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Summation Theorems Involving Appell's Double Hypergeometric Function of Third Kind

M. I. Qureshi[1], **M. S. Baboo**[2], **and Ashfaq Ahmad**[3]

[1] *Department of Applied Sciences and Humanities, Faculty of Engineering and Technology, Jamia Millia Islamia (A Central University), New Delhi -110025, India.*

[2]*School of Basic Sciences and Research, Sharda University, Greater Noida, Uttar Pradesh, 201306, India.*

[3] *Department of Applied Sciences and Humanities, Faculty of Engineering and Technology, Jamia Millia Islamia (A Central University), New Delhi -110025, India.*

Abstract

In this paper, we find summation theorems of $F_3[A, B; C, D; G; x, y]$, where x takes form $1, 2y-1, \frac{1+y}{2}, \frac{8+y}{9}, \frac{8y+1}{9}, \frac{9y-1}{8}, \frac{3y+1}{4}, \frac{4y-1}{3}, \frac{3y-1}{2}, \frac{3+y}{4}, \frac{2y+1}{3}$ and other rational functions of y with suitable convergence conditions.

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1. INTRODUCTION

For definitions of Pochhammer symbol, generalized hypergeometric function ${}_pF_q$ and basic theorems, we refer monumental work of Srivastava and Manocha [14], Rainville [10] and other notation have their usual meanings.

2. METHODOLOGY

The Appell's function: The Appell's function of third kind is defined as

$$F_3[A,B;C,D;G;x,y]=$$
$$=\sum_{r,s=0}^{\infty}\frac{(A)_r(B)_s(C)_r(D)_s}{(G)_{r+s}}\frac{x^r}{r!}\frac{y^s}{s!}$$
$$=\sum_{s=0}^{\infty}\frac{(B)_s(D)_s}{(G)_s}\frac{y^s}{s!}\ {}_2F_1\left[\begin{array}{c}A,C;\\G+s;\end{array}\ x\right], \tag{1}$$
$$max\{|x|,|y|\}<1.$$

The Appell's function of third kind can also be written as

$$F_3\left[A,C-A;B,C-B;C;x,\frac{-y}{1-y}\right]= \tag{2}$$
$$=(1-y)^{C-A-B}{}_2F_1\left[\begin{array}{cc}A,B;&\frac{x-y}{1-y}\\C;&\end{array}\right],$$
$$\left(|y|<1,\ \left|\tfrac{x-y}{1-y}\right|<1\right),$$

see Appell 1926 [2]; Bailey [3, p,80 Equation (5)] and Slater [12, p.220, Entry (8.3.1.6)]. We use the equation (2) along with Gauss hypergeometric summation theorems to find the solution of Appell's function F_3.

3. APPLICATION OF APPELL'S FUNCTION F_3 IN SUMMATION THEOREMS

In the equation (2) put $A=a, B=b, C=1+a-b$ and $x=1$, use the Gauss's summation theorem, we get

$$F_3\left[a,1-b;b,1+a-2b;1+a-b;1,\frac{-y}{1-y}\right]=$$

$$= \frac{(1-y)^{1-2b}\Gamma(1-2b)\Gamma(1+a-b)}{\Gamma(1-b)\Gamma(1+a-2b)}, \tag{3}$$

$$\left(|y| < 1;\ \Re(b) < \tfrac{1}{2}\ ;\ 1+a-b \in \mathbb{C}\backslash\mathbb{Z}_0^- \right).$$

In the equation (2) put $A = a, B = b, C = 1 + a - b$ and $x = 2y - 1$, use the Kummer's first summation theorem, we get

$$F_3\left[a, 1-b; b, 1+a-2b; 1+a-b; 2y-1, \frac{-y}{1-y}\right] =$$

$$= \frac{(1-y)^{1-2b}\Gamma(1+\frac{a}{2})\Gamma(1+a-b)}{\Gamma(1+\frac{a}{2}-b)\Gamma(1+a)}, \tag{4}$$

$$\left(|y| < 1\ ; \Re(b) < 1\ ;\ 1+a-b \in \mathbb{C}\backslash\mathbb{Z}_0^- \right).$$

