Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 15, Number 4 (2019), pp. 505-516

https://dx.doi.org/10.37622/GJPAM/15.4.2019.505-516

# On Fuzzy Z - Ideals in Z - Algebras

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**Abstract**

In this paper, we introduce the notion of fuzzy Z-ideal of a Z-algebra and investigate their properties. We study the homomorphic image and pre-image of fuzzy Z-ideals under Z-homomorphisms. We have also proved that the Cartesian product of fuzzy Z-ideals is a fuzzy Z-ideal.
**Keywords:** Z-algebra, Z-ideal, Z-homomorphism, Level Z-ideals, Fuzzy Z-Ideals, Cartesian Product of Z-algebras.

**AMS Classification 2010:** 03B47, 03B52.

## 1. INTRODUCTION

To deal with the uncertainty in the real physical world problem, the binary valued logic could not be applied. To handle this situation, Zadeh [11], in the year 1965, introduced the notion of fuzzy sets in which every member of the universal set under consideration can be assigned a value in the real interval [0,1], called the membership grading of the element. Imai and Iseki [4, 5], introduced two new classes of algebras that arise from the propositional logic. The algebras that was developed from the BCK and BCI logics are known as BCK-algebras and BCI-algebras. Since then many new algebras were developed. One such class of algebra developed from the propositional logic is the Z-algebras by Chandramouleeswaran et al.[2] in the year 2017.

In the year 1975, Rosenfeld [8] fuzzified the groups. Following the idea of fuzzy groups, Xi [10] introduced the concept of fuzzy BCK-algebras. Jun et al. [7] studied the concept of fuzzy ideals in BCK-algebras. In our earlier paper [9] we introduced the notion of fuzzy Z-Subalgebras in Z-algebras and studied some of their properties. In this paper, we introduce the concept of fuzzy Z-ideals of Z-algebras and prove some simple but elegant properties.

## 2. PRELIMINARIES

In this section we recall some basic definitions that are needed for our work.

**Definition 2.1.** *[6]* A BCK- algebra $(X, *, 0)$ is a nonempty set X with constant $0$ and a binary $*$ operation satisfying the following conditions:
(i) $(x * y) * (x * z) \leq (z * y)$
(ii)$x * (x * y) \leq y$
(iii)$x \leq x$
(iv) $x \leq y$ and $y \leq x$ implies $x = y$
(v) $0 \leq x \Rightarrow$ x=0 is defined by $x * y = 0$ for all $x, y \in X$.

**Definition 2.2.** *[5]* A BCI- algebra $(X, *, 0)$ is a nonempty set X with constant 0 and a binary operation $*$ satisfying the following conditions:
(i) $(x * y) * (x * z) \leq (z * y)$
(ii) $x * (x * y) \leq y$
(iii) $x \leq x$
(iv) $x \leq y$ and $y \leq x$ implies $x = y$
(v) $x \leq 0$ implies x=0 is defined by $x * y = 0$ for all $x, y \in X$.

**Definition 2.3.** *[2]* A Z-algebra $(X, *, 0)$ is a nonempty set X with constant 0 and a binary operation $*$ satisfying the following conditions:
(Z1) $x * 0 = 0$
(Z2) $0 * x = x$
(Z3) $x * x = x$
(Z4) $x * y = y * x$ when $x \neq 0$ and $y \neq 0$ for all $x, y \in X$.

**Definition 2.4.** *[2]* Let X be a Z-algebra and I be a subset of X. Then, I is called a Z-ideal of X, if it satisfies the following conditions: For all x, y in X,
(i) $0 \in I$
(ii) $x * y \in I$ and $y \in I$ implies $x \in I$.

**Definition 2.5.** *[2]* Let $(X, *, 0)$ and $(Y, *', 0')$ be two Z-algebras. A mapping $h : (X, *, 0) \rightarrow (Y, *', 0')$ is said to be a Z-homomorphism of Z-algebras if $h(x * y) = h(x) *' h(y)$ for all $x, y \in X$.

**Definition 2.6.** *[9]* Let h be a Z-homomorphism from the Z-algebra $(X, *, 0)$ to the Z-algebra $(Y, *^{'}, 0^{'})$.Then
1. h is called
(i) a Z-monomorphism of Z-algebras if h is 1-1.
(ii) an Z-epimorphism of Z-algebras if h is onto.
2. h is called an Z-endomorphism of Z-algebras if h is a mapping from $(X, *, 0)$ into itself.

