(f \circ g)_{\Delta}(x) = (f \wedge g)_{\Delta}(x)$.

Case 3, let $\inf(\Delta) < f(x) < \sup(\Delta)$. Then

$$\begin{aligned} \max\{(f \circ g)(x), \inf(\Delta)\} &\geq \min\{(f \wedge g)(x), \sup(\Delta)\} \\ &= (f \wedge g)(x) \\ &= \max\{(f \wedge g)(x), \inf(\Delta)\} \\ &\geq \min\{(f \circ g)(x), \sup(\Delta)\} \end{aligned}$$

and thus $(f \circ g)(x) = (f \wedge g)(x)$ which implies that $(f \circ g)_{\Delta}(x) = (f \wedge g)_{\Delta}(x)$.

By Case 1-3, we have $(f \circ g)_{\Delta}(x) = (f \wedge g)_{\Delta}(x)$. In the case that $(f \wedge g)(x) = g(x)$, we can similarly prove that $(f \circ g)_{\Delta}(x) = (f \wedge g)_{\Delta}(x)$. Hence $f \circ g =_{\Delta} f \wedge g$. Therefore (ii) is true.

Conversely, assume that (ii) holds. Let L and R be a left ideal and a right ideal of S, respectively. By Theorem 3.2, we get the characteristic functions C_R and C_L are a Δ-fuzzy right ideal and a Δ-fuzzy left ideal of S, respectively. By assumption (ii) and Proposition 2.1, we have

$$C_{RL} = C_R \circ C_L =_\Delta C_R \wedge C_L = C_{R\cap L}.$$

Hence by Proposition 2.3, we have $RL = R \cap L$ and using Theorem 2.1, we get S is regular that is (i) is true. □

If we take $\Delta = (\gamma, \delta]$ in Theorem 3.4, then it is reduced to the following corollary.

Corollary 3.1. [15] *Let* S *be a semigroup. The following conditions are equivalent:*

(i) S *is regular.*

(ii) $f * g = f \wedge^* g$ *for every* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy right ideal* f *and* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy left ideal* g *of* S.

Definition 3.3 A fuzzy subset f of a semigroup S is called a Δ**-fuzzy semiprime subset** of S if it satisfies the following condition:

for all $x \in S$, if $x \in S[f;\Delta]$ or $x^2 \in S[f;\Delta]$, then $\max\{f(x), \inf(\Delta)\} \geq \min\{f(x^2), \sup(\Delta)\}$.

Next, intra-regular semigroups will be characterized in terms of Δ*-fuzzy ideals.*

Theorem 3.5. *Let* S *be a semigroup. The following conditions are equivalent:*

(i) S *is intra-regular.*

(ii) *Every* Δ*-fuzzy ideal of* S *is* Δ*-fuzzy semiprime.*

(iii) $f_\Delta(x) = f_\Delta(x^2)$ *for all* Δ*-fuzzy ideal* f *of* S *and for all* $x \in S$.

Proof. (i)$\Rightarrow$(ii). First assume that (i) holds. Let f be a Δ-fuzzy ideal of S and let $x \in S$ be such that $x \in S[f;\Delta]$ or $x^2 \in S[f;\Delta]$. Then $x = ax^2b$ for some $a, b \in S$. Now, we consider in the case: $x \in S[f;\Delta]$.

Case 1, let $\inf(\Delta) \geq f(x)$. Then

$$\inf(\Delta) = \max\{f(x), \inf(\Delta)\} = \max\{f(ax^2b), \inf(\Delta)\} \geq \min\{f(ax^2), \sup(\Delta)\}.$$

Thus $\inf(\Delta) \geq f(ax^2)$ implies that $ax^2 \in S[f;\Delta]$. Hence

$$\max\{f(x), \inf(\Delta)\} = \inf(\Delta) = \max\{f(ax^2), \inf(\Delta)\} \geq \min\{f(x^2), \sup(\Delta)\}.$$

Case 2, let $f(x) \geq \sup(\Delta)$ and then it is clear.

