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An Optimal Three-Step Iterative Method Free From Derivative for Solving Nonlinear Equations

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Abstract

This article discusses a three-step iterative method in which no derivative is required for solving nonlinear equations. The method analitically shows that it has order eight and requires four evaluation functions for each iteration. The proposed method is optimal in the sense of Kung and Traub's conjecture and has the efficiency index 1.682. Numerical experiments show that the new method is comparable with other discussed methods.

Mathematics Subject Classification: 65H05, 65H99

Keywords: Iterative method, derivative-free method, nonlinear equation, efficiency index, order of convergence

1. INTRODUCTION

Numerical method has an important role for solving several mathematical problems. One of the most basic problems in mathematics is finding the root of nonlinear equation of the form

$$f(x) = 0. \tag{1}$$

Newton's method is a famous method in solving equation (1). The method needs to compute a function and its derivative for each iteration and it has quadratically convergence [19]. If the derivative of f in Newton's method is estimated by a forward difference, then the Newton's method becomes Steffensen's method [17] having the same order of convergence and the number of function evaluations as the Newton's method. Based on Kung and Traub's conjecture [9] Newton's and Steffensen's methods are optimal iterative method because both have quadratically convergence and require two evaluation functions for each iteration.

In the recent years, a large number of iterative methods have been modified to obtain a derivative-free method with higher order convergence and optimal, for example Soleymani et al. [16] and Solaimani et al. [15].

In this article, a new iterative method is established based on the optimal eight-order iterative method by Sharma and Arora [11] as follows:

$$\left.\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{f'(x_n)f(y_n)}{f[y_n,x_n]^2}, \\ x_{n+1} &= z_n - \frac{f[z_n,y_n]}{f[z_n,x_n]}\frac{f(z_n)}{2f[z_n,y_n]-f[z_n,x_n]}, \end{aligned}\right\} \tag{2}$$

by approximating the first derivative in (2) using divided difference of order one with one parameter. The discussion begin in section 2 by constructing the method and it is followed by the convergence analysis of the method. In section 3, the computational tests of the proposed method is performed to see the effective of the proposed method by comparing with some other optimal eighth-order derivative free iterative methods.

2. AN OPTIMAL THREE-STEP ITERATIVE METHOD FREE FROM DERIVATIVE

If the first derivative of f in the first and second steps of (2) are approximated using divided difference with one-parameter β, that is

$$f'(x_n) \approx f[w_n, x_n]$$

where $w_n = x_n + \beta f(x_n)^3$ and $\beta \neq 0$ [1], then the following new iteration method is obtained

$$w_n = x_n + \beta f(x_n)^3, \tag{3}$$

$$\left.\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f[w_n,x_n]}, \\ z_n &= y_n - \frac{f[w_n,x_n]f(y_n)}{f[y_n,x_n]^2}, \\ x_{n+1} &= z_n - \frac{f[z_n,y_n]}{f[z_n,x_n]}\frac{f(z_n)}{2f[z_n,y_n]-f[z_n,x_n]}. \end{aligned}\right\} \tag{4}$$

In the following, the analysis of convergence of the proposed method (4) is performed as stated in Theorem 1.

Theorem 1 Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a sufficiently differentiable function for the open interval I and $\alpha \in I$ be a simple root of $f(x) = 0$. If x_0 is sufficiently close to α then the method defined by (4) has an eight-order convergence, and satisfies the following error equation:

$$e_{n+1} = -A_2(f'(\alpha)^3 A_2^2 \beta A_3 + A_2 A_4 A_3 + 2A_2^2 A_3^2 - A_3^3 - 2A_2^3 A_4 - 2f'(\alpha)^3 A_2^4 \beta)e_n^8 + O(e_n^9), \tag{5}$$

where $A_j = \frac{f^{(j)}(\alpha)}{j! f'(\alpha)}$, $j = 2, 3, \ldots, 8$ and $e_n = x_n - \alpha$.