**Note:** If $h : (X, *, 0) \to (Y, *^{'}, 0^{'})$ is a Z-homomorphism then $h(0) = 0^{'}$.

**Definition 2.7.** *[11]* Let X be a nonempty set. A fuzzy set A in X is characterized by a membership function $\mu_A$ which associates with each point x in X, a real number in the interval [0,1] with the value of $\mu_A(x)$ at x representing the "grade of membership" of x in A. That is, a fuzzy set A in X is characterized by a membership function $\mu_A : X \to [0, 1]$.

**Definition 2.8.** *[9]* Let $(X, *, 0)$ be a Z-algebra. A fuzzy set A in X with a membership function $\mu_A$ is said to be a fuzzy Z- Subalgebra of a Z-algebra X if, for all x, y in X the following condition is satisfied: $\mu_A(x * y) \geq min\{\mu_A(x), \mu_A(y)\}$.

**Definition 2.9.** *[7]* A fuzzy set A in BCK-algebra X with membership function $\mu_A$ is called a fuzzy ideal of X if it satisfies the following conditions:
(i) $\mu_A(0) \geq \mu_A(x)$
(ii) $\mu_A(x) \geq min\{\mu_A(x * y), \mu_A(y)\}$ for all $x, y \in X$.

**Definition 2.10.** *[3]* Let A be a fuzzy set of X. For a fixed $t \in [0, 1]$, the set $U(A; t) = \{x \in X | \mu_A(x) \geq t\}$ is called an upper level subset ( upper level cut, upper t-level subset) of A.

**Definition 2.11.** *[3]* Let A be a fuzzy set of X. For a fixed $t \in [0, 1]$, the set $L(A; t) = \{x \in X | \mu_A(x) \leq t\}$ is called a lower level subset (lower level cut, lower t-level subset) of A.

**Note:** (i) $t_1 \leq t_2, U(A; t_2) \subseteq U(A; t_1)$ and $L(A; t_1) \subseteq L(A; t_2)$.
(ii) $U(A; t) \cup L(A; t) = X$ for all $t \in [0, 1]$.

**Definition 2.12.** *[8]* A fuzzy set A in X with a membership function $\mu_A$ is said to have the sup property if for any subset $T \subset X$ there exists $x_0 \in X$ such that $\mu_A(x_0) = \sup\limits_{t \in T} \mu_A(t)$.

**Definition 2.13.** *[1]* A fuzzy relation A on a nonempty set X is a fuzzy set A with a membership function $\mu_A : X \times X \to [0, 1]$.

**Definition 2.14.** *[1]* If A is a fuzzy relation with a membership function $\mu_A$ on a set X and B is a fuzzy set of X with a membership function $\mu_B$ then A is a fuzzy relation on B if for all $x, y \in X$, $\mu_A(x, y) \leq min\{\mu_B(x), \mu_B(y)\}$.

**Definition 2.15.** *[1]* Let B be a fuzzy set on a set X with a membership function $\mu_B$ then the strongest fuzzy relation $A_B$ on X, that is, a fuzzy relation A on B whose membership function $\mu_{A_B} : X \times X \to [0, 1]$ is given by $\mu_{A_B}(x, y) = min\{\mu_B(x), \mu_B(y)\}$.

**Theorem 2.16.** Let $(X, *, 0)$ and $(Y, *^{'}, 0^{'})$ be two Z-algebras. Then $(X \times Y, *^{''}, 0^{''})$ is a Z-algebra where $(x_1, y_1) *^{''} (x_2, y_2) = (x_1 * x_2, y_1 *^{'} y_2)$ for all $(x_1, y_1), (x_2, y_2) \in X \times Y$, with $0^{''} = (0, 0^{'})$ as constant element.

## 3. FUZZY Z -IDEALS IN Z-ALGEBRAS:

In this section, we introduce the notion of Fuzzy Z-Ideals of Z-algebras and prove some simple but elegant results.

**Definition 3.1.** Let $(X, *, 0)$ be a Z-algebra. A fuzzy set A in X with a membership function $\mu_A$ is said to be fuzzy Z- ideal of a Z-algebra X if it satisfies the following conditions: For all x, y in X,
(i)$\mu_A(0) \geq \mu_A(x)$
(ii) $\mu_A(x) \geq min\{\mu_A(x * y), \mu_A(y)\}$

**Example 3.2.** Consider a Z-algebra X={0,1,2,3} with the following Cayley table as in [9]:

**Table 1**

| $*$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 0 | 1 | 3 | 2 |
| 2 | 0 | 3 | 2 | 1 |
| 3 | 0 | 2 | 1 | 3 |

Then $(X, *, 0)$ is a Z-algebra.