Case 3, let $\sup(\Delta) > f(x) > \inf(\Delta)$ and then

$$\max\{f(ax^2), \inf(\Delta)\} \geq \min\{f(x), \sup(\Delta)\} = \max\{f(ax^2b), \inf(\Delta)\} \geq \min\{f(ax^2), \sup(\Delta)\}.$$

Thus $f(ax^2) = f(x)$ and so

$$\max\{f(x), \inf(\Delta)\} = \max\{f(ax^2), \inf(\Delta)\} \geq \min\{f(x^2), \sup(\Delta)\}.$$

By Case 1-3, we show that $\max\{f(x), \inf(\Delta)\} \geq \min\{f(x^2), \sup(\Delta)\}$. In the case that $x^2 \in S[f;\Delta]$, we can similarly prove that $\max\{f(x), \inf(\Delta)\} \geq \min\{f(x^2), \sup(\Delta)\}$. Hence f is Δ-fuzzy semiprime and then (i) implies (ii).

(ii) $\Rightarrow$ (iii). Assume that (ii) holds and let $x \in S$. If $x, x^2 \notin S[f;\Delta]$, then $f_\Delta(x) = f_\Delta(x^2)$. Suppose that $x \in S[f;\Delta]$ or $x^2 \in S[f;\Delta]$. Since f is Δ-fuzzy ideal and a Δ-fuzzy semiprime subset of S, we have

$$\max\{f(x), \inf(\Delta)\} \geq \min\{f(x^2), \sup(\Delta)\}$$

and

$$\max\{f(x^2), \inf(\Delta)\} \geq \min\{f(x), \sup(\Delta)\}.$$

Next, we consider in the case: $x \in S[f;\Delta]$.

Case 1, let $\inf(\Delta) \geq f(x)$ and so

$$\inf(\Delta) = \max\{f(x), \inf(\Delta)\} \geq \min\{f(x^2), \sup(\Delta)\}.$$

Then $\inf(\Delta) \geq f(x^2)$ which implies that $f_\Delta(x) = \inf(\Delta) = f_\Delta(x^2)$.

Case 2, let $f(x) \geq \sup(\Delta)$. Then

$$\max\{f(x^2), \inf(\Delta)\} \geq \min\{f(x), \sup(\Delta)\} = \sup(\Delta).$$

Thus $f(x) \geq \sup(\Delta)$ and so $f_\Delta(x) = \sup(\Delta) = f_\Delta(x^2)$.

Case 3, let $\sup(\Delta) > f(x) > \inf(\Delta)$. Then

$$\begin{aligned}\max\{f(x^2), \inf(\Delta)\} &\geq \min\{f(x), \sup(\Delta)\} \\ &= f(x) \\ &= \max\{f(x), \inf(\Delta)\} \\ &\geq \min\{f(x^2), \sup(\Delta)\}.\end{aligned}$$

Thus $f(x) = f(x^2)$ and so $f_\Delta(x) = f_\Delta(x^2)$.

By Case 1-3, we show that $f_\Delta(x) = f_\Delta(x^2)$. In the case that $x^2 \in S[f;\Delta]$, we can similarly prove that $f_\Delta(x) = f_\Delta(x^2)$. Hence $f_\Delta(x) = f_\Delta(x^2)$ when $x \in S[f;\Delta]$ or $x^2 \in S[f;\Delta]$. Therefore (ii) implies (iii).

(iii) $\Rightarrow$ (i). Assume that (iii) holds. Let L and R be a left ideal and a right ideal of S, respectively. Then LR is an ideal of S and by Theorem 3.2, we have C_{LR} is a Δ-fuzzy ideal of S. By assumption, we get

$$(C_{LR})_\Delta(x) = (C_{LR})_\Delta(x^2) = \sup(\Delta) \text{ for all } x \in L \cap R.$$

Hence $L \cap R \subseteq LR$. By Theorem 2.2, we have S is intra-regular and so (iii) implies (i).

□

If we take $\Delta = (\gamma, \delta]$ in Theorem 3.5, then it is reduced to the following corollary.

Corollary 3.2. *Let S be a semigroup. The following conditions are equivalent.*

(i) S *is intra-regular.*

(ii) $f^*(x) = f^*(x^2)$ *for all* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy ideal* f *of* S *and for all* $x \in S$.

The final theorem, we characterize semisimple semigroups by using Δ-fuzzy ideals.

Theorem 3.6. *For a semigroup S, the following conditions are equivalent.*

(i) S *is semisimple.*

(ii) $f \circ f =_\Delta f$ *for every* Δ*-fuzzy ideal* f *of* S.

