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Vague Soft Matrix-Based Decision-Making

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Abstract

Soft set theory, introduced by Molodtsov, is an efficient mathematical tool makes it easy to deal with uncertainty. However, using it to represent problem parameters vagueness is difficult. So, using the vague soft set (VSS) which is an extension of the soft set is more general and flexible than using the soft set. In this paper, we, first, define vague soft matrix (VSM) from another point of view for representing VSS in a matrix form. Furthermore, different types of matrices in VSS theory and the operations of them are discussed. Moreover, according to these operations, new results on VSMs are investigated. We, then, introduce decision-making methods based on VSMs that allow constructing more effective decision processes. Finally, applied real-life problems in medicine and education are given to show that the proposed methods successfully work for problems that contain uncertain data.
Keywords: Decision-making, Fuzzy set, Soft set, Vague set, Vague soft set.
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1. MOTIVATION AND INTRODUCTION

Uncertain data are inherent and pervasive in many important applications in the areas such as economics, engineering, medical science, environmental science, sociology, business management and many other fields. Uncertain data in these applications can be caused by information incompleteness, data randomness, delayed data updates, measuring instruments limitations, etc. Due to the amount of uncertain rapidly increasing collected and accumulated data and the importance of these applications, research on effective techniques that are dedicated to model uncertain data and tackle uncertainties has attracted much interest in recent years and yet remained challenging at large. Black [4] first introduced the vagueness concept as follows: the vagueness of a term is shown by producing "borderline cases", i.e., individuals to which it seems impossible either to

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apply or not to apply the term. In the literature, there have been a big amount of research and applications concerning some special mathematical tools like probability theory, fuzzy sets [32], intuitionistic fuzzy set (IFS) theory [2], vague set theory [11] and other mathematical tools which are useful approaches for describing uncertainty. However, all of these theories have their own difficulties which have been pointed out in [21]. Molodtsov [21] suggested that one of the reasons for those difficulties may be the parametrization tools inadequacy of these theories, so he introduced the soft set concept. The soft set is a mathematical tool, free from those above difficulties, for dealing easier with uncertainties. It associates a set with a set of parameters. It brings about a rich potential for applications in many fields such as Riemann integration, measure theory, decision-making, game theory and many other fields. After that, Maji et al. [18], [19] considered and studied the theory of soft set initiated by Molodtsov. They introduced several notions in soft set theory, made a clear theoretical study on soft set theory in more details and applied it in a decision-making problem. Soft matrix theory was established by Çağman et al. [6]. They constructed a soft decision-making model on the soft set theory. Using matrices to represent the concept of the soft set made it easy to handle and also provided us with more applications, especially those involve decision-making. After that, Maji et al. [17] have studied the theory of fuzzy soft sets (FSSs) by embedding the ideas of fuzzy sets. Yang et al.[31] interpolated a matrix representation of FSSs. Later on, Çağman et al.[8] extended the study of the fuzzy soft matrices (FSMs) and several algebraic operations which are more functional to make theoretical studies in the FSS theory. Then, Basu et al. [3] and Kumar and Kaur [14] studied FSM and established some operations and notions on FSMs. They and many others (see [1], [7], [9], [22], [23] and [29]) introduced several results on FSSs and FSMs and made applications of these concepts in decision-making problems in many fields of science. When they combined the concept of fuzzy set and the concept of soft set, they discovered more applications in real-life decision-making problems. In later years, Intuitionistic fuzzy soft sets (IFSSs) were initiated as an extension of soft sets by Maji et al.[20]. Later, Chetia et al.[10] proposed the concepts of intuitionistic fuzzy soft matrices (IFSMs) which are a representation of IFSSs. In real world, the difficulty is that objects in the universal set may precisely not satisfy the problem's parameters, which usually illustrate some attributes, properties or characteristics of objects in the universal set. The concept of FSSs partially resolves this difficulty, but falls short in dealing with additional complexity, i.e., the mapping may be too vague. It is, therefore, desirable to extend the theory of soft set and FSS theory using the concept of vague set theory. Vague set theory is actually an extension of fuzzy set theory. As a parameterization of the vague set, Xu et al.[30] introduced the vague soft set (VSS) theory which is an extension of soft set by combining the notions of the vague sets and the soft sets. In addition, they presented the basic properties of VSSs. VSS theory makes the object world descriptions more realistic, accurate and practical making it a widely promising tool at least in some cases. Recently, Wang and many others (see [24]-[28]) introduced many results on VSSs and studied their properties. In present, concurrently with our work, Inthumathi and Pavithra [13] extended the concept of soft matrix by introducing the notion of VSM. In this paper, continuing of their work, which has appeared simultaneously during our work, we, firstly, defining VSM from another point of view, deriving its basic properties and illustrating its potential applications. The rest of the paper is arranged as the following. Section(2) introduces the basic concepts and definitions for each of the following: fuzzy set, vague set, soft set, FSS and VSS. Section(3) presents the concept of VSMs and some types, operations and new results related to it. Furthermore, in section(3), medical and educational applications of VSMs are introduced by applying the current method using decision-making to illustrate the concepts in a friendly way. Section(4) provides conclusions and open questions for further investigation.

2. DEFINITIONS AND PRELIMINARIES

This section is devoted to introduce main definitions and preliminaries which will be needed in the sequel in the following set theory's extensions, respectively: fuzzy set theory, vague set theory, soft set theory, FSS theory and VSS theory.

Definition 2.1. ***(Fuzzy set)****[32] Let U be a universal set (space of points or objects). A fuzzy set (class) X over U is a set characterized by a function f_X, $f_X : U \to [0, 1]$. f_X is called the membership, characteristic or indicator function of the fuzzy set X and the value $f_X(u)$ is called the grade of membership of $u \in U$ in X. A fuzzy set X over a universal set U can be represented as follows: $X = \{(f_X(u)/u) : u \in U,\ f_X(u) \in [0, 1]\}$.*

Definition 2.2. ***(Vague set)****[11] Let U be a universal set, $U = \{u_1, u_2, ..., u_n\}$, say. A vague set V over U is a set characterized by a truth membership function t_V and a false membership function f_V. For $u \in U$, $t_V(u)$ is a lower bound on the grade of membership of u derived from the evidence for u and $f_V(u)$ is a lower bound on the negation of u derived from the evidence against u. $t_V(u)$ and $f_V(u)$ both associate a real number in the interval $[0, 1]$ with each point in U, where $t_V(u)+f_V(u) \leq 1$. That is $t_V : U \to [0, 1]$, and $f_V : U \to [0, 1]$. This approach bounds the grade of membership of u to an interval $[t_V(u), 1 - f_V(u)] \subseteq [0, 1]$, i.e., the exact grade of membership $\mu_V(u)$ of u may be unknown, but is bounded by $t_V(u) \leq \mu_V(u) \leq 1 - f_V(u)$, where $t_V(u) + f_V(u) \leq 1$. For a continuous universal set U, a vague set V can be written* *as follows:* $V = \int_U [t_V(u_i), 1 - f_V(u_i)]/u_i$, $u_i \in U$. *For a discrete universal set* U*, a vague set* V *can be written as follows:* $V = \sum_{i=1}^{n} [t_V(u_i), 1 - f_V(u_i)]/u_i$, $u_i \in U$.

Remark 2.1. *[11] Interval containment is defined in the natural way*
$[a, b] \leq [c, d] \Leftrightarrow a \geq c$ *and* $b \leq d$.

Definition 2.3. *[12] Let* $\star$ *denote any of the four arithmetic operations on closed intervals: addition* $(+)$*, subtraction* $(-)$*, multiplication* $(\cdot)$ *and division* $(/)$*. Then,* $[a, b] \star [c, d] = \{x \star y : a \leq x \leq b, c \leq y \leq d\}$. *It is a general property of all arithmetic operations on closed intervals, except that* $[a, b]/[c, d]$ *isn't defined when* $0 \in [c, d]$. *The four arithmetic operations on closed intervals are defined, using the end points of the two intervals, as the following: Addition:* $[a, b] + [c, d] = [a + c, b + d]$*, Subtraction:* $[a, b] - [c, d] = [a - d, b - c]$,
Multiplication: $[a, b] \cdot [c, d] = [\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)]$
and Division: $[a, b]/[c, d] = [\min(a/c, a/d, b/c, b/d), \max(a/c, a/d, b/c, b/d)]$.

Definition 2.4. ***(Soft set)**([18],[21]) Let* U *be a universal set, E be a set of parameters and* $A \subseteq E$. *The power set of U is defined by* $P(U) = 2^U$. *A pair* (F, A) *is called a soft set over* U *and is defined as a set of ordered pairs* $F_A = \{(e, F_A(e)) : e \in E, F_A(e) \in P(U)\}$*, where* F *is a mapping given by* $F : A \rightarrow P(U)$. A *is called the support of* F_A *and we have* $F_A(e) \neq \phi$ *for all* $e \in A$ *and* $F_A(e) = \phi$ *for all* $e \notin A$. *In other words, a soft set* (F, A) *over* U *is a parameterized family of the set* U.