Define a fuzzy set $A_1$ and $A_2$ in X with a membership function $\mu_{A_1}$ and $\mu_{A_2}$ are given by $\mu_{A_1}(x) = 0.9$ for all x=0,1,2,3. and

$$\mu_{A_2}(x) = \begin{cases} 0.8 & \textit{if}\, x = 0 \\ 0.6 & \textit{if}\, x = 1 \\ 0.5 & \textit{if}\, x = 2, 3. \end{cases}$$

Then $A_1$ is a fuzzy Z-ideal of X , while $A_2$ is not.For,
$\mu_{A_2}(2) = 0.5 \ngeq 0.8 = min\{0.8, 0.8\} = min\{\mu_{A_2}(0), \mu_{A_2}(0)\} = min\{\mu_{A_2}(2 * 0), \mu_{A_2}(0)\}$.

**Theorem 3.3.** Arbitrary intersection of fuzzy Z-ideals of Z-algebra X is also a fuzzy Z-ideal.
**Proof:**Let $\{A_i | i \in \Omega\}$ be a family of fuzzy Z-ideals of Z-algebra X.
***To prove:*** $\cap_{i\in\Omega} A_i$ is a fuzzy Z-ideal of X.
For any $x, y \in X$,

$$\begin{aligned}
&(i)\mu_{\cap_{i\in\Omega}A_i}(0) = \inf_{i\in\Omega}(\mu_{A_i}(0)) \geq \inf_{i\in\Omega}(\mu_{A_i}(x)) = \mu_{\cap_{i\in\Omega}A_i}(x) \\
&(ii)\mu_{\cap_{i\in\Omega}A_i}(x) = \inf_{i\in\Omega}(\mu_{A_i}(x)) \\
&\qquad\qquad\qquad \geq \{\inf_{i\in\Omega}(\mu_{A_i}(x * y)), \inf_{i\in\Omega}(\mu_{A_i}(y))\} \\
&\qquad\qquad\qquad = min\{\mu_{\cap_{i\in\Omega}A_i}(x * y), \mu_{\cap_{i\in\Omega}A_i}(y)\}
\end{aligned}$$

From (i) and (ii) we get,
$\cap_{i\in\Omega} A_i$ is a fuzzy Z-ideal of X.
Hence the proof.

**Theorem 3.4.** A fuzzy set A of a Z-algebra $(X, *, 0)$ is a fuzzy Z-ideal if and only if for any $t \in [0, 1]$, $U(A; t) = \{x \in X | \mu_A(x) \geq t\}$ is an Z-ideal of X where $U(A; t) \neq \phi$.
***Proof:***Suppose A is a fuzzy Z-ideal of X and $U(A; t) \neq \phi$ for $t \in [0, 1]$.
Let $x \in U(A; t)$, then $\mu_A(x) \geq t$ .
By definition of fuzzy Z-ideal, we have $\mu_A(0) \geq \mu_A(x) \geq t$. Then $0 \in U(A; t)$.
If $x * y \in U(A; t)$ and $y \in U(A; t)$, then $\mu_A(x * y) \geq t$ and $\mu_A(y) \geq t$.
By definition, we have $\mu_A(x) \geq min\{\mu_A(x * y), \mu_A(y)\} \geq min\{t, t\} = t$.
Therefore $x \in U(A; t)$. Hence U(A;t) is an Z-ideal of X.
Conversely, suppose that for each $t \in [0, 1]$, U(A;t) is either empty or an Z-ideal of X.
For any $x \in X$, let $\mu_A(x) = t$. Then $x \in U(A; t)$
Since $U(A; t) \neq \phi$ is an Z-ideal of X, we have $0 \in U(A; t)$
and hence $\mu_A(0) \geq t = \mu_A(x)$.
Thus $\mu_A(0) \geq \mu_A(x)$, for all $x \in X$.
Assume that $\mu_A(x) \geq min\{\mu_A(x * y), \mu_A(y)\}$ for all $x, y \in X$ is not true.
Then there exists $x_0, y_0 \in X$ such that $\mu_A(x_0) < min\{\mu_A(x_0 * y_0), \mu_A(y_0)\}$.
Let $t_0 = \frac{1}{2}[\mu_A(x_0) + min\{\mu_A(x_0 * y_0), \mu_A(y_0)\}]$.