Proof. (i) $\Rightarrow$ (ii). Assume that (i) holds. Let f be a Δ-fuzzy ideal of S and x be an element of S such that $x \in S[f;\Delta]$. By Theorem 2.3, there exist $a,b,c,d \in S$ such that $x = (axb)(cxd)$. Since f is a Δ-fuzzy ideal of S and by Theorem 3.3, we have

$$\max\{f(x), \inf(\Delta)\} \geq \min\{(f \circ C_S)(x), \sup(\Delta)\} \geq \min\{(f \circ f)(x), \sup(\Delta)\}$$

and

$$\begin{aligned}\max\{(f \circ f)(x), \inf(\Delta)\} &\geq \max\{\min\{f(axb), f(cxd)\}, \inf(\Delta)\} \\ &= \min\{\max\{f(axb), \inf(\Delta)\}, \max\{f(cxd), \inf(\Delta)\}\} \\ &\geq \min\{f(x), \sup(\Delta)\}.\end{aligned}$$

Next, we consider three cases as follows.

Case 1, let $\inf(\Delta) \geq f(x)$. Then

$$\inf(\Delta) = \max\{f(x), \inf(\Delta)\} \geq \min\{(f \circ f)(x), \sup(\Delta)\}$$

which implies that $\inf(\Delta) \geq (f \circ f)(x)$. Thus $f_\Delta(x) = (f \circ f)_\Delta(x)$.

Case 2, let $f(x) \geq \sup(\Delta)$. Then

$$\max\{(f \circ f)(x), \inf(\Delta)\} \geq \min\{f(x), \sup(\Delta)\} = \sup(\Delta)$$

and so $(f\circ f)(x)\geq \sup(\Delta)$. Thus $f_{\Delta}(x)=(f\circ f)_{\Delta}(x)$.

Case 3, let $\sup(\Delta)>f(x)>\inf(\Delta)$. Then

$$\begin{aligned}\max\{(f\circ f)(x),\inf(\Delta)\} &\geq \min\{f(x),\sup(\Delta)\}\\ &= f(x)\\ &= \max\{f(x),\inf(\Delta)\}\\ &\geq \min\{(f\circ f)(x),\sup(\Delta)\}\end{aligned}$$

and so $(f\circ f)(x)=f(x)$. Thus $(f\circ f)_{\Delta}(x)=f_{\Delta}(x)$.

By Case 1-3, we get $(f\circ f)_{\Delta}(x)=f_{\Delta}(x)$. Hence $f\circ f=_{\Delta} f$ and therefore (i) implies (ii).

(ii) $\Rightarrow$ (i). Assume that (iii) holds. Let I be an ideal of S. By Theorem 3.2, we get the characteristic function C_I is a Δ-fuzzy ideal of S. By assumption (ii) and Proposition 2.1, we have

$$C_{I^2}=C_I\circ C_I=_{\Delta} C_I.$$

Hence by Proposition 2.3, we see that $I^2=I$. Therefore S is semisimple and so (ii) implies (i). □

If we take $\Delta=(\gamma,\delta]$ in Theorem 3.6, then it is reduced to the following corollary.

Corollary 3.3. [15] *For a semigroup S, the following conditions are equivalent.*

(i) *S is semisimple.*

(ii) *$f*f=f^*$ for every $(\in_{\gamma},\in_{\gamma}\vee q_{\delta})$-fuzzy ideal f of S.*

4. Conclusions and Future Work

Fuzzy set theory is an important mathematical notion, which easily handles uncertainties and has applications in real-life problems. In this paper, we introduce notions of a Δ-fuzzy left ideal, a Δ-fuzzy right ideal and a Δ-fuzzy ideal of a semigroup which are an $(\in_{\gamma},\in_{\gamma}\vee q_{\delta})$-fuzzy left ideal, an $(\in_{\gamma},\in_{\gamma}\vee q_{\delta})$-fuzzy right ideal and an $(\in_{\gamma},\in_{\gamma}\vee q_{\delta})$-fuzzy ideal, respectively and characterize semigroups by such fuzzy sets. We hope that the results given in this paper will have an impact on the upcoming research in this area and other aspects of fuzzy algebraic structures so that this leads to new horizons of interest and innovations. Our results can also be applied to other algebraic structures such as Γ-semigroups, LA-semigroups, ternary semigroups and near ring. Our future plans are to apply the Δ-fuzzy structures for characterizing Γ-semigroups, LA-semigroup, ternary semigroups and near ring.

Acknowledgments: We are very thankful to the reviewers for careful detailed reading and helpful comments/suggestions that improve the overall presentation of this paper.

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