Proof. Let α be a simple root of $f(x) = 0$, then $f(\alpha) = 0$ and $f'(\alpha) \neq 0$. Using Taylor's expansion [4] of $f(x)$ about $x = \alpha$, we have

$$\begin{aligned} f(x) = f(\alpha) + f'(\alpha)(x-\alpha) + f^{(2)}(\alpha)\frac{(x-\alpha)^2}{2!} + f^{(3)}(\alpha)\frac{(x-\alpha)^3}{3!} \\ + f^4(\alpha)\frac{(x-\alpha)^4}{4!} + f^5(\alpha)\frac{(x-\alpha)^5}{5!} + f^{6)}(\alpha)\frac{(x-\alpha)^6}{6!} \\ + f^{(7)}(\alpha)\frac{(x-\alpha)^7}{7!} + f^{8)}(\alpha)\frac{(x-\alpha)^8}{8!} + O((x-\alpha)^9). \end{aligned} \tag{6}$$

Since $f(\alpha) = 0$, then by evaluating (6) in $x = x_n$ and considering $e_n = x_n - \alpha$, we obtain

$$f(x_n) = f'(\alpha)(e_n + A_2 e_n^2 + A_3 e_n^3 + A_4 e_n^4 + A_5 e_n^5 + A_6 e_n^6 + A_7 e_n^7 + A_8 e_n^8) + O(e_n^9), \tag{7}$$

where

$$A_j = \frac{f^{(j)}(\alpha)}{j! f'(\alpha)}, \qquad j = 2, 3, \ldots, 8.$$

Using (7), we have

$$\begin{aligned} f(x_n)^3 = f'(\alpha)^3 \Big(& e_n^3 + 3A_2 e_n^4 + (3A_3 + 3A_2^2)e_n^5 + (3A_4 + 6A_2A_3 + A_2^3)e_n^6 \\ & + (3A_3^2 + 3A_5 + 3A_2^2 A_3 + 6A_2A_4)e_n^7 \\ & + (6A_2A_5 + 3A_2^2 A_4 + 6A_3A_4 + 3A_2A_3^2 + 3A_6)e_n^8 \Big) + O(e_n^9). \end{aligned} \tag{8}$$

Since $x_n = e_n + \alpha$ and by substituting (8) into (3), we obtain

$$\begin{aligned} w_n = e_n + \alpha + \beta f'(\alpha)^3 \Big(& e_n^3 + 3A_2 e_n^4 + (3A_3 + 3A_2^2)e_n^5 + (3A_4 + 6A_2A_3 + A_2^3)e_n^6 \\ & + (3A_3^2 + 3A_5 + 3A_2^2 A_3 + 6A_2A_4)e_n^7 + (6A_2A_5 + 3A_2^2 A_4 \\ & + 6A_3A_4 + 3A_2A_3^2 + 3A_6)e_n^8 \Big) + O(e_n^9). \end{aligned} \tag{9}$$

By evaluating (6) in $x = w_n$ in a similar way and using (9), we get

$$\begin{aligned} f(w_n) = f'(\alpha)\Big(& e_n + A_2 e_n^2 + (A_3 + \beta f'(\alpha)^3)e_n^3 + \cdots + (9\beta f'(\alpha)^3 A_6 + 15A_2^3 f'(\alpha)^6 \beta^2 \\ & + 24A_2 f'(\alpha)^6 \beta^2 A_3 + \cdots + 27\beta f'(\alpha)^3 A_3 A_4)e_n^8 \Big) + O(e_n^9). \end{aligned} \tag{10}$$