In the equation (2) put $A = a, B = \frac{a-2-\sqrt{2-a}}{2}, C = \frac{a+4-\sqrt{2-a}}{2}$ and $x = 2y - 1$ and using a result of Brychkov [4, p.579, Equation (100)], we get
$F_3\left[a, \frac{4-a-\sqrt{2-a}}{2}; \frac{a-2-\sqrt{2-a}}{2}, 3; \frac{a+4-\sqrt{2-a}}{2}; 2y-1, \frac{-y}{1-y}\right] =$

$$= (1-y)^{3-a}\, \frac{2 + a(3+\sqrt{2-a})}{2^{a+1}}, \tag{5}$$

$$\left(|y| < 1\ ;\ \ a < 4 \right).$$

In the equation (2) put $A = a, B = \frac{a-3-\sqrt{7-3a}}{2}, C = \frac{a+5-\sqrt{7-3a}}{2}$ and $x = 2y - 1$ and using a result of Brychkov [4, p.579, Equation (101)], we get
$F_3\left[a, \frac{5-a-\sqrt{7-3a}}{2}; \frac{a-3-\sqrt{7-3a}}{2}, 4; \frac{a+5-\sqrt{7-3a}}{2}; 2y-1, \frac{-y}{1-y}\right] =$

$$= (1-y)^{4-a}\, \frac{6 + a(15 - a + 4\sqrt{7-3a})}{6 \times 2^{a+1}}, \tag{6}$$

$$\left(|y| < 1\ ;\ \ a < 5 \right).$$

In the equation (2) put $A = a, B = b, C = \frac{1+a+b}{2}$ and $x = \frac{1+y}{2}$, use the Kummer's second summation theorem, we get

$$F_3\left[a, \frac{1-a+b}{2}; b, \frac{1+a-b}{2}; \frac{1+a+b}{2}; \frac{1+y}{2}, \frac{-y}{1-y}\right] =$$

$$= \frac{(1-y)^{\frac{1-a-b}{2}}\Gamma(\frac{1+a+b}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{1+a}{2})\Gamma(\frac{1+b}{2})}, \tag{7}$$

$$\left(|y| < 1;\ \tfrac{1+a+b}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right).$$

In the equation (2) put $A = a, B = b, C = \frac{1+a+b-m}{2}$ and $x = \frac{1+y}{2}$, use the summation theorem recorded by Prudnikov *et al.* [9, p.491, Entry (7.3.7.2)], we get
$F_3\left[a, \frac{1-a+b-m}{2}; b, \frac{1+a-b-m}{2}; \frac{1+a+b-m}{2}; \frac{1+y}{2}, \frac{-y}{1-y}\right] =$

$$= \frac{2^{b-1}(1-y)^{\frac{1-a-b-m}{2}}\Gamma(\frac{1+a+b-m}{2})}{\Gamma(b)}\times$$

$$\times \sum_{r=0}^{m}\left\{\binom{m}{r}\frac{\Gamma(\frac{b+r}{2})}{\Gamma(\frac{a+1+r-m}{2})}\right\}, \tag{8}$$

$$\left(|y| < 1;\ b, \tfrac{1+a+b-m}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-;\quad m \in \mathbb{N}_0\right).$$