Example 2.1. *Consider a soft set* (F, E) *which describes "attractiveness of cars" under the consideration of a decision maker to purchase. Suppose that there are five cars to be considered in the universal set* U*, denoted by* $U = \{c_1, c_2, c_3, c_4, c_5\}$ *and* $E = \{e_1, e_2, e_3\}$*, where* $e_i (i = 1, 2, 3)$ *stands for the parameters in a word of "beautiful", "expensive", and "luxurious", respectively. We can write the soft set* (F, E) *over* U *by the following relation:* $(F, E) = \{(e_1, \{c_1, c_2\}), (e_2, \{c_1, c_3, c_4\}), (e_3, \{c_1, c_2, c_5\})\}$.
This soft set can be represented in the following tabular form:

U	e_1	e_2	e_3
c_1	1	1	1
c_2	1	0	1
c_3	0	1	0
c_4	0	1	0
c_5	0	0	1

Table (1): Tabular representation of the soft set (F, E).

Definition 2.5. ***(Soft matrix)[5]*** *Let (F_A, E) be a soft set over a universal set U. Then one defines a subset of $U \times E$ uniquely by $R_A = \{(u,e) : e \in A, u \in F_A(e)\}$ which is called a relation from of (F_A, E). We can write the characteristic function of R_A as $\eta_{R_A} : U \times E \to \{0,1\}$, where*

$$\eta_{R_A} = \begin{cases} 1, \ if \ (u,e) \in R_A \\ 0, \ if \ (u,e) \notin R_A. \end{cases}$$

If $U = \{u_1, u_2, ..., u_n\}$ and $A \subseteq E = \{e_1, e_2, ..., e_m\}$, we present R_A by a table as follows:

R_A	e_1	e_2	$\cdots$	e_m
u_1	$\eta_{R_A}(u_1, e_1)$	$\eta_{R_A}(u_1, e_2)$	$\cdots$	$\eta_{R_A}(u_1, e_m)$
u_2	$\eta_{R_A}(u_2, e_1)$	$\eta_{R_A}(u_2, e_2)$	$\cdots$	$\eta_{R_A}(u_2, e_m)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
u_n	$\eta_{R_A}(u_n, e_1)$	$\eta_{R_A}(u_n, e_2)$	$\cdots$	$\eta_{R_A}(u_n, e_m)$

Table (2): Tabular representation of R_A.

Let $a_{ij} = \eta_{R_A}(u_i, e_j)$, for $i = 1, 2, ..., n$, $j = 1, 2, ..., m$, we can define the soft matrix of order $n \times m$ of the soft set (F_A, E) over U in the following form:

$$[a_{ij}]_{n \times m} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}$$

Example 2.2. *According to Example (2.1), the soft matrix of the soft set is as following:*

$$[a_{ij}]_{5 \times 3} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Definition 2.6. ***(Fuzzy soft set)[17]*** *Let U be a universal set, E be a set of parameters and $A \subseteq E$. A pair (F, A) is called a fuzzy soft set (FSS) over U, where F is a mapping given by $F : A \to \mathcal{F}(U)$, $\mathcal{F}(U)$ is the family of all fuzzy subsets of U (the power set of fuzzy sets on U) and the fuzzy subset of U is defined as a map f from U to $[0,1]$.*

Example 2.3. *Example (2.1) can be characterized by a membership function instead of* $0, 1$ *numbers which associates each element with a real number in the interval* $[0, 1]$*, then we can write the FSS* (F, E) *over* U *as follows:* $(F, E) = \{F(e_1) = \{(c_1, 0.2), (c_2, 0.7)\}, F(e_2) = \{(c_1, 0.6), (c_3, 0.8), (c_4, 0.4)\}, F(e_3) = \{(c_1, 0.3), (c_2, 0.5), (c_5, 0.9)\}\}$*. This FSS can be represented in the following tabular form:*

U	e_1	e_2	e_3
c_1	0.2	0.6	0.3
c_2	0.7	0	0.5
c_3	0	0.8	0
c_4	0	0.4	0
c_5	0	0	0.9

Table (3): Tabular representation of the FSS (F, E)*.*

Definition 2.7. ***(Fuzzy soft matrix)****[3] Let* (F_A, E) *be a FSS over a universal set* U*. Then one defines a subset of* $U \times E$ *uniquely by* $R_A = \{(u, e) : e \in A, u \in F_A(e)\}$ *which is called a relation from of* (F_A, E)*. We can write the characteristic function of* R_A *as* $\psi_{R_A} : U \times E \to [0, 1]$*, and* $\psi_{R_A}(u, e) = f(u, e)$*, where* $f(u, e)$ *is the membership value of* $u \in U$ *for all* $e \in E$*, stated previously. If* $U = \{u_1, u_2, ..., u_n\}$ *and* $A \subseteq E = \{e_1, e_2, ..., e_m\}$*, then we present* R_A *by a table in the following form:*

R_A	e_1	e_2	$\cdots$	e_m
u_1	$\psi_{R_A}(u_1, e_1)$	$\psi_{R_A}(u_1, e_2)$	$\cdots$	$\psi_{R_A}(u_1, e_m)$
u_2	$\psi_{R_A}(u_2, e_1)$	$\psi_{R_A}(u_2, e_2)$	$\cdots$	$\psi_{R_A}(u_2, e_m)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
u_n	$\psi_{R_A}(u_n, e_1)$	$\psi_{R_A}(u_n, e_2)$	$\cdots$	$\psi_{R_A}(u_n, e_m)$

Table (4): Tabular representation of R_A*.*

Let $\psi_{ij} = \psi_{R_A}(u_i, e_j)$*, for* $i = 1, 2, ..., n, j = 1, 2, ..., m$*, we can define the fuzzy soft matrix (FSM) of order* $n \times m$ *of the FSS* (F_A, E) *over* U *in the following form:*

$$[\psi_{ij}]_{n\times m} = \begin{pmatrix} \psi_{11} & \psi_{12} & \cdots & \psi_{1m} \\ \psi_{21} & \psi_{22} & \cdots & \psi_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{n1} & \psi_{n2} & \cdots & \psi_{nm} \end{pmatrix}$$

Example 2.4. *According to Example (2.3), the FSM of the FSS is as following:*

$$[\psi_{ij}]_{5\times 3} = \begin{pmatrix} 0.2 & 0.6 & 0.3 \\ 0.7 & 0 & 0.5 \\ 0 & 0.8 & 0 \\ 0 & 0.4 & 0 \\ 0 & 0 & 0.9 \end{pmatrix}$$

Definition 2.8. ***(Vague soft set)****[30] Let U be a universal set, E be a set of parameters and $A \subseteq E$. A pair (F, A) is called a vague soft set (VSS) over U, where F is a mapping given by $F : A \to \mathcal{V}(U)$, $\mathcal{V}(U)$ is the family of all vague subsets of U (the power set of vague sets on U) and the vague subset of U is defined by its truth and false membership functions t and f, respectively, as stated previously.*

Example 2.5. *Example (2.3) can be characterized by the truth and false membership functions instead of a membership function which associates each element with an interval subset of $[0, 1]$, then we can write the VSS (F, E) over U as follows: $(F, E) = \{F(e_1) = \{(c_1, [0.2, 0.3]), (c_2, [0.7, 0.9])\}, F(e_2) = \{(c_1, [0.6, 0.8]), (c_3, [0.8, 0.9]), (c_4, [0.4, 0.7])\}, F(e_3) = \{(c_1, [0.3, 0.8]), (c_2, [0.5, 0.6]), (c_5, [0.9, 1])\}\}$. This VSS can be represented in the following tabular form:*

U	e_1	e_2	e_3
c_1	[0.2,0.3]	[0.6,0.8]	[0.3,0.8]
c_2	[0.7,0.9]	[0,0]	[0.5,0.6]
c_3	[0,0]	[0.8,0.9]	[0,0]
c_4	[0,0]	[0.4,0.7]	[0,0]
c_5	[0,0]	[0,0]	[0.9,1]

Table (5): Tabular representation of the VSS (F, E).

3. MAIN RESULTS

The aim of this section is to introduce the concept of VSM from another point of view. This section splits into two subsections. The first one is devoted to introduce the definitions and properties involving VSMs. Furthermore, medical and educational applications using VSMs are discussed in the second subsection.

3.1 Vague Soft Matrices (VSMs)

In this section, we define the VSM from another point of view and introduce new results on it.

Definition 3.1. ***(Vague soft matrix)*** *Let (F_A, E) be a VSS over a universal set U. Then one defines a subset of $U \times E$ uniquely by $R_A = \{(u, e) : e \in A, u \in F_A(e)\}$ which is called a relation from of (F_A, E). Define the characteristic function χ_{R_A} of R_A as $t_{R_A} \leq \chi_{R_A} \leq 1 - f_{R_A}$, where $t_{R_A} : U \times E \to [0, 1]$, $f_{R_A} : U \times E \to [0, 1]$, and $\chi_{R_A}(u, e) = \mu(u, e)$, where $t(u, e) \leq \mu(u, e) \leq 1 - f(u, e)$, and $t(u, e)$, $f(u, e)$ and $\mu(u, e)$ are the truth membership function, the false membership function and the exact grade of membership value of $u \in U$ for all $e \in E$, respectively, stated previously. If $U = \{u_1, u_2, ..., u_n\}$ and $A \subseteq E = \{e_1, e_2, ..., e_m\}$, then we present R_A by a table in the following form:*

R_A	e_1	e_2	$\cdots$	e_m
u_1	$[t_{R_A}(u_1, e_1), 1 - f_{R_A}(u_1, e_1)]$	$[t_{R_A}(u_1, e_2), 1 - f_{R_A}(u_1, e_2)]$	$\cdots$	$[t_{R_A}(u_1, e_m), 1 - f_{R_A}(u_1, e_m)]$
u_2	$[t_{R_A}(u_2, e_1), 1 - f_{R_A}(u_2, e_1)]$	$[t_{R_A}(u_2, e_2), 1 - f_{R_A}(u_2, e_2)]$	$\cdots$	$[t_{R_A}(u_2, e_m), 1 - f_{R_A}(u_2, e_m)]$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
u_n	$[t_{R_A}(u_n, e_1), 1 - f_{R_A}(u_n, e_1)]$	$[t_{R_A}(u_n, e_2), 1 - f_{R_A}(u_n, e_2)]$	$\cdots$	$[t_{R_A}(u_n, e_m), 1 - f_{R_A}(u_n, e_m)]$

Table (6): Tabular representation of R_A.