From (7) and (10), we obtain

$$
\begin{aligned}
f(w_n) - f(x_n) = {} & \beta f'(\alpha)^4 e_n^3 + 5\beta f'(\alpha)^4 A_2 e_n^4 + (9\beta f'(\alpha)^4 A_2^2 + 6\beta f'(\alpha)^4 A_3) e_n^5 + \cdots \\
& + (6A_4 f'(\alpha)^7 \beta^2 + 27\beta f'(\alpha)^4 A_3 A_4 + 9\beta f'(\alpha)^4 A_6 + \cdots \\
& + 24A_2 f'(\alpha)^7 \beta^2 A_3 + 9A_2^3 f'(\alpha)^4 \beta A_3 + 27\beta f'(\alpha)^4 A_2 A_3^2) e_n^8 \\
& + O(e_n^9),
\end{aligned}
\tag{11}
$$

and from (9), we have

$$
\begin{aligned}
w_n - x_n = {} & \beta f'(\alpha)^3 e_n^3 + 3\beta A_2 f'(\alpha)^3 e_n^4 + \beta(3A_3 + 3A_2^2) f'(\alpha)^3 e_n^5 + \beta(3A_4 + 6A_2 A_3 \\
& + A_2^3) f'(\alpha)^3 e_n^6 + \beta(3A_5 + 3A_3^2 + 3A_2^2 A_3 + 6A_2 A_4) f'(\alpha)^3 e_n^7 \\
& + \beta(3A_2^2 A_4 + 3A_6 + 6A_3 A_4 + 6A_2 A_5 + 3A_2 A_3^2) f'(\alpha)^3 e_n^8 \\
& + O(e_n^9).
\end{aligned}
\tag{12}
$$

Applying divided difference formula [10] and using (11) and (12) gives rise to

$$
f[w_n, x_n] = \frac{f'(\alpha)\Big(1 + 5A_2 e_n + B e_n^2 + C e_n^3 + D e_n^4 + E e_n^5\Big)}{1 + 3A_2 e_n + F e_n^2 + G e_n^3 + H e_n^4 + I e_n^5}, \tag{13}
$$

where

$$
\begin{aligned}
B = {} & 9A_2^2 + 6A_3, \\
C = {} & 7A_2^3 + 21A_2 A_3 + \beta A_2 f'(\alpha)^3 + 7A_4, \\
D = {} & 3f'(\alpha)^3 A_3 \beta + 6f'(\alpha)^3 A_2^2 \beta + 12A_3^2 + 24A_2 A_4 + 2A_2^4 + 8A_5 + 24A_2^2 A_3, \\
E = {} & 6f'(\alpha)^3 A_4 \beta + 15f'(\alpha)^3 A_2^3 \beta + 24f'(\alpha)^3 A_2 \beta A_3 + 27A_3 A_4 + 9A_6 + 27A_2^2 A_4 \\
& + 27A_2 A_5 + 27A_2 A_3^2 + 9A_2^3 A_3, \\
F = {} & 3A_3 + 3A_2^2, \\
G = {} & 3A_4 6A_2 A_3 + A_2^3, \\
H = {} & 3A_3^2 + 3A_5 + 3A_2^2 A_3 + 6A_2 A_4, \\
I = {} & 6A_2 A_5 + 3A_2^2 A_4 + 6A_3 A_4 + 3A_2 A_3^2 + 3A_6.
\end{aligned}
$$

Equation (13) can be simplified by using a geometric series formula [18], and we get

$$
\begin{aligned}
f[w_n, x_n] = {} & f'(\alpha) + 2f'(\alpha) A_2 e_n + 3f'(\alpha) A_3 e_n^2 + (\beta f'(\alpha)^4 A_2 + 4A_4 f'(\alpha)) e_n^3 + \cdots \\
& + (-18A_4^2 f'(\alpha)^4 \beta - 135 f'(\alpha)^4 A_2^4 \beta A_3 + 18A_3^3 f'(\alpha)^4 \beta - \cdots \\
& - 15 f'(\alpha) A_5^2 + 117 f'(\alpha) A_6 A_2 A_3 + 240 f'(\alpha) A_2 A_3^2 A_4) e_n^8 + O(e_n^9).
\end{aligned}
\tag{14}
$$