Then $\mu_A(x_0) < t_0 < min\{\mu_A(x_0 * y_0), \mu_A(y_0)\}$. This implies $x_0 * y_0, y_0 \in U(A; t_0)$ and $x_0 \notin U(A; t_0)$ .
But $U(A; t_0)$ is an Z-ideal of X. So $x_0 \in U(A; t_0)$ by the definition of Z-ideal. This implies $\mu_A(x_0) \geq t_0$.This is a contradiction.
Therefore $\mu_A(x) \geq min\{\mu_A(x * y), \mu_A(y)\}$. Hence A is a fuzzy Z-ideal of Z-algebra X.

**Definition 3.5.** Let A be a fuzzy Z-ideal of X. For any $t \in [0, 1]$, Z-ideals U(A;t) are called Upper level Z-ideals of A.

**Remark 3.6.** Henceforth, the Upper level Z-ideals will be referred as level Z-ideals.

**Theorem 3.7.** A fuzzy set A of a Z-algebra $(X, *, 0)$ is a fuzzy Z-ideal if and only if every nonempty level subset of U(A;q), $q \in Im(A)$ is a Z-ideal.
***Proof:*** Let A is a fuzzy Z-ideal.
***Claim:*** U(A;q), $q \in Im(A)$ is a Z-ideal.
Since $U(A; q) \neq \phi$ there exists $x \in U(A; q)$ such that $\mu_A(x) \geq q$ .
Since A is a fuzzy Z-ideal, $\mu_A(0) \geq \mu_A(x)$ for all $x \in X$.
Hence for this $x \in U(A; q)$, $\mu_A(0) \geq q$, which shows that $0 \in U(A; q)$.
Now, for any $x, y \in X$, assume that $x * y \in U(A; q)$ and $y \in U(A; q)$.
Then $\mu_A(x * y) \geq q$ and $\mu_A(y) \geq q$. This shows that, $min\{\mu_A(x * y), \mu_A(y)\} \geq q$.
Since A is a fuzzy Z-ideal, $\mu_A(x) \geq min\{\mu_A(x * y), \mu_A(y)\} \geq q$.
Thus $x \in U(A; q)$, this proves that U(A;q) is a Z-ideal of X.
Conversely, let U(A;q), $q \in Im(A)$ be a Z-ideal of X.
**Claim:** A is a fuzzy Z-ideal.
Let $x, y \in X$. For any $q \in Im(A)$, let $q = min\{\mu_A(x * y), \mu_A(y)\}$.
Therefore, $\mu_A(x * y) \geq q$ and $\mu_A(y) \geq q$.
This shows that $x * y, y \in U(A; q)$.
Since U(A;q) is a Z-ideal we have $x \in U(A; q)$.
This proves that $\mu_A(x) \geq q = min\{\mu_A(x * y), \mu_A(y)\}$.
This shows that A is a fuzzy Z-ideal of X.

**Theorem 3.8.** Let A be a fuzzy Z-ideal of Z-algebra X and let $x \in X$. Then $\mu_A(x) = t$ if and only if $x \in U(A; t)$ but $x \notin U(A; q)$ for all $q > t$.
***Proof:*** Let A be a fuzzy Z-ideal of X and let $x \in X$.
Assume $\mu_A(x) = t$, so that $x \in U(A; t)$.
If possible, let $x \in U(A; q)$ for $q > t$.
Then $\mu_A(x) \geq q > t$.
This contradicts the fact that $\mu_A(x) = t$. Hence $x \neq U(A; q)$ for all $q > t$.
Conversely, let $x \in U(A; t)$, but $x \notin U(A; q)$ for all $q > t$.
$x \in U(A; t) \Rightarrow \mu_A(x) \geq t$.
Since $x \notin U(A; q)$ for all $q > t$, $\mu_A(x) = t$.

## 4. Z-HOMOMORPHISM ON FUZZY Z-IDEALS IN Z-ALGEBRAS

In this section we prove some theorems on fuzzy Z-ideals under Z-homomorphisms in Z-algebras.