Let $\chi_{ij} = \mu_{R_A}(u_i, e_j)$, $t_{ij} = t_{R_A}(u_i, e_j)$ and $f_{ij} = f_{R_A}(u_i, e_j)$, for $i = 1, 2, ..., n$, $j = 1, 2, ..., m$, where $t_{R_A}(u_i, e_j) \leq \mu_{R_A}(u_i, e_j) \leq 1 - f_{R_A}(u_i, e_j)$. That is $t_{ij} \leq \chi_{ij} \leq 1 - f_{ij}$, i.e.,
$\chi_{ij} \in [t_{ij}, 1 - f_{ij}]$. Thus, we can define a matrix in the following form:

$$[\chi_{ij}]_{n\times m} = \begin{pmatrix} [t_{11}, 1 - f_{11}] & [t_{12}, 1 - f_{12}] & \cdots & [t_{1m}, 1 - f_{1m}] \\ [t_{21}, 1 - f_{21}] & [t_{22}, 1 - f_{22}] & \cdots & [t_{2m}, 1 - f_{2m}] \\ \vdots & \vdots & \ddots & \vdots \\ [t_{n1}, 1 - f_{n1}] & [t_{n2}, 1 - f_{n2}] & \cdots & [t_{nm}, 1 - f_{nm}] \end{pmatrix}$$

which is called a vague soft matrix (VSM) of order $n \times m$ of the VSS (F_A, E) over U.

Example 3.1. *According to Example (2.5), the VSM of the VSS is as following:*

$$[\chi_{ij}]_{5\times 3} = \begin{pmatrix} [0.2, 0.3] & [0.6, 0.8] & [0.3, 0.8] \\ [0.7, 0.9] & [0, 0] & [0.5, 0.6] \\ [0, 0] & [0.8, 0.9] & [0, 0] \\ [0, 0] & [0.4, 0.7] & [0, 0] \\ [0, 0] & [0, 0] & [0.9, 1] \end{pmatrix}$$

Definition 3.2. *Different types of matrices in VSS theory are introduced as follows:*

1. *A VSM of order: (a)* $1 \times m$ *with a single row and any number of columns is called a row-VSM, i.e, the row-VSM is a VSM whose universal set contains only one object.*
   (b) $n \times 1$ *with a single column and any number of rows is called a column-VSM, i.e, the column-VSM is a VSM whose parameter set contains only one parameter.*
   (c) $n \times n$ *with equal columns and rows is called a square VSM, i.e, the square VSM is a VSM whose parameters' number is equal to its objects' number.*
   (d) $n \times m$ *is called a null VSM if all of its elements are zero vague value, i.e.,* $[0, 0]$ *and it is denoted by* $\widetilde{\Phi}$*. So, the VSS associate with a null VSM must be a null VSS.*
   (e) $n \times m$ *is called an absolute or complete VSM if all of its elements are unit vague value, i.e.,* $[1, 1]$ *and it is denoted by* $\widetilde{C}_A$*. So, the VSS associate with a complete VSM must be a complete (absolute) VSS.*
   (f) $n \times n$ *is called a diagonal VSM if all of its non-diagonal elements are zero vague value, i.e.,* $[0, 0]$*.*

2. *The transpose of a VSM* $[\tilde{a}_{ij}]$ *of order* $n \times m$ *is another VSM of order* $m \times n$*, denoted by* $[\tilde{a}_{ij}]^T$*, obtained from* $[\tilde{a}_{ij}]$ *by interchanging its rows and columns.*

3. *A diagonal VSM of order* $n \times n$ *is called a scalar VSM if all of its non-diagonal elements are zero vague value, i.e.,* $[0, 0]$*, and all elements of its main diagonal have the same true and false membership values.*

4. *A scalar VSM is called an identity VSM, denoted by* I_F*, if all elements of its main diagonal are unit vague value, i.e.,* $[1, 1]$*.*

5. *A square VSM of order* $n \times n$ *is called: (a) an upper triangular VSM if all elements under the main diagonal are zero vague value, i.e.,* $[0, 0]$*.*
   (b) a lower triangular VSM if all elements above the main diagonal are zero vague value, i.e., $[0, 0]$*.*

6. *A square VSM* $[\tilde{a}_{ij}]$ *of order* $n \times m$ *is called a symmetric VSM if* $\tilde{a}_{ij} = \tilde{a}_{ji}$*, for all* $i = 1, 2, ..., n$ *and* $j = 1, 2, ..., m$*.*

7. *A VSM* $[\tilde{a}_{ij}]_{n\times m}$ *is: (a) a vague soft submatrix of a VSM* $[\tilde{b}_{ij}]_{n\times m}$*, denoted by* $[\tilde{a}_{ij}]_{n\times m}\tilde{\subseteq}[\tilde{b}_{ij}]_{n\times m}$*, if* $\tilde{a}_{ij} \leq \tilde{b}_{ij}$*, for all* $i = 1, 2, ..., n$ *and* $j = 1, 2, ..., m$*.*
   (b) a proper vague soft submatrix of a VSM $[\tilde{b}_{ij}]_{n\times m}$*, denoted by* $[\tilde{a}_{ij}]_{n\times m}\tilde{\subset}[\tilde{b}_{ij}]_{n\times m}$*, if* $\tilde{a}_{ij} \leq \tilde{b}_{ij}$ *for at least one term* $\tilde{a}_{ij} < \tilde{b}_{ij}$*, for all* $i = 1, 2, ..., n$ *and* $j = 1, 2, ..., m$*.*

8. *Two VSMs* $[\tilde{a}_{ij}]_{n\times m}$ *and* $[\tilde{b}_{ij}]_{n\times m}$ *are called vague soft equal matrices, denoted by* $[\tilde{a}_{ij}]_{n\times m}\tilde{=}[\tilde{b}_{ij}]_{n\times m}$*, if* $\tilde{a}_{ij} = \tilde{b}_{ij}$*, for all* $i = 1, 2, ..., n$ *and* $j = 1, 2, ..., m$*.*

Definition 3.3. ***(The complement of a VSM)*** *If* $\tilde{A} = [\tilde{a}_{ij}]_{n\times m}$ *be a VSM, where* $\tilde{a}_{ij} = [t_{ij}, 1 - f_{ij}]$*,* t_{ij} *and* f_{ij} *are the truth and false membership functions at* i^{th} *row and* j^{th} *column, then the complement of* $\tilde{A}$*, denoted by* $\tilde{A}^o = [\tilde{a}_{ij}]^o_{n\times m}$*, is defined by* $[\tilde{a}_{ij}]^o_{n\times m} = [\tilde{c}_{ij}]_{n\times m}$*, where* $\tilde{c}_{ij} = [f_{ij}, 1 - t_{ij}]$*, for all* $i = 1, 2, ..., n$ *and* $j = 1, 2, ..., m$*.*

Definition 3.4. ***(Addition of two VSMs)*** *Let* $\tilde{A} = [\tilde{a}_{ij}]_{n\times m}$ *and* $\tilde{B} = [\tilde{b}_{ij}]_{n\times m}$ *be two VSMs of the same order, then* $\tilde{A}$ *and* $\tilde{B}$ *are said to be compatible for addition and we define the addition of* $\tilde{A}$ *and* $\tilde{B}$ *by* $[\tilde{a}_{ij}]_{n\times m}\dot{+}[\tilde{b}_{ij}]_{n\times m} = [\tilde{c}_{ij}]_{n\times m}$*, where for all* $i = 1, 2, ..., n$ *and* $j = 1, 2, ..., m$*,* $\tilde{c}_{ij} = [a_{ij}, b_{ij}] + [c_{ij}, d_{ij}] = [\min\{1, a_{ij} + c_{ij}\}, \min\{1, b_{ij} + d_{ij}\}]$*.*

Definition 3.5. ***(Subtraction of two VSMs)*** *Let* $\tilde{A} = [\tilde{a}_{ij}]_{n\times m}$ *and* $\tilde{B} = [\tilde{b}_{ij}]_{n\times m}$ *be two VSMs of the same order, then* $\tilde{A}$ *and* $\tilde{B}$ *are said to be compatible for subtraction and we define the subtraction of* $\tilde{A}$ *and* $\tilde{B}$ *by* $[\tilde{a}_{ij}]_{n\times m}\dot{-}[\tilde{b}_{ij}]_{n\times m} = [\tilde{c}_{ij}]_{n\times m}$*, where for all* $i = 1, 2, ..., n$ *and* $j = 1, 2, ..., m$*,* $\tilde{c}_{ij} = [a_{ij}, b_{ij}] - [c_{ij}, d_{ij}] = [\max\{0, a_{ij} + c_{ij} - 1\}, \max\{0, b_{ij} + d_{ij} - 1\}]$*.*