On substituting (7), (14) and $x_n = e_n + \alpha$ in the first step of (4) yields

$$y_n = \alpha + A_2^2 e_n^2 + (2A_3 - 2A_2^2)e_n^3 + \cdots + (236A_2^4 A_4 - 45A_2^2 A_4 A_3 + 170A_6 A_2^2 + \cdots - 6\beta^2 A_2 f'(\alpha)^6 A_3 + 36f'(\alpha)^3 A_2 \beta A_3^2 + 294f'(\alpha)^3 A_2^3 \beta A_3)e_n^8 + O(e_n^9). \tag{15}$$

On substituting $x = y_n$ in (6) and using (15), after simplifying it is obtained that

$$f(y_n) = A_2 f'(\alpha)e_n^2 + (2A_3 f'(\alpha) - 2f'(\alpha)A_2^2)e_n^3 + \cdots + (45f'(\alpha)A_3^2 A_4 - 12\beta f'(\alpha)^4 A_2 A_5 + 121f'(\alpha)A_2 A_4^2 - \cdots - 78\beta f'(\alpha)^4 A_3 A_4 - 102f'(\alpha)A_2 A_3^3 + 126f'(\alpha)^4 A_2^5 \beta)e_n^8 + O(e_n^9). \tag{16}$$

Similar to (14), using (16), (7), (15) and $x_n = e_n + \alpha$, we obtain

$$f[y_n, x_n] = f'(\alpha) + f'(\alpha)A_2 e_n + (A_3 f'(\alpha) + f'(\alpha)A_2^2)e_n^2 + (-2f'(\alpha)A_2^3 + A_4 f'(\alpha) + 3f'(\alpha)A_2 A_3)e_n^3 + \cdots + (6A_4^2 f'(\alpha)^4 \beta - 3A_2^4 f'(\alpha)^7 \beta^2 + f'(\alpha)A_7 A_3 + 228f'(\alpha)^4 A_2^4 \beta A_3 - \cdots - f'(\alpha)A_7 A_2^2 + 464f'(\alpha)A_2^6 A_3)e_n^8 + O(e_n^9). \tag{17}$$

Using (17), we have

$$f[y_n, x_n]^2 = f'(\alpha)^2 \Big(1 + 2A_2 e_n + (3A_2^2 + 2A_3)e_n^2 + (8A_2 A_3 + 2A_4 - 2A_2^3)e_n^3 + \cdots + (9A_5^2 - 14A_3^4 - 240A_2^8 + \cdots - 286f'(\alpha)^3 A_2^2 \beta A_3^2 + 302f'(\alpha)^3 A_2^4 \beta A_3 - 24f'(\alpha)^3 A_4 \beta A_2^3)e_n^8\Big) + O(e_n^9). \tag{18}$$

On substituting (15), (16), (17) and (14) into the second step of (4), and using geometric series then we obtain

$$z_n = \alpha + (2A_2^3 - A_2 A_3)e_n^4 + (-2A_2 A_4 + 14A_2^2 A_3 - 10A_2^4 - f'(\alpha)^3 A_2^2 \beta - 2A_3^2)e_n^5 + \cdots + (145A_2^7 + 86A_3^2 A_4 + 2A_7 A_2 + \cdots + 88f'(\alpha)^3 A_4 \beta A_2^2 - 7f'(\alpha)^3 A_2 \beta A_5 - 33f'(\alpha)^3 A_4 \beta A_3)e_n^8 + O(e_n^9). \tag{19}$$

Similar to (7), evaluating (6) in $x = z_n$ and using (19), we get

$$f(z_n) = f'(\alpha)\Big((2A_2^3 - A_2 A_3)e_n^4 + (-2A_2 A_4 - 10A_2^4 - 2A_3^2 + 14A_2^2 A_3 - \beta f'(\alpha)^3 A_2^2)e_n^5 + \cdots + (149A_2^7 - 652A_2^5 A_3 - 7\beta f'(\alpha)^3 A_2 A_5 + \cdots - 6A_2 f'(\alpha)^6 \beta^2 A_3 - 202A_2 A_3^3 - 13A_2^3 f'(\alpha)^3 \beta A_3)e_n^8\Big) + O(e_n^9). \tag{20}$$