**Example 4.1.** Consider the Z-algebras $(X, *, 0)$ and $(Y, *^{'}, 0^{'})$ with the following Cayley table:

**Table 2**

| $*$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 0 | 1 | 3 | 2 |
| 2 | 0 | 3 | 2 | 1 |
| 3 | 0 | 2 | 1 | 3 |

**Table 3**

| $*^{'}$ | $0^{'}$ | 1 | 2 | 3 |
|---|---|---|---|---|
| $0^{'}$ | $0^{'}$ | 1 | 2 | 3 |
| 1 | $0^{'}$ | 1 | 1 | 3 |
| 2 | $0^{'}$ | 1 | 2 | 1 |
| 3 | $0^{'}$ | 3 | 1 | 3 |

Then the function $h : (X, *, 0) \to (Y, *^{'}, 0^{'})$ such that

$$h(x) = \begin{cases} 0 & \textit{if } x = 0 \\ 3 & \textit{if } x = 1, 2, 3. \end{cases}$$

is a Z-homomorphism.

Define a fuzzy set A in X with membership function $\mu_A$ is given by $\mu_A(x) = 0.4$ for all $x \in X$ is a fuzzy Z-ideal of X.
Then the homomorphic image of A, h(A) with a membership function $\mu_{h(A)}$ defined by

$$\mu_{h(A)}(y) = \begin{cases} \sup\limits_{z \in h^{-1}(y)} \mu_A(z) & \text{if } h^{-1}(y) = \{x | h(x) = y \neq \phi\} \\ 0 & \text{if } otherwise \end{cases}$$

is a fuzzy set in Y.
This implies, $\mu_{h(A)}(y) = 0.4$ for all $y \in Y$ is a fuzzy Z-ideal of Y.

**Theorem 4.2.** Let h be a Z-homomorphism from a Z –algebra $(X, *, 0)$ onto a Z-algebra $(Y, *^{'}, 0^{'})$ and A be a fuzzy Z-ideal of X with the supremum property. Then image of A denoted by h(A) is a fuzzy Z-ideal of Y.
***Proof:*** Let $a, b \in Y$ with $x_0 \in h^{-1}(a)$ and $y_0 \in h^{-1}(b)$ such that $\mu_A(x_0) = \sup_{t \in h^{-1}(a)} \mu_A(t)$; $\mu_A(y_0) = \sup_{t \in h^{-1}(b)} \mu_A(t)$ .
$(i) \mu_{h(A)}(0^{'}) = \sup_{t \in h^{-1}(0')} \mu_A(t) \geq \mu_A(0) \geq \mu_A(x_0) = \sup_{t \in h^{-1}(a)} \mu_A(t) = \mu_{h(A)}(a)$

$$\begin{aligned} (ii) min\{\mu_{h(A)}(a * b), \mu_{h(A)}(b)\} &= min\{\sup_{t \in h^{-1}(a*b)} \mu_A(t), \sup_{t \in h^{-1}(b)} \mu_A(t)\} \\ &= min\{\mu_A(x_0 * y_0), \mu_A(y_0)\} \\ &\leq \mu_A(x_0) \\ &= \sup_{t \in h^{-1}(a)} \mu_A(t) \\ &= \mu_{h(A)}(a) \end{aligned}$$

This implies, $\mu_{h(A)}(a) \geq min\{\mu_{h(A)}(a * b), \mu_{h(A)}(b)\}$.
Hence h(A) is a fuzzy Z-ideal of Y.

**Theorem 4.3.** Let $h : X \to Y$ be a homomomorphism of Z-algebra. If B is a fuzzy Z-ideal of Y, then $h^{-1}(B)$ is a fuzzy Z-ideal of X.
**Proof:**Since B is a fuzzy Z-ideal of Y. For any $x \in X$, we have
$(i) \mu_{h^{-1}(B)}(x) = \mu_B(h(x)) \leq \mu_B(0^{'}) = \mu_B(h(0)) = \mu_{h^{-1}(B)}(0)$
(ii) Let $x, y \in X$. Then

$$\begin{aligned} min\{\mu_{h^{-1}(B)}(x * y), \mu_{h^{-1}(B)}(y)\} &= min\{\mu_B(h(x * y)), \mu_B(h(y))\} \\ &= min\{\mu_B(h(x) *^{'} h(y)), \mu_A(h(y))\} \\ &\leq \mu_B(h(x)) \\ &= \mu_{h^{-1}(B)}(x) \end{aligned}$$
$$\Rightarrow \mu_{h^{-1}(B)}(x) \geq min\{\mu_{h^{-1}(B)}(x * y), \mu_{h^{-1}(B)}(y)\}$$

From (i) and (ii) we get, $h^{-1}(B)$ is a fuzzy Z-ideal of X.