In the equation (2) put $A = a, B = b, C = \frac{1+a+b+m}{2}$ and $x = \frac{1+y}{2}$, use the summation theorem given by Rakha-Rathie [11, p.827, Theorem (1)], we get
$F_3\left[a, \frac{1-a+b+m}{2}; b, \frac{1+a-b+m}{2}; \frac{1+a+b+m}{2}; \frac{1+y}{2}, \frac{-y}{1-y}\right] =$

$$= \frac{2^{a-1}(1-y)^{\frac{1-a-b+m}{2}}\Gamma(\frac{1+a+b-m}{2})\Gamma(\frac{b-a+1-m}{2})}{\Gamma(a)\Gamma(\frac{b-a+1+m}{2})}\times$$

$$\times \sum_{r=0}^{m}\left\{\binom{m}{r}\frac{(-1)^r\Gamma(\frac{a+r}{2})}{\Gamma(\frac{b+1+r-m}{2})}\right\}, \tag{9}$$

$$\left(|y| < 1;\ a, \tfrac{a+b+1+m}{2}, \tfrac{b-a+1-m}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-;\quad m \in \mathbb{N}_0\right).$$

In the equation (2) put $A = a, B = 1-a, C = c$ and $x = \frac{1+y}{2}$, use the Kummer's third summation theorem, we get

$$F_3\left[a, c-a; 1-a, c+a-1; c; \frac{1+y}{2}, \frac{-y}{1-y}\right] =$$

$$= \frac{(1-y)^{c-1}\Gamma(\frac{c}{2})\Gamma(\frac{c+1}{2})}{\Gamma(\frac{c+a}{2})\Gamma(\frac{c+1-a}{2})}, \tag{10}$$

$$\left(|y| < 1;\ c \in \mathbb{C}\backslash\mathbb{Z}_0^-\right).$$

In the equation (2) put $A = \frac{a}{2}, B = \frac{a+1}{2}, C = \frac{2a+2}{3}$ and $x = \frac{8+y}{9}$, using a result of Andrews *et al.* [1, p.131, Entry 3.1.20], Kummer [6, p.136, Article 25(6)] see also Prudnikov *et al.* [9, p.495, Equation (38)], we get

$$F_3\left[\frac{a}{2}, \frac{a+4}{6}; \frac{a+1}{2}, \frac{a+1}{6}; \frac{2a+2}{3}; \frac{8+y}{9}, \frac{-y}{1-y}\right] =$$
$$= \left(\frac{3}{2}\right)^a \frac{(1-y)^{\frac{1-2a}{6}}\sqrt(\pi)\Gamma(\frac{2a+2}{3})}{\Gamma(\frac{a+4}{6})\Gamma(\frac{a+1}{2})}, \tag{11}$$
$$\left(|y| < 1;\ \frac{2a+2}{3} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right).$$

In the equation (2) put $A = 2a+1, B = -a, C = \frac{2}{3}$ and $x = \frac{8+y}{9}$, using a result of Heymann, see Per W. Karlsson [5, p.335, Equation (3.2)], see also Brychkov [4, p.584, Equation (144)], we get

$$F_3\left[2a+1, -\frac{6a+1}{3}; -a, a+\frac{2}{3}; \frac{2}{3}; \frac{8+y}{9}, \frac{-y}{1-y}\right] =$$
$$= \frac{2\ \times 3^a \sin(\pi a + \frac{5\pi}{6})}{(1-y)^{a+\frac{1}{3}}}. \tag{12}$$

In the equation (2) put $A = 2a+2, B = -a, C = \frac{4}{3}$ and $x = \frac{8+y}{9}$, using a result of Heymann, see Per W. Karlsson [5, p.335, Equation (3.3)], see also Brychkov [4, p.584, Equation (145)], we get

$$F_3\left[2a+2, -\frac{6a+2}{3}; -a, \frac{3a+4}{3}; \frac{4}{3}; \frac{y+8}{9}, \frac{-y}{1-y}\right] =$$
$$= \frac{3^a\Gamma(\frac{3}{2})\Gamma(\frac{1}{6})}{(1-y)^{a+\frac{2}{3}}\Gamma(\frac{1}{6}-a)\Gamma(a+\frac{3}{2})}, \tag{13}$$
$$\left(|y| < 1\right).$$