Definition 3.6. ***(Product of two VSMs)*** *Let* $\tilde{A} = [\tilde{a}_{ij}]_{n\times m}$ *and* $\tilde{B} = [\tilde{b}_{ij}]_{m\times p}$ *be two VSMs of order* $n \times m$ *and* $m \times p$*, respectively. Then* $\tilde{A}$ *and* $\tilde{B}$ *are said to be compatible for product and the product of* $\tilde{A}$ *and* $\tilde{B}$ *is defined as follows:* $[\tilde{a}_{ij}]_{n\times m}\dot{\times}[\tilde{b}_{ij}]_{m\times p} = [\tilde{c}_{ik}]_{n\times p}$*, where* $[\tilde{c}_{ik}]_{n\times p}$ *is a VSM such that for* $i = 1, 2, ..., n$*,* $j = 1, 2, ..., m$ *and* $p = 1, 2, ..., k$*,* $\tilde{c}_{ik} = [a_{ij}, b_{ij}] \times [c_{ij}, d_{ij}] = \max\{[\min(a_{ij}c_{jk}, a_{ij}d_{jk}, b_{ij}c_{jk}, b_{ij}d_{jk}), \max(a_{ij}c_{jk}, a_{ij}d_{jk}, b_{ij}c_{jk}, b_{ij}d_{jk})]\}$*.*

Definition 3.7. ***(Union of two VSMs)*** *The union of two VSMs* $\tilde{A} = [\tilde{a}_{ij}]_{n\times m}$ *and* $\tilde{B} = [\tilde{b}_{ij}]_{n\times m}$ *is a VSM* $\tilde{C} = [\tilde{c}_{ij}]_{n\times m}$*, denoted by* $\tilde{C} = \tilde{A}\tilde{\cup}\tilde{B}$*, if for all* $i = 1, 2, ..., n$ *and* $j = 1, 2, ..., m$*, we have* $t_{\tilde{c}_{ij}} = \max[t_{\tilde{a}_{ij}}, t_{\tilde{b}_{ij}}]$*, and* $1 - f_{\tilde{c}_{ij}} = \max[1 - f_{\tilde{a}_{ij}}, 1 - f_{\tilde{b}_{ij}}]$*.*

Definition 3.8. ***(Intersection of two VSMs)*** *The intersection of two VSMs* $\tilde{A} = [\tilde{a}_{ij}]_{n\times m}$ *and* $\tilde{B} = [\tilde{b}_{ij}]_{n\times m}$ *is a VSM* $\tilde{C} = [\tilde{c}_{ij}]_{n\times m}$*, denoted by* $\tilde{C} = \tilde{A}\tilde{\cap}\tilde{B}$*, if for all* $i = 1, 2, ..., n$ *and* $j = 1, 2, ..., m$*, we have* $t_{\tilde{c}_{ij}} = \min[t_{\tilde{a}_{ij}}, t_{\tilde{b}_{ij}}]$*, and* $1 - f_{\tilde{c}_{ij}} = \min[1 - f_{\tilde{a}_{ij}}, 1 - f_{\tilde{b}_{ij}}]$*.*

Theorem 3.1. *Let* $\tilde{A} = [\tilde{a}_{ij}]_{n\times m}$ *and* $\tilde{B} = [\tilde{b}_{ij}]_{n\times m}$ *be two VSMs of the same order, then the following results are satisfied:*

1. $[\tilde{a}_{ij}]_{n\times m}\dot{+}\widetilde{C}_A = \widetilde{C}_A$ *and* $[\tilde{a}_{ij}]_{n\times m}\dot{-}\widetilde{C}_A = [\tilde{a}_{ij}]_{n\times m}$*.*

2. $\widetilde{\Phi}\dot{+}\widetilde{C}_A = \widetilde{C}_A$ *and* $\widetilde{\Phi}\dot{-}\widetilde{C}_A = \widetilde{\Phi}$*.*

3. $[\tilde{a}_{ij}]_{n\times m}\dot{+}\widetilde{\Phi} = [\tilde{a}_{ij}]_{n\times m}$ *and* $[\tilde{a}_{ij}]_{n\times m}\dot{-}\widetilde{\Phi} = \widetilde{\Phi}$.

4. $[\tilde{a}_{ij}]_{n\times m}\dot{+}[\tilde{a}_{ij}]^{o}_{n\times m} = [\tilde{d}_{ij}]_{n\times m}$, *and* $[\tilde{a}_{ij}]_{n\times m}\dot{-}[\tilde{a}_{ij}]^{o}_{n\times m} = [\tilde{g}_{ij}]_{n\times m}$, *where* $[\tilde{d}_{ij}]_{n\times m}$ *and* $[\tilde{g}_{ij}]_{n\times m}$ *are VSMs whose elements are in the form of* $[t_{ij} + f_{ij}, 1]$ *and* $[0, 1-(f_{ij}+t_{ij})]$, *respectively, for* $i = 1, 2, ..., n$ *and* $j = 1, 2, ..., m$, *where* t_{ij} *and* f_{ij} *are the truth and false membership functions, respectively.*

Proof. Let $[\tilde{c}_{ij}]_{n\times m} = [\tilde{a}_{ij}]_{n\times m}\dot{+}\widetilde{C}_A$, then we have, for $i = 1, 2, ..., n$ and $j = 1, 2, ..., m$,

$$\tilde{c}_{ij} = [a_{ij}, b_{ij}] + [c_{ij}, d_{ij}],$$

where $[a_{ij}, b_{ij}]$ and $[c_{ij}, d_{ij}]$ are elements of $[\tilde{a}_{ij}]_{n\times m}$ and $\widetilde{C}_A$, respectively. Thus

$$\begin{aligned}\tilde{c}_{ij} &= [t_{ij}, 1-f_{ij}] + [1, 1]\\ &= [\min\{1, t_{ij}+1\}, \min\{1, 1-f_{ij}+1\}]\\ &= [\min\{1, 1+t_{ij}\}, \min\{1, 2-f_{ij}\}]\\ &= [1, 1].\end{aligned}$$

Therefore, we have every element of $[\tilde{c}_{ij}]_{n\times m}$ is $[1, 1]$, i.e, an element of $\widetilde{C}_A$. Hence $[\tilde{c}_{ij}]_{n\times m} = \widetilde{C}_A$. The other statements of this theorem can be proved in a similar manner.

Proposition 3.1. *[22] Let* $\tilde{A} = [\tilde{a}_{ij}]_{n\times m}$ *and* $\tilde{B} = [\tilde{b}_{ij}]_{n\times m}$ *be two VSMs of the same order with* $t_{ij} = 1 - f_{ij}$ *for* $i = 1, 2, ..., n$ *and* $j = 1, 2, ..., m$, *where* t_{ij} *and* f_{ij} *are the truth and false membership functions, respectively. i.e. for two FSMs of the same order, then* $[\tilde{a}_{ij}]_{n\times m}\dot{+}[\tilde{a}_{ij}]^{o}_{n\times m} = \widetilde{C}_A$ *and* $[\tilde{a}_{ij}]_{n\times m}\dot{-}[\tilde{a}_{ij}]^{o}_{n\times m} = \widetilde{\Phi}$.

Remark 3.1. *The above two results aren't satisfied for any two VSMs in general (i.e., if* $t_{ij} \neq 1 - f_{ij}$ *for* $i = 1, 2, ..., n$ *and* $j = 1, 2, ..., m$, *where* t_{ij} *and* f_{ij} *are the truth and false membership functions, respectively).*

Proof. To show that the above two results aren't satisfied for any two VSMs in general, consider the following example:

$$[\tilde{a}_{ij}]_{3\times 3} = \begin{pmatrix} [0.2, 0.3] & [0.3, 0.6] & [0.1, 0.3]\\ [0.8, 1] & [0, 0] & [0.6, 0.7]\\ [0.4, 0.5] & [0.7, 0.8] & [0.5, 0.7] \end{pmatrix}$$

$$[\tilde{a}_{ij}]^{o}_{3\times 3} = \begin{pmatrix} [0.7, 0.8] & [0.4, 0.7] & [0.7, 0.9]\\ [0, 0.2] & [0, 0] & [0.3, 0.4]\\ [0.5, 0.6] & [0.2, 0.3] & [0.3, 0.5] \end{pmatrix}$$

$$[\tilde{a}_{ij}]_{3\times3}\dot{+}[\tilde{a}_{ij}]^o_{3\times3} = \begin{pmatrix} [0.9,1] & [0.7,1] & [0.8,1] \\ [0.8,1] & [0,0] & [0.9,1] \\ [0.9,1] & [0.9,1] & [0.8,1] \end{pmatrix} \neq (\widetilde{C}_A)_{3\times3}.$$

$$[\tilde{a}_{ij}]_{3\times3}\dot{-}[\tilde{a}_{ij}]^o_{3\times3} = \begin{pmatrix} [0,0.1] & [0,0.3] & [0,0.2] \\ [0,0.2] & [0,0] & [0,0.1] \\ [0,0.1] & [0,0.1] & [0,0.2] \end{pmatrix} \neq \widetilde{\Phi}_{3\times3}.$$

Theorem 3.2. *Let $\tilde{A} = [\tilde{a}_{ij}]_{n\times m}$ and $\tilde{B} = [\tilde{b}_{ij}]_{n\times m}$ be two VSMs of the same order, then De Morgan's laws are valid as follows $([\tilde{a}_{ij}]_{n\times m}\tilde{\cup}[\tilde{b}_{ij}]_{n\times m})^o = [\tilde{a}_{ij}]^o_{n\times m}\tilde{\cap}[\tilde{b}_{ij}]^o_{n\times m}$ and $([\tilde{a}_{ij}]_{n\times m}\tilde{\cap}[\tilde{b}_{ij}]_{n\times m})^o = [\tilde{a}_{ij}]^o_{n\times m}\tilde{\cup}[\tilde{b}_{ij}]^o_{n\times m}$.*