**Theorem 4.4.** Let $h : X \to Y$ be an Z-epimorphism of Z-algebras. Let B be a fuzzy set of Y. If $h^{-1}(B)$ is a fuzzy Z-ideal of X then B is a fuzzy Z-ideal of Y.
**Proof:**Assume that $h^{-1}(B)$ is a fuzzy Z-ideal of X.
**To prove:**B is a fuzzy Z-ideal of Y.
Let $y \in Y$, there exists $x \in X$ such that h(x)=y . Then
$(i) \mu_B(y) = \mu_B(h(x)) = \mu_{h^{-1}(B)}(x) \leq \mu_{h^{-1}(B)}(0) = \mu_B(h(0)) = \mu_B(0^{'})$
This implies $\mu_B(0^{'}) \geq \mu_B(y)$.
(ii) Let $x, y \in Y$. Then there exists $a, b \in X$ such that h(a)=x and h(b)=y. It follows that

$$\begin{aligned}\mu_B(x) = \mu_B(h(a)) &= \mu_{h^{-1}(B)}(a) \\ &\geq min\{\mu_{h^{-1}(B)}(a * b), \mu_{h^{-1}(B)}(b)\} \\ &= min\{\mu_B(h(a * b)), \mu_B(h(b))\} \\ &= min\{\mu_B(h(a) *' h(b)), \mu_B(h(b))\} \\ &= min\{\mu_B(x *' y), \mu_B(y)\}\end{aligned}$$

This implies, $\mu_B(x) \geq min\{\mu_B(x *' y), \mu_B(y)\}$

From (i) and (ii) we get, B is a fuzzy Z-ideal of Y.

**Definition 4.5.** Let h be an Z-endomorphism of Z-algebras and A be a fuzzy set in X. We define a new fuzzy set $A^h$ in X as $\mu_{A^h}(x) = \mu_A(h(x))$ for all $x \in X$.

**Theorem 4.6.** Let h be an Z-endomorphism of Z-algebra X and A be a fuzzy set in X. Then $A^h$ is a fuzzy Z-ideal of X if A is a fuzzy Z-ideal.
**Proof:**Obvious.

## 5. CARTESIAN PRODUCT OF FUZZY Z-IDEALS IN Z-ALGEBRAS

In this section we discuss the concept of Cartesian product of fuzzy Z-ideals in Z-algebras.

**Theorem 5.1.** If A and B be fuzzy Z-ideals in a Z-algebra X then $A \times B$ is a fuzzy Z-ideals in $X \times X$.
**Proof:** Let A and B be fuzzy Z-ideals in a Z-algebra X.
**To prove:** $A \times B$ is a fuzzy Z-ideals in $X \times X$.
(i) Let $(x_1, x_2) \in X \times X$,
$\mu_{A\times B}(0, 0) = min\{\mu_A(0), \mu_B(0)\} \geq min\{\mu_A(x_1), \mu_B(x_2)\} = \mu_{A\times B}(x_1, x_2)$
Hence $\mu_{A\times B}(0, 0) \geq \mu_{A\times B}(x_1, x_2)$
(ii) Let $(x_1, x_2), (y_1, y_2) \in X \times X$. Then,

$$\begin{aligned}\mu_{A\times B}(x_1, x_2) &= min\{\mu_A(x_1), \mu_B(x_2)\} \\ &\geq min\{min\{\mu_A(x_1 * y_1), \mu_A(y_1)\}, min\{\mu_B(x_2 * y_2), \mu_B(y_2)\}\} \\ &= min\{min\{\mu_A(x_1 * y_1), \mu_B(x_2 * y_2)\}, min\{\mu_A(y_1), \mu_B(y_2)\}\} \\ &= min\{\mu_{A\times B}((x_1 * y_1), (x_2 * y_2), \mu_{A\times B}(y_1, y_2)\} \\ &= min\{\mu_{A\times B}((x_1, x_2) * (y_1, y_2)), \mu_{A\times B}(y_1, y_2)\}\end{aligned}$$

Hence $\mu_{A\times B}(x_1, x_2) \geq min\{\mu_{A\times B}((x_1, x_2) * (y_1, y_2)), \mu_{A\times B}(y_1, y_2)\}$

By (i) and (ii) we get, $A \times B$ is a fuzzy Z-ideal in $X \times X$.