Then using (20), (16), (7), (19), (15) and noting $x_n = e_n + \alpha$, we have

$$\begin{aligned}\frac{f[z_n, y_n]}{f[z_n, x_n]} &= 1 - A_2 e_n + (2A_2^2 - A_3)e_n^2 + (-A_4 + 4A_2A_3 - 4A_2^3)e_n^3 + \cdots \\ &\quad + (-171A_2^8 - 281f'(\alpha)^3 A_2^3 \beta A_4 - 76 f'(\alpha)^3 A_2^2 \beta A_3^2 - \cdots \\ &\quad - 272 f'(\alpha)^3 A_2^4 \beta A_3 + \frac{48 A_3 A_4 A_5}{A_2} - 73 f'(\alpha)^3 A_3^3 \beta)e_n^8 + O(e_n^9), \quad (21)\end{aligned}$$

and

$$\begin{aligned}\frac{f(z_n)}{2f[z_n, y_n] - f[z_n, x_n]} &= (2A_2^3 - A_2A_3)e_n^4 + (13A_2^2A_3 - f'(\alpha)^3 A_2^2 \beta - 8A_2^4 - 2A_3^2 \\ &\quad - 2A_2A_4)e_n^5 + \cdots + (-149A_2A_3^3 + 29 f'(\alpha)^3 A_2^3 \beta A_3 \\ &\quad - 7 f'(\alpha)^3 A_2 \beta A_5 - \cdots + 76 f'(\alpha)^3 A_4 \beta A_2^2 \\ &\quad - 33 f'(\alpha)^3 A_4 \beta A_3 + 126 A_2 A_5 A_3)e_n^8 + O(e_n^9). \quad (22)\end{aligned}$$

On substituting (19), (21) and (22) in the third step of (4), and since $e_{n+1} = x_{n+1} - \alpha$ we obtain the following error equation:

$$\begin{aligned}e_{n+1} &= -A_2(f'(\alpha)^3 A_2^2 \beta A_3 + A_2A_4A_3 + 2A_2^2A_3^2 - A_3^3 - 2A_2^3A_4 - 2f'(\alpha)^3 A_2^4 \beta)e_n^8 \\ &\quad + O(e_n^9). \quad (23)\end{aligned}$$

From the definition of order of convergence [10] the method defined by equation (4) has an eight-order convergence and Theorem 1 proved. □

Equation (4) is a derivative free iterative method that requires four evaluation functions, namely $f(w_n)$, $f(x_n)$, $f(y_n)$ and $f(z_n)$. This new iterative method is optimal in the sense of Kung and Traub conjecture [9] and has the efficiency index $8^{1/4} \approx 1.682$.

3. NUMERICAL EXPERIMENTS

In this section, the effectiveness of the proposed method is discussed by applying it to the given test functions. The proposed method (DFM) in (4) for $\beta = 1.5 \times 10^{-3}$ is compared with the existing optimal eight-order derivative-free iterative methods, Soleymani et al. (SVPM) [16], Cordero et al. (CHMTM) [7], Soleimani et al. (SSSM) [15], Choubey and Jaiswal (CJM) [5] and Sharma and Arora (SAM) [12].

We consider the following test functions for comparing the proposed method

(i) $f_1(x) = e^{-x^2+x+2} - \cos(x+1) + x^3 + 1, \quad \alpha \in [-1, 0],$ [11],

(ii) $f_2(x) = x^2 - e^x - 3x + 2, \quad \alpha \in [0, 1],$ [14],

(iii) $f_3(x) = 3x + \sin(x) - e^x, \quad \alpha \in [0, 2],$ [15],

(iv) $f_4(x) = x^3 + 4x^2 - 15,$ $\alpha \in [1, 2],$ [13],

(v) $f_5(x) = xe^{-x} - 0.1,$ $\alpha \in [0, 5],$ [15].