Proof. For all $i = 1, 2, ..., n$, $j = 1, 2, ..., m$ and $\tilde{c}_{ij} = [t_{\tilde{c}_{ij}}, 1 - f_{\tilde{c}_{ij}}] \in [\tilde{c}_{ij}]_{n\times m} = [\tilde{a}_{ij}]_{n\times m}\tilde{\cup}[\tilde{b}_{ij}]_{n\times m}$, where $t_{\tilde{c}_{ij}}$ and $f_{\tilde{c}_{ij}}$ are the truth and false membership functions of the vague soft element $\tilde{c}_{ij} \in [\tilde{c}_{ij}]_{n\times m}$, respectively, we have

$$\begin{aligned}([\tilde{a}_{ij}]_{n\times m}\tilde{\cup}[\tilde{b}_{ij}]_{n\times m})^o &\ni [t_{\tilde{c}_{ij}}, 1 - f_{\tilde{c}_{ij}}]^o \\ &= [\max\{t_{\tilde{a}_{ij}}, t_{\tilde{b}_{ij}}\}, \max\{1 - f_{\tilde{a}_{ij}}, 1 - f_{\tilde{b}_{ij}}\}]^o \\ &= [\max\{t_{\tilde{a}_{ij}}, t_{\tilde{b}_{ij}}\}, 1 - \min\{f_{\tilde{a}_{ij}}, f_{\tilde{b}_{ij}}\}]^o \\ &= [\min\{f_{\tilde{a}_{ij}}, f_{\tilde{b}_{ij}}\}, \min\{1 - t_{\tilde{a}_{ij}}, 1 - t_{\tilde{b}_{ij}}\}] \\ &= [t_{\tilde{a}_{ij}}, 1 - f_{\tilde{a}_{ij}}]^o\tilde{\cap}[t_{\tilde{b}_{ij}}, 1 - f_{\tilde{b}_{ij}}]^o \in [\tilde{a}_{ij}]^o_{n\times m}\tilde{\cap}[\tilde{b}_{ij}]^o_{n\times m}.\end{aligned}$$

Therefore, $([\tilde{a}_{ij}]_{n\times m}\tilde{\cup}[\tilde{b}_{ij}]_{n\times m})^o \subseteq [\tilde{a}_{ij}]^o_{n\times m}\tilde{\cap}[\tilde{b}_{ij}]^o_{n\times m}$ and $[\tilde{a}_{ij}]^o_{n\times m}\tilde{\cap}[\tilde{b}_{ij}]^o_{n\times m} \subseteq ([\tilde{a}_{ij}]_{n\times m}\tilde{\cup}[\tilde{b}_{ij}]_{n\times m})^o$ which completes the proof. The other statement can be proved similarly.

3.2 Applications

In this section, applied real-life problems in medicine and education are introduced by applying the current method using decision-making to illustrate the concepts in a friendly way and give more accurate results. The first part is to introduce the methodology and the algorithm used in general. The second part is to discuss examples from our real-life.

3.2.1 Methodology and algorithm

In this section, an algorithm for medical diagnosis, educational evaluation or methods... etc. using VSM is presented. Assume that there is a set of n patients, students or methods... etc. $P = \{p_1, p_2, ..., p_n\}$, say, with a set of m symptoms or exams... etc. $S = \{s_1, s_2, ..., s_m\}$, say, related to a set of k diseases, levels or intervals... etc. $D = \{d_1, d_2, ..., d_k\}$, say. We apply VSMs to diagnose which patient is suffering from what disease or which student is qualified to take what education level... etc. To do this end, we first construct VSS (F_1, P) over S, where F_1 is a mapping $F_1 : P \to V(S)$ which gives a relation VSM $\tilde{A}$, called (patient-symptom matrix) or (student-exam matrix)... etc. Thus the general form of $\tilde{A}$ is

$$\tilde{A} = \begin{array}{c} \\ p_1 \\ p_2 \\ \vdots \\ p_n \end{array} \begin{array}{c} \begin{array}{cccc} s_1 & s_2 & \cdots & s_m \end{array} \\ \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1m} \\ \tilde{a}_{21} & \tilde{a}_{22} & \cdots & \tilde{a}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{n1} & \tilde{a}_{n2} & \cdots & \tilde{a}_{nm} \end{pmatrix} \end{array}.$$

Then, we construct VSS (F_2, S) over D, where F_2 is a mapping $F_2 : S \to V(D)$ which gives a relation VSM $\tilde{B}$, called (symptom-disease matrix) or (exam-level matrix)... etc. Thus the general form of $\tilde{B}$ is

$$\tilde{B} = \begin{array}{c} \\ s_1 \\ s_2 \\ \vdots \\ s_m \end{array} \begin{array}{c} \begin{array}{cccc} d_1 & d_2 & \cdots & d_k \end{array} \\ \begin{pmatrix} \tilde{b}_{11} & \tilde{b}_{12} & \cdots & \tilde{b}_{1k} \\ \tilde{b}_{21} & \tilde{b}_{22} & \cdots & \tilde{b}_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{b}_{m1} & \tilde{b}_{m2} & \cdots & \tilde{b}_{mk} \end{pmatrix} \end{array}.$$

(Depending on the case study itself, we can take the transpose of $\tilde{B}$ if it is needed to perform the product). Now, performing the product operation of two VSMs $\tilde{A}$ and $\tilde{B}$ (or $\tilde{A}$ and $\tilde{B}^T$ as needed), we get VSM $\tilde{C}$, called diagnosis matrix (patient-disease matrix or student-level matrix... etc.), where $\tilde{C} = [\tilde{c}_{ij}]_{n\times k} = \tilde{A}\dot{\times}\tilde{B}$
(or $\tilde{C} = [\tilde{c}_{ij}]_{n\times m} = \tilde{A}\dot{\times}\tilde{B}^T$). Therefore, the general form of $\tilde{C}$ is

$$\tilde{C} = \begin{array}{c} \\ p_1 \\ p_2 \\ \vdots \\ p_n \end{array} \begin{array}{c} \begin{array}{cccc} d_1 & d_2 & \cdots & d_k \end{array} \\ \begin{pmatrix} \tilde{c}_{11} & \tilde{c}_{12} & \cdots & \tilde{c}_{1k} \\ \tilde{c}_{21} & \tilde{c}_{22} & \cdots & \tilde{c}_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{c}_{n1} & \tilde{c}_{n2} & \cdots & \tilde{c}_{nk} \end{pmatrix} \end{array}.$$

Finally, to make the decision, if $\max \tilde{c}_{il} = \tilde{c}_{is}$ (as intervals), for $l = 1, 2, ..., k$, then we are able to say that the patient p_i is suffering from the disease d_s or the student s_i is qualified to take the education level l_s... etc. Instead of this, we can use different methods to transform vague values into fuzzy values [16] as follows. Let $\mathcal{V}(U)$ be the set of all vague sets of the universal set U, then, for all $V \in \mathcal{V}(U)$, $u \in U$ and its vague value $[t_V(u), 1 - f_V(u)]$, the fuzzy membership function of u to V^F, where V^F is the fuzzy set corresponding to the vague set V, is defined as:
Method (1)**:** $f_{V^F} = \frac{1+t_V(u)-f_V(u)}{2}$.
Method (2)**:** $f_{V^F} = \frac{t_V(u)}{t_V(u)+f_V(u)}$.
Method (3)**:** $f_{V^F} = (1-\lambda) \times t_V(u) + \lambda \times (1 - f_V(u))$, where $\lambda \in [0, 1]$ is the degree of optimism of the evaluator. For instance, we can take $\lambda = 0.6$.
Method (4)**:** $f_{V^F} = t_V(u) + \frac{1}{2} \times [1 + \frac{t_V(u)-f_V(u)}{t_V(u)+f_V(u)+2\lambda}][1 - t_V(u) - f_V(u)]$ (general model), take $\lambda = 1$ for simplicity.

Thus, we can write the matrix of fuzzy values came from vague values $\tilde{V}^F$ as following

$$\tilde{V}^F = \begin{array}{c} \\ p_1 \\ p_2 \\ \vdots \\ p_n \end{array} \begin{array}{c} \begin{array}{cccc} d_1 & d_2 & \cdots & d_k \end{array} \\ \begin{pmatrix} \tilde{v}_{11} & \tilde{v}_{12} & \cdots & \tilde{v}_{1k} \\ \tilde{v}_{21} & \tilde{v}_{22} & \cdots & \tilde{v}_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{v}_{n1} & \tilde{v}_{n2} & \cdots & \tilde{v}_{nk} \end{pmatrix} \end{array}.$$

Now, if $\max \tilde{v}_{il} = \tilde{v}_{is}$, for $l = 1, 2, ..., k$, then we are able to make the decision that the patient p_i is suffering from the disease d_s or the student s_i is qualified to take the education level l_s... etc. It should be noted that if $\max \tilde{v}_{il} = \tilde{v}_{is}$ occurs for more than one value of l, $l = 1, 2, ..., k$, we can reassess the symptoms or exams... etc. to break the tie.