**Theorem 5.2.** Let A and B be fuzzy sets in a Z-algebra X such that $A \times B$ is a fuzzy Z-ideal of $X \times X$. Then,
(i) Either $\mu_A(0) \geq \mu_A(x)$ (or) $\mu_B(0) \geq \mu_B(x)$ for all $x \in X$.

(ii) If $\mu_A(0) \geq \mu_A(x)$ for all $x \in X$, then either $\mu_B(0) \geq \mu_A(x)$ (or) $\mu_B(0) \geq \mu_B(x)$.
(iii) If $\mu_B(0) \geq \mu_B(x)$ for all $x \in X$, then either $\mu_A(0) \geq \mu_A(x)$ (or) $\mu_A(0) \geq \mu_B(x)$.
**Proof:** Let A and B be fuzzy sets in a Z-algebra X such that $A \times B$ is a fuzzy Z-ideal of $X \times X$.
(i) If $\mu_A(0) < \mu_A(x_1)$ and $\mu_B(0) < \mu_B(x_2)$ for some $x \in X$.

$$\begin{aligned} \text{Then } \mu_{A\times B}(x_1, x_2) &= min\{\mu_A(x_1), \mu_B(x_2)\} \\ &> min\{\mu_A(0), \mu_B(0)\} \\ &= \mu_{A\times B}(0,0), \text{ which is a contradiction.} \end{aligned}$$

Hence, either $\mu_A(0) \geq \mu_A(x)$ (or) $\mu_B(0) \geq \mu_B(x)$ for all $x \in X$.
(ii) Let $\mu_A(0) \geq \mu_A(x)$ for all $x \in X$.
**To prove:**Either $\mu_B(0) \geq \mu_A(x)$ (or) $\mu_B(0) \geq \mu_B(x)$.
Assume that there exists $x_1, x_2 \in X$ such that $\mu_B(0) < \mu_A(x_1)$ and $\mu_B(0) < \mu_B(x_2)$.

$$\begin{aligned} \text{Then, } \mu_{A\times B}(0,0) &= min\{\mu_A(0), \mu_B(0)\} \\ &= \mu_B(0) \end{aligned}$$

$$\begin{aligned} \mu_{A\times B}(x_1, x_2) &= min\{\mu_A(x_1), \mu_B(x_2)\} \\ &> \mu_B(0) \\ &= \mu_{A\times B}(0,0) \end{aligned}$$

$\Rightarrow \mu_{A\times B}(x_1, x_2) > \mu_{A\times B}(0,0)$, which is a contradiction.

Hence, either $\mu_B(0) \geq \mu_A(x)$ (or) $\mu_B(0) \geq \mu_B(x)$.
(iii) will obtain by interchanging the roles of A and B in part (ii).
Hence the proof.

**Theorem 5.3.** Let A and B be fuzzy sets in a Z-algebra X and $A \times B$ is fuzzy Z-ideal of $X \times X$ then either A or B is a fuzzy Z-ideal of X.
**Proof:** Let A and B be fuzzy sets in a Z-algebra X and $A \times B$ is fuzzy Z-ideal of $X \times X$.
**To prove:**B is a fuzzy Z-ideal of X.
By Theorem 5.2(i), we can assume that $\mu_B(0) \geq \mu_B(x)$ for all $x \in X$. then by Theorem 5.2 (iii), either $\mu_A(0) \geq \mu_A(x)$ (or) $\mu_A(0) \geq \mu_B(x)$.
Let $\mu_A(0) \geq \mu_B(x)$ for any $x \in X$, then

$$\begin{aligned} \mu_B(x) &= min\{\mu_A(0), \mu_B(x)\} \\ &= \mu_{A\times B}(0, x) \\ &\geq min\{\mu_{A\times B}((0, x) * (0, y)), \mu_{A\times B}(0, y)\} \\ &= min\{\mu_{A\times B}((0 * 0), (x * y)), \mu_{A\times B}(0, y)\} \\ &= min\{\mu_{A\times B}(0, (x * y)), \mu_{A\times B}(0, y)\} \\ &= min\{min\{\mu_A(0), \mu_B(x * y)\}, min\{\mu_A(0), \mu_B(y)\}\} \\ &= min\{\mu_B(x * y), \mu_B(y)\} \end{aligned}$$

Therefore, $\mu_B(x) \geq min\{\mu_B(x * y), \mu_B(y)\}$
Hence B is a fuzzy Z-ideal of X.
By Theorem 5.2 (i), assume that $\mu_A(0) \geq \mu_A(x)$ for all $x \in X$.
By Theorem 5.2 (ii), assume that $\mu_B(0) \geq \mu_A(x)$ for any $x \in X$.
Then A is a fuzzy Z-ideal of X.
This completes the proof.