To perform computational tests we use $\epsilon = 1.0 \times 10^{-200}$, and we stop the iteration if $|f(x_{n+1})| < \epsilon$ and the maximum iteration is reaching 100. The computational order of convergence (COC) [8] is approximated using the formula

$$\text{COC} \approx \frac{\log |f(x_n)/f(x_{n-1})|}{\log |f(x_{n-1})/f(x_{n-2})|}.$$

Table 1 shows the comparison of computational results of the six methods where f_i denote the functions of the nonlinear equations, *n* is the number iteration, COC is computational order of convergence and $|f(x_{n+1})|$ is the absolute value of the function.

Based on numerical comparison in Table 1, all of the discussed methods successfully found the approximate root for the given test functions and there is no significant difference among the discussed methods in terms of number of iteration needed to obtain an approximate root. On computational test for function f_1, DFM take less number of iterations than those of the existing optimal eight-order derivative-free iterative methods. For the function f_4, DFM requires the same number iteration with CJM and take less number of iterations than those of SVPM, CHMTM, SSSM and SAM. As for functions f_2, f_3 and f_5, DFM requires the same number of iterations with the other existing optimal eight-order derivative-free iterative methods and have accuracy higher than the existing methods. It can be observed that the computational order of convergence (COC) is in accordance with the theoritical order of convergence.

It can be seen from Table 1 that the proposed method shows better performance in terms of number of iterations as compared with the existing optimal derivative-free eight-order methods, especially for the functions f_1 and f_4. However, the proposed method has equal performance as compared with the existing optimal derivative-free eight-order methods as can be seen in functions f_2, f_3 and f_5. Therefore the proposed method can be said to be competitive or used as an alternative method to solve nonlinear equations.

Table 1: Comparison of computational results of the discussed iterative methods

Methods	n	COC	$\lvert f(x_{n+1})\rvert$
$f_1(x)$			
$x_0 = 1.475$			
SVPM	15	8.00	$1.66e-1318$
CHMTM	11	8.00	$1.54e-689$
SSSM	16	8.00	$7.10e-595$
CJM	8	7.97	$1.77e-638$
SAM	8	7.94	$1.88e-337$
DFM	6	7.99	$3.00e-644$
$f_2(x)$			
$x_0 = -2.6$			
SVPM	4	8.00	$1.84e-706$
CHMTM	4	8.00	$1.58e-1003$
SSSM	3	7.11	$1.82e-206$
CJM	4	8.00	$1.56e-1007$
SAM	3	7.11	$1.39e-265$
DFM	4	8.00	$4.59e-1597$
$f_3(x)$			
$x_0 = 0.5$			
SVPM	3	8.23	$1.56e-402$
CHMTM	3	8.26	$2.64e-376$
SSSM	3	8.10	$4.12e-555$
CJM	3	8.08	$2.11e-613$
SAM	3	8.09	$2.75e-530$
DFM	3	8.06	$3.52e-613$

Methods	n	COC	$\lvert f(x_{n+1})\rvert$
$f_4(x)$			
$x_0 = 0.0$			
SVPM	9	7.36	$1.68e-243$
CHMTM	8	7.31	$7.28e-240$
SSSM	13	7.61	$1.87e-341$
CJM	4	8.14	$8.64e-491$
SAM	14	7.96	$2.95e-615$
DFM	4	8.09	$3.06e-512$
$f_5(x)$			
$x_0 = -1.0$			
SVPM	4	7.84	$8.60e-376$
CHMTM	4	7.71	$1.28e-290$
SSSM	4	7.96	$3.85e-738$
CJM	4	8.01	$1.00e-934$
SAM	4	7.95	$2.72e-574$
DFM	4	7.99	$1.16e-936$

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