Algorithm :

Step(1): Input VSS (F_1, P) and VSS (F_2, S) to obtain the corresponding VSM $\tilde{A}$ and VSM $\tilde{B}$, respectively.
Step(2): Compute the product $\tilde{A} \dot{\times} \tilde{B}$ (or $\tilde{A} \dot{\times} \tilde{B}^T$) to get the diagnosis matrix $\tilde{C}$.
Step(3): Find s for which $\tilde{c}_{is} = \max \tilde{c}_{il}$ (as intervals) directly from the matrix $\tilde{C}$, or compute f_{V^F} $\forall$ $[t_V(u), 1 - f_V(u)] \in$ VSM $\tilde{C}$ using the Methods (1), (2), (3) and (4) to find the matrix $\tilde{V}^F$ from which we can find s, where $\tilde{v}_{is} = \max \tilde{v}_{il}$.
Step(4): Conclude that the patient p_i is suffering from the disease d_s or the student s_i

is qualified to take the education level l_s... etc. for each of these different methods and compare the results of them.

3.2.2 Case study

The aim of this section is to apply the general method stated in the previous section on three real-life decision-making medical and educational examples using VSMs. In the following examples, we used *Mathematica Program* to generate data and made all calculations and mathematical operations using *Mathematica Program*.

Example 3.2. ***(Medical diagnosis)*** *Suppose that there are four patients: Joseph, David, Samuel and John in a hospital with five symptoms: headache, temperature, cough, nausea and diarrhea. Let the possible three diseases relating to the above symptoms be malaria, typhoid and viral fever. Now, take $P = \{p_1, p_2, p_3, p_4\}$ as the universal set, where p_1, p_2, p_3 and p_4 represent patients Joseph, David, Samuel and John, respectively. Next, consider the set $S = \{s_1, s_2, s_3, s_4, s_5\}$ as a universal set, where s_1, s_2, s_3, s_4 and s_5 represent symptoms headache, temperature, cough, nausea and diarrhea, respectively. Finally, let the set $D = \{d_1, d_2, d_3\}$, where d_1, d_2 and d_3 represent the diseases malaria, typhoid and viral fever, respectively. Suppose that*

$$F_1(p_1) = \{[0.1, 0.19]/s_1, [0.77, 0.81]/s_2, [0.7, 0.84]/s_3, [0.07, 0.25]/s_4, [0.89, 0.93]/s_5\},$$

$$F_1(p_2) = \{[0.3, 0.86]/s_1, [0.12, 0.61]/s_2, [0.09, 0.22]/s_3, [0.42, 0.9]/s_4, [0.62, 0.76]/s_5\},$$

$$F_1(p_3) = \{[0.87, 0.94]/s_1, [0.08, 0.37]/s_2, [0.2, 0.66]/s_3, [0.35, 0.91]/s_4, [0.67, 0.98]/s_5\},$$

$$F_1(p_4) = \{[0.3, 0.89]/s_1, [0.47, 0.99]/s_2, [0.24, 0.62]/s_3, [0.3, 0.52]/s_4, [0.02, 0.95]/s_5\}.$$

Then, the VSS (F_1, P) represents the relation matrix (patient-symptom matrix) $\tilde{A}$ and is given by

$$\tilde{A} = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} \begin{array}{c} \begin{array}{ccccc} s_1 & s_2 & s_3 & s_4 & s_5 \end{array} \\ \begin{pmatrix} [0.1, 0.19] & [0.77, 0.81] & [0.7, 0.84] & [0.07, 0.25] & [0.89, 0.93] \\ [0.3, 0.86] & [0.12, 0.61] & [0.09, 0.22] & [0.42, 0.9] & [0.62, 0.76] \\ [0.87, 0.94] & [0.08, 0.37] & [0.2, 0.66] & [0.35, 0.91] & [0.67, 0.98] \\ [0.3, 0.89] & [0.47, 0.99] & [0.24, 0.62] & [0.3, 0.52] & [0.02, 0.95] \end{pmatrix} \end{array}$$

Next, suppose

$F_2(s_1) = \{[0.27, 0.96]/d_1, [0.22, 0.96]/d_2, [0.7, 0.92]/d_3\},$

$F_2(s_2) = \{[0.89, 0.91]/d_1, [0.1, 0.73]/d_2, [0.04, 0.78]/d_3\},$

$F_2(s_3) = \{[0.64, 0.92]/d_1, [0.54, 0.7]/d_2, [0.28, 0.29]/d_3\},$

$F_2(s_4) = \{[0.21, 0.76]/d_1, [0.49, 0.66]/d_2, [0.22, 0.59]/d_3\},$

$F_2(s_5) = \{[0.31, 0.61]/d_1, [0.31, 0.61]/d_2, [0.89, 0.95]/d_3\}.$

Then, the VSS (F_2, S) *(*$F_2 : S \to V(D)$ *is determined from expert medical documentation) is represented by the relation matrix (symptom-disease matrix)* $\tilde{B}$ *and is given by*

$$\tilde{B} = \begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \end{array} \begin{array}{c} \begin{array}{ccc} d_1 & d_2 & d_3 \end{array} \\ \begin{pmatrix} [0.27, 0.96] & [0.22, 0.96] & [0.7, 0.92] \\ [0.89, 0.91] & [0.1, 0.73] & [0.04, 0.78] \\ [0.64, 0.92] & [0.54, 0.7] & [0.28, 0.29] \\ [0.21, 0.76] & [0.49, 0.66] & [0.22, 0.59] \\ [0.31, 0.61] & [0.31, 0.61] & [0.89, 0.95] \end{pmatrix} \end{array}.$$

Then, performing the product operation $\tilde{A}\dot{\times}\tilde{B}$*, we get the VSM* $\tilde{C}$*, called the patient-disease matrix (patient-diagnosis matrix) as*

$$\tilde{C} = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} \begin{array}{c} \begin{array}{ccc} d_1 & d_2 & d_3 \end{array} \\ \begin{pmatrix} [0.45, 0.77] & [0.38, 0.6] & [0.79, 0.89] \\ [0.08, 0.84] & [0.06, 0.83] & [0.21, 0.79] \\ [0.24, 0.91] & [0.19, 0.9] & [0.6, 0.94] \\ [0.42, 0.9] & [0.06, 0.86] & [0.01, 0.92] \end{pmatrix} \end{array}.$$

From the above matrix, by comparing the intervals, we can say that patients p_1*,* p_3 *and* p_4 *are suffering from disease* d_3 *and patient* p_2 *is suffering from disease* d_1*, i.e., Joseph, Samuel and John are suffering from viral fever and David is suffering from malaria. Or, we have:*

$$\tilde{V_1}^F = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} \begin{array}{c} \begin{array}{ccc} d_1 & d_2 & d_3 \end{array} \\ \begin{pmatrix} 0.61 & 0.49 & 0.84 \\ 0.46 & 0.45 & 0.5 \\ 0.57 & 0.54 & 0.77 \\ 0.66 & 0.46 & 0.47 \end{pmatrix} \end{array}, \quad \tilde{V_2}^F = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} \begin{array}{c} \begin{array}{ccc} d_1 & d_2 & d_3 \end{array} \\ \begin{pmatrix} 0.67 & 0.49 & 0.88 \\ 0.34 & 0.29 & 0.51 \\ 0.72 & 0.67 & 0.91 \\ 0.81 & 0.33 & 0.2 \end{pmatrix} \end{array},$$

$$\tilde{V_3}^F = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} \begin{array}{c} \begin{array}{ccc} d_1 & d_2 & d_3 \end{array} \\ \begin{pmatrix} 0.14 & 0.09 & 0.28 \\ 0.02 & 0.02 & 0.06 \\ 0.08 & 0.07 & 0.22 \\ 0.15 & 0.02 & 0.007 \end{pmatrix} \end{array}, \quad \tilde{V_4}^F = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} \begin{array}{c} \begin{array}{ccc} d_1 & d_2 & d_3 \end{array} \\ \begin{pmatrix} 0.62 & 0.49 & 0.85 \\ 0.44 & 0.43 & 0.5 \\ 0.59 & 0.56 & 0.81 \\ 0.69 & 0.45 & 0.45 \end{pmatrix} \end{array},$$

where $\tilde{V_1}^F$, $\tilde{V_2}^F$, $\tilde{V_3}^F$ *and* $\tilde{V_4}^F$ *represent the matrices obtained by Method* (1), *Method* (2), *Method* (3) *and Method* (4), *respectively, to transform vague values into fuzzy values. Therefore, there is an agreement in the diagnosis' decision according to the Methods* (1), (2), (3) *and* (4) *which is that patients* p_1, p_2 *and* p_3 *are suffering from disease* d_3 *and patient* p_4 *is suffering from disease* d_1, *i.e., Joseph, David and Samuel are suffering from viral fever and John is suffering from malaria. We can compare the results of the five methods in a table as the following:*

The patient	The disease which he is suffering from				
	Vague value	Transformed fuzzy value			
		$Method(1)$	$Method(2)$	$Method(3)$	$Method(4)$
p_1	d_3	d_3	d_3	d_3	d_3
p_2	d_1	d_3	d_3	d_3	d_3
p_3	d_3	d_3	d_3	d_3	d_3
p_4	d_3	d_1	d_1	d_1	d_1

Table (7): Comparative results of the used methods.