**Theorem 5.4.** Let $A_B$ be the strongest fuzzy relation on Z-algebra X, where B is a fuzzy set of X. If B is a fuzzy Z-ideal of a Z-algebra X, then $A_B$ is a fuzzy Z-ideal of $X \times X$.
**Proof:**Let B be a fuzzy Z-ideal of a Z-algebra X.
Let $(x_1, x_2), (y_1, y_2) \in X \times X$.
Then $\mu_{A_B}(0,0) = min\{\mu_B(0), \mu_B(0)\}$

$$\begin{aligned} &\geq min\{\mu_B(x_1), \mu_B(x_2)\} \\ &= \mu_{A_B}(x_1, x_2) \end{aligned}$$

and also $\mu_{A_B}(x_1, x_2) = min\{\mu_B(x_1), \mu_B(x_2)\}$

$$\begin{aligned} &\geq min\{min\{\mu_B(x_1 * y_1), \mu_B(y_1)\}, min\{\mu_B(x_2 * y_2), \mu_B(y_2)\}\} \\ &= min\{min\{\mu_B(x_1 * y_1), \mu_B(x_2 * y_2)\}, min\{\mu_B(y_1), \mu_B(y_2)\}\} \\ &= min\{\mu_{A_B}((x_1 * y_1), (x_2 * y_2), \mu_{A_B}(y_1, y_2)\} \\ &= min\{\mu_{A_B}((x_1, x_2) * (y_1, y_2)), \mu_{A_B}(y_1, y_2)\} \end{aligned}$$

Hence $A_B$ is a fuzzy Z-ideal of $X \times X$.

**Theorem 5.5.** If the strongest fuzzy relation $A_B$ is a fuzzy Z-ideal of $X \times X$, then B is a fuzzy Z-ideal of a Z-algebra X.
**Proof:** Let $A_B$ is a fuzzy Z-ideal of $X \times X$. Then for all $(x_1, x_2), (y_1, y_2) \in X \times X$.
$min\{\mu_B(0), \mu_B(0)\} = \mu_B(0, 0) \geq \mu_{A_B}(x_1, x_2) = min\{\mu_B(x_1), \mu_B(x_2)\}$
Then, $\mu_B(0) \geq min\{\mu_B(x_1), \mu_B(x_2)\}$
$\Rightarrow \mu_B(0) \geq \mu_B(x_1)$ or $\mu_B(0) \geq \mu_B(x_2)$ for all $x_1, x_2 \in X$.

Also,

$$\begin{aligned} min\{\mu_B(x_1), \mu_B(x_2)\} &= \mu_{A_B}(x_1, x_2) \\ &\geq min\{\mu_{A_B}((x_1, x_2) * (y_1, y_2)), \mu_{A_B}(y_1, y_2)\} \\ &= min\{\mu_{A_B}((x_1 * y_1), (x_2 * y_2), \mu_{A_B}(y_1, y_2)\} \\ &= min\{min\{\mu_B(x_1 * y_1), \mu_B(x_2 * y_2)\}, min\{\mu_B(y_1), \mu_B(y_2)\}\} \\ &= min\{min\{\mu_B(x_1 * y_1), \mu_B(y_1)\}, min\{\mu_B(x_2 * y_2), \mu_B(y_2)\}\} \end{aligned}$$

Put $x_2 = y_2 = 0$, we get $\mu_B(x_1) \geq min\{\mu_B(x_1 * y_1), \mu_B(y_1)\}$
Hence B is a fuzzy Z-ideal of a Z-algebra X.

## 6. CONCLUSION

In this article, we have introduced fuzzy Z-ideals in Z-algebras and discussed their properties. We extend this concept in our research work.

**Acknowedgement**
Authors wish to thank **Dr.M.Chandramouleeswaran**, Professor and Head, PG Department of Mathematics, Sri Ramanas College of Arts and Science for Women, Aruppukottai, for his valuable suggestions to improve this paper a successful one.

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