There is an agreement in the final diagnosis' decision of the first patient (Joseph) and the third patient (Samuel) by using the vague value method and the Methods (1), (2), (3) *and* (4) *(transformed fuzzy value) which makes us sure that they are suffering from the third disease (viral fever), but we aren't sure about the final diagnosis' decision of the second patient (David) and the fourth patient (John) if they are suffering from the first disease (malaria) or the third disease (viral fever). In this case, the decision obtained by the four methods (most methods) is recommended or we can reassess the symptoms (if needed) to break the tie.*

Example 3.3. ***(Medical methods)*** *According to Example (3.2.3) page* 90 *in [15] (the data has been collected from real patients by researchers in medicine), there are four medical methods:* $Z_1(1)$, $Z_2(2)$, $Z_2(3)$ *and* LR *in using Drug* $6-MP$ *for two intervals:* $S(10+)$ *and* $S(20+)$ *compared with those for Placebo effect for two groups of leukemia patients in a clinical trial. Now, take* $M^D = \{m_1^D, m_2^D, m_3^D, m_4^D\}$ *as the universal set of the medical methods using Drug* $6 - MP$, *where* m_1^D,m_2^D,m_3^D *and* m_4^D *represent* *medical methods $Z_1(1)$, $Z_2(2)$, $Z_2(3)$ and LR using Drug $6-MP$. Next, consider the set $M^P = \{m_1^P, m_2^P, m_3^P, m_4^P\}$ as a universal set of the medical methods for Placebo effect, where m_1^P,m_2^P,m_3^P and m_4^P represent medical methods $Z_1(1)$, $Z_2(2)$, $Z_2(3)$ and LR for Placebo effect, respectively. Finally, let the set $I = \{i_1, i_2\}$, where i_1 and i_2 represent the intervals $S(10+)$ and $S(20+)$, respectively. According to the given data in the Example, we have:*

$F_1(m_1^D) = \{[0.56, 0.94]/i_1, [0.4, 0.85]/i_2\}$, $F_1(m_2^D) = \{[0.5, 0.89]/i_1, [0.37, 0.81]/i_2\}$,
$F_1(m_3^D) = \{[0.53, 0.89]/i_1, [0.39, 0.82]/i_2\}$, $F_1(m_4^D) = \{[0.54, 0.9]/i_1, [0.4, 0.82]/i_2\}$.

In addition, we have:

$F_2(m_1^P) = \{[0.17, 0.59]/i_1, [0, 0.22]/i_2\}$, $F_2(m_2^P) = \{[0.18, 0.58]/i_1, [0.016, 0.26]/i_2\}$,
$F_2(m_3^P) = \{[0.2, 0.6]/i_1, [0.024, 0.31]/i_2\}$,
$F_2(m_4^P) = \{[0.2, 0.59]/i_1, [0.016, 0.27]/i_2\}$.

Then, the VSSs (F_1, M^D) and (F_2, M^P) represent the relation matrices (Drug $6-MP$'s methods-interval matrix) $\tilde{A}$ and (Placebo's methods-interval matrix) $\tilde{B}$, respectively. They are, respectively, given by

$$\tilde{A} = \begin{array}{c} \\ m_1^D \\ m_2^D \\ m_3^D \\ m_4^D \end{array} \begin{array}{c} \begin{array}{cc} i_1 & i_2 \end{array} \\ \begin{pmatrix} [0.56, 0.94] & [0.4, 0.85] \\ [0.5, 0.89] & [0.37, 0.81] \\ [0.53, 0.89] & [0.39, 0.82] \\ [0.54, 0.9] & [0.4, 0.82] \end{pmatrix} \end{array},$$

$$\tilde{B} = \begin{array}{c} \\ m_1^P \\ m_2^P \\ m_3^P \\ m_4^P \end{array} \begin{array}{c} \begin{array}{cc} i_1 & i_2 \end{array} \\ \begin{pmatrix} [0.17, 0.59] & [0, 0.22] \\ [0.18, 0.58] & [0.016, 0.26] \\ [0.2, 0.6]] & [0.024, 0.31] \\ [0.2, 0.59] & [0.016, 0.27] \end{pmatrix} \end{array}.$$

Now, take the transpose of the VSM $\tilde{B}$ to make it possible to find the product.

$$\tilde{B}^T = \begin{array}{c} \\ i_1 \\ i_2 \end{array} \begin{array}{c} \begin{array}{cccc} m_1^P & m_2^P & m_3^P & m_4^P \end{array} \\ \begin{pmatrix} [0.17, 0.59] & [0.18, 0.58] & [0.2, 0.6]] & [0.2, 0.59] \\ [0, 0.22] & [0.016, 0.26] & [0.024, 0.31] & [0.016, 0.27] \end{pmatrix} \end{array}.$$

Then, performing the product operation $\tilde{A}\dot{\times}\tilde{B}^T$, we get the VSM $\tilde{C}$, called the diagnosis matrix (Drug $6-MP$'s methods-Placebo's methods matrix) as

$$\tilde{C} = \begin{array}{c} \\ m_1^D \\ m_2^D \\ m_3^D \\ m_4^D \end{array} \begin{array}{c} \begin{array}{cccc} m_1^P & m_2^P & m_3^P & m_4^P \end{array} \\ \begin{pmatrix} [0.0951, 0.5546] & [0.1007, 0.5452] & [0.1119, 0.564] & [0.1119, 0.5546] \\ [0.0849, 0.5251] & [0.0899, 0.5162] & [0.0999, 0.534] & [0.0999, 0.5251] \\ [0.09, 0.5251] & [0.0953, 0.5162] & [0.1059, 0.534] & [0.1059, 0.5251] \\ [0.0917, 0.531] & [0.0971, 0.522] & [0.1079, 0.54] & [0.1079, 0.531] \end{pmatrix} \end{array}.$$

From the above matrix, by comparing the intervals, we can say that the method m_2^D is the least related to the method m_2^P (the most free of Placebo effect) and the method m_3^D is the most related to the method m_3^P (the most one has Placebo effect), but we aren't sure about the methods m_1^D and m_4^D if they are very related to the methods m_1^P and m_4^P or not, respectively. So, the best method (the most effective method) to use the Drug $6-MP$ is the second one ($Z_2(2)$ method) and the worst method (the least effective method) to use the Drug $6-MP$ is the third one ($Z_2(3)$ method), but we aren't sure about the efficiency of the first method ($Z_1(1)$ method) and the fourth method (LR method). Or, using the four different methods to transform vague values into fuzzy values, we have:

$\mathbf{Method(1)}:$

$$\tilde{V_1}^F = \begin{array}{c} \\ m_1^D \\ m_2^D \\ m_3^D \\ m_4^D \end{array} \begin{array}{c} \begin{array}{cccc} m_1^P & m_2^P & m_3^P & m_4^P \end{array} \\ \begin{pmatrix} 0.3249 & 0.323 & 0.338 & 0.3333 \\ 0.305 & 0.3031 & 0.317 & 0.3125 \\ 0.3076 & 0.3058 & 0.32 & 0.3155 \\ 0.3114 & 0.3096 & 0.324 & 0.3195 \end{pmatrix} \end{array}.$$

Thus, our decision according to Method (1) is the same as the above method.

$\mathbf{Method(2)}$:

$$\tilde{V_2}^F = \begin{matrix} & m_1^P & m_2^P & m_3^P & m_4^P \\ \begin{matrix} m_1^D \\ m_2^D \\ m_3^D \\ m_4^D \end{matrix} & \multicolumn{4}{c}{\begin{pmatrix} 0.1761 & 0.1814 & 0.2043 & 0.2009 \\ 0.1518 & 0.1568 & 0.1766 & 0.1739 \\ 0.1594 & 0.1647 & 0.1853 & 0.1824 \\ 0.1636 & 0.1689 & 0.1901 & 0.1871 \end{pmatrix}} \end{matrix}.$$

According to Method (2)*, the best method to use the Drug* $6-MP$ *is the first one (*$Z_1(1)$ *method) and the worst method to use the Drug* $6-MP$ *is the third one (*$Z_2(3)$ *method).*

$\mathbf{Method(3)}$:

$$\tilde{V_3}^F = \begin{matrix} & m_1^P & m_2^P & m_3^P & m_4^P \\ \begin{matrix} m_1^D \\ m_2^D \\ m_3^D \\ m_4^D \end{matrix} & \multicolumn{4}{c}{\begin{pmatrix} 0.3708 & 0.0219 & 0.0252 & 0.0248 \\ 0.0178 & 0.0185 & 0.0213 & 0.021 \\ 0.0189 & 0.0196 & 0.0226 & 0.0222 \\ 0.0194 & 0.0202 & 0.0233 & 0.0229 \end{pmatrix}} \end{matrix}$$

According to Method (3)*, the worst methods to use the Drug* $6-MP$ *are the first one (*$Z_1(1)$ *method), the third one (*$Z_2(3)$ *method) and the fourth one (*LR *method).*

$\mathbf{Method(4)}$:

$$\tilde{V_4}^F = \begin{matrix} & m_1^P & m_2^P & m_3^P & m_4^P \\ \begin{matrix} m_1^D \\ m_2^D \\ m_3^D \\ m_4^D \end{matrix} & \multicolumn{4}{c}{\begin{pmatrix} 0.2932 & 0.2922 & 0.3092 & 0.3044 \\ 0.2715 & 0.2704 & 0.286 & 0.2816 \\ 0.2749 & 0.2741 & 0.29 & 0.2855 \\ 0.279 & 0.2781 & 0.2943 & 0.2898 \end{pmatrix}} \end{matrix}.$$

Therefore, our diagnosis' decision according to Method (4) *is the same as Method* (1)*. We can compare the results of the five methods in a table as the following:*

Our conclusion from the table is that the method m_2^D *is the least related to the method* m_2^P *(the most free of Placebo effect) and the method* m_3^D *is the most related to the method* m_3^P *(the most one has Placebo effect) followed by the method* m_4^D *which is*

The drug methods	The best (✓) and the worst (×) methods to use the drug				
	Vague value	Transformed fuzzy value			
		$Method(1)$	$Method(2)$	$Method(3)$	$Method(4)$
m_1^D	−	−	✓	×	−
m_2^D	✓	✓	−	−	✓
m_3^D	×	×	×	×	×
m_4^D	−	−	−	×	−

Table (8): Comparative results of the used methods.

also related to the method m_4^P (also has Placebo effect). But, we aren't sure about the method m_1^D if it is very related to the method m_1^P or not. So, the best method (the most effective method) to use the Drug $6-MP$ is the second one ($Z_2(2)$ method) and the worst methods (the least effective methods) to use the Drug $6-MP$ is the third one ($Z_2(3)$ method) followed by the fourth one (LR method), but we aren't sure about the efficiency of the first method ($Z_1(1)$ method).

Example 3.4. ***(Educational evaluation)*** *Take $S=\{s_1,s_2,s_3,s_4,s_5\}$ as the universal set, where s_1, s_2, s_3, s_4 and s_5 represent students Diana, Marina, Sandra, Sara and Tina, respectively. Next, consider the set $E=\{e_1,e_2,e_3,e_4,e_5\}$ as a universal set, where e_1, e_2, e_3, e_4 and e_5 represent exams English, Computer, Mathematics, Science and History, respectively, which they participated in. Finally, let the set $L=\{l_1,l_2,l_3\}$, where l_1, l_2 and l_3 represent the levels of education normal, advanced and highly advanced, respectively, which we have to place them in. Suppose that the relation matrices (student-exam matrix) $\tilde{A}$ and (exam-level matrix) $\tilde{B}$ are given, respectively, by (depending on a given VSSs (F_1,S) and (F_2,E),respectively):*

$$\tilde{A}=\begin{array}{c} \\ s_1\\ s_2\\ s_3\\ s_4\\ s_5\end{array}\begin{array}{c}\begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5\end{array}\\ \begin{pmatrix}[0.04,0.49] & [0.31,0.45] & [0.07,0.59] & [0.35,0.55] & [0.3,0.35]\\ [0.31,0.84] & [0.86,0.94] & [0.13,0.95] & [0.39,0.42] & [0.09,0.46]\\ [0.67,0.73] & [0.55,0.94] & [0.22,0.44] & [0.23,0.55] & [0.18,0.82]\\ [0.16,0.7] & [0.18,0.94] & [0.71,0.89] & [0.18,0.57] & [0.25,0.39]\\ [0.75,0.94] & [0.37,0.79] & [0.72,0.73] & [0.03,0.67] & [0.1,0.74]\end{pmatrix}\end{array}.$$

$$\tilde{B}=\begin{array}{c} \\ e_1\\ e_2\\ e_3\\ e_4\\ e_5\end{array}\begin{array}{c}\begin{array}{ccc} l_1 & l_2 & l_3\end{array}\\ \begin{pmatrix}[0.74,0.9] & [0.37,0.64] & [0.76,0.8]\\ [0.06,0.93] & [0.16,0.74] & [0.6,0.89]\\ [0.16,0.38] & [0.37,0.9] & [0.37,0.81]\\ [0.31,0.65] & [0.23,0.24] & [0.08,0.63]\\ [0.46,0.99] & [0.61,0.65] & [0.24,0.66]\end{pmatrix}\end{array}.$$

Then, the student-level matrix (student-evaluation matrix) $\tilde{C} = \tilde{A}\dot{\times}\tilde{B}$ is as follows:

$$\tilde{C} = \begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \end{array} \begin{array}{c} \begin{array}{ccc} l_1 & l_2 & l_3 \end{array} \\ \begin{pmatrix} [0.03, 0.44] & [0.02, 0.53] & [0.02, 0.48] \\ [0.05, 0.88] & [0.05, 0.86] & [0.52, 0.84] \\ [0.03, 0.88] & [0.09, 0.7] & [0.33, 0.85] \\ [0.01, 0.88] & [0.26, 0.81] & [0.1, 0.84] \\ [0.56, 0.85] & [0.27, 0.66] & [0.57, 0.75] \end{pmatrix} \end{array}.$$

From the above matrix, by comparing the intervals, we can say that Diana is qualified to take the education level advanced and Marina, Sandra, Sara and Tina are qualified to take the education level normal. Or, using the four different methods to transform vague values into fuzzy values, where $\tilde{V_1}^F$, $\tilde{V_2}^F$, $\tilde{V_3}^F$ and $\tilde{V_4}^F$ represent the matrices obtained by Method (1), Method (2), Method (3) and Method (4), respectively, as follows:

$$\tilde{V_1}^F = \begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \end{array} \begin{array}{c} \begin{array}{ccc} l_1 & l_2 & l_3 \end{array} \\ \begin{pmatrix} 0.23 & 0.28 & 0.25 \\ 0.46 & 0.45 & 0.68 \\ 0.46 & 0.39 & 0.59 \\ 0.44 & 0.54 & 0.47 \\ 0.7 & 0.46 & 0.66 \end{pmatrix} \end{array}, \quad \tilde{V_2}^F = \begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \end{array} \begin{array}{c} \begin{array}{ccc} l_1 & l_2 & l_3 \end{array} \\ \begin{pmatrix} 0.05 & 0.06 & 0.04 \\ 0.32 & 0.28 & 0.77 \\ 0.24 & 0.23 & 0.69 \\ 0.09 & 0.58 & 0.41 \\ 0.79 & 0.44 & 0.7 \end{pmatrix} \end{array},$$

$$\tilde{V_3}^F = \begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \end{array} \begin{array}{c} \begin{array}{ccc} l_1 & l_2 & l_3 \end{array} \\ \begin{pmatrix} 0.0056 & 0.0058 & 0.0052 \\ 0.02 & 0.01 & 0.17 \\ 0.01 & 0.02 & 0.11 \\ 0.004 & 0.08 & 0.03 \\ 0.19 & 0.07 & 0.17 \end{pmatrix} \end{array},$$

$$\tilde{V_4}^F = \begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \end{array} \begin{array}{c} \begin{array}{ccc} l_1 & l_2 & l_3 \end{array} \\ \begin{pmatrix} 0.19 & 0.23 & 0.21 \\ 0.45 & 0.44 & 0.7 \\ 0.44 & 0.37 & 0.61 \\ 0.42 & 0.54 & 0.47 \\ 0.72 & 0.46 & 0.67 \end{pmatrix} \end{array}.$$

There is an agreement in the evaluation's decision according to the Methods (1), (2), (3) and (4) which is that Diana and Sara are qualified to take the education level advanced, Marina and Sandra are qualified to take the education level highly advanced *and Tina is qualified to take the education level normal. We can compare the results as follows:*

The student	The level of education which she is qualified to take				
	Vague value	Transformed fuzzy value			
		$Method(1)$	$Method(2)$	$Method(3)$	$Method(4)$
s_1	l_2	l_2	l_2	l_2	l_2
s_2	l_1	l_3	l_3	l_3	l_3
s_3	l_1	l_3	l_3	l_3	l_3
s_4	l_1	l_2	l_2	l_2	l_2
s_5	l_1	l_1	l_1	l_1	l_1

Table (9): Comparative results of the used methods.

From the above table, we are sure that Diana and Tina are qualified to take the advanced education level and the normal education level, respectively.

4. CONCLUSIONS AND FUTURE WORK

Introducing only one of fuzzy set, vague set or soft set, separately (the basic extensions of set theory) has been studied by many mathematicians. On the other hand, combining any two basic extensions of them isn't only more general than using only one of them but also gives us more extended and accurate results. Many researchers have studied some of those general extensions such as FSS and VSS and have applied them in real-life decision-making problems in many fields. It is clear that some of these extensions are considered as extensions for each other not only for set theory. Using matrices to represent any of these extensions facilitates the task of applying them in examples. In our study, we defined the most general matrix related to the most general extension of set theory, recently defined, which is the VSM from another point of view. Furthermore, A case study has been taken to exhibit the simplicity of the technique. Three applied real-life decision-making medical problems were discussed by using the current method based on the most general tool which is the VSM theory to illustrate the concepts in a friendly way and give more general accurate results. In the first medical example, we used *Mathematica Program* to generate data and made all calculations and mathematical operations using *Mathematica Program*. The second medical example's data has been collected from real patients by researchers in medicine. The third example was about educational evaluation and *Mathematica Program* was also used in it. We made the final decision results by using five different methods and compared the final results obtained by them. The advantages of the proposed methods are that they are more flexible and more intelligent than the previous methods due to the fact that we use vague sets rather than fuzzy sets to represent the vague degree of symptom or the vague mark in the exam... etc., where the evaluator can use vague values to indicate the degree of the evaluator's satisfaction for each item. Especially, the proposed methods are particularly useful when the assessment involves subjective evaluation. In addition, the proposed methods are more stable to evaluate patients' symptoms or students' answers... etc. than the previous methods. They can evaluate patients' diagnoses or students' answers... etc. in a more flexible and more intelligent manner. This type of investigations fills some gaps in the literature. The authors can introduce new results by using similar techniques in this paper. There are other extensions of set theory (non-classical sets), defined by many authors other than those we have mentioned such as rough sets, hard sets and multisets, etc. which could be also combined together. Finally, we can make several applications using any of these combined extensions in many fields of science.

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CONFLICT OF INTEREST

The authors declare that they have no conflict of interest.

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