In the equation (2) put $A = a, B = \frac{2a+1}{2}, C = 4a$ and $x = \frac{8+y}{9}$ and using a result of Prudnikov *et al.* [9, p.496, Equation (41)], we get

$$F_3\left[a, 3a; \frac{2a+1}{2}, \frac{6a-1}{2}; 4a; \frac{8+y}{9}, \frac{-y}{1-y}\right] =$$

$$= \frac{(3)^{2a}(1-y)^{\frac{4a-1}{2}}\Gamma(\frac{1}{2})\Gamma(4a)}{2^{6a-1}\Gamma(3a)\Gamma(a+\frac{1}{2})}, \tag{14}$$

$$\left(|y| < 1\ ;\ \ a \in \mathbb{C}\backslash\mathbb{Z}_0^-\right).$$

In the equation (2) put $A = 2a, B = \frac{1-2a}{2}, C = \frac{6a+5}{6}$ and $x = \frac{8y+1}{9}$ and using a result of Heymann, see Per.W. Karlsson [5, p.330, Equation (1.2)] and Brychkov [4, p.580, Equation 116], we get

$$F_3\left[2a, \frac{5-6a}{6}; \frac{1-2a}{2}, \frac{6a+1}{3}; \frac{6a+5}{6}; \frac{8y+1}{9}, \frac{-y}{1-y}\right] =$$

$$= \frac{3^a(1-y)^{\frac{1}{3}}\Gamma(\frac{6a+5}{6})\Gamma(\frac{2}{3})}{4^a\Gamma(\frac{3a+2}{3})\Gamma(\frac{5}{6})}, \tag{15}$$

$$\left(|y| < 1;\ \frac{6a+5}{6} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right).$$

In the equation (2) put $A = 2a, B = 1-a, C = \frac{3a+2}{3}$ and $y = \frac{8y+1}{9}$, using a result of Heymann, see Per.W. Karlsson [5, p.330, Equation (1.3)] and Brychkov [4, p.581, Equation 117], we get

$$F_3\left[2a, \frac{2-3a}{3}; 1-a, \frac{6a-1}{3}; \frac{3a+2}{3}; \frac{8y+1}{9}, \frac{-y}{1-y}\right] =$$

$$= \frac{3^a\Gamma(\frac{3a+2}{3})\Gamma(\frac{1}{2})}{4^a(1-y)^{\frac{1}{3}}\Gamma(\frac{2a+1}{2})\Gamma(\frac{2}{3})}, \tag{16}$$

$$\left(|y| < 1;\ \frac{3a+2}{3} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right).$$

In the equation (2) put $A = a, B = 1-2a, C = \frac{4-3a}{3}$ and $x = \frac{8y+1}{9}$ and using a result of Prudnikov *et al.* [9, p.495, Equation (27)], we get

$$F_3\left[a, \frac{4-6a}{3}; 1-2a, \frac{1+3a}{3}; \frac{4-3a}{3}; \frac{8y+1}{9}, \frac{-y}{1-y}\right] =$$

$$= (1-y)^{\frac{1}{3}}\frac{\Gamma(\frac{2-3a}{3})\Gamma(\frac{4-3a}{3})}{3^a\Gamma(\frac{2}{3})\Gamma(\frac{4-6a}{3})}, \tag{17}$$

$$\left(|y| < 1;\quad \frac{4-3a}{3} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right).$$

In the equation (2) put $A = \frac{a}{2}, B = \frac{2-a}{6}, C = \frac{2a+5}{6}$ and $x = \frac{9y-1}{8}$, using a result of Andrews *et al.* [1, p.177, Question 3(a)], see also Prudnikov *et al.* [9, p.494, Equation (19)], we get

$$F_3\left[\frac{a}{2}, \frac{5-a}{6}; \frac{2-a}{6}, \frac{a+1}{2}; \frac{2a+5}{6}; \frac{9y-1}{8}, \frac{-y}{1-y}\right] =$$

$$= \frac{\sqrt(\pi)(1-y)^{\frac{1}{2}}\Gamma(\frac{2a+2}{3})}{2^{\frac{a}{2}}\Gamma(\frac{a+4}{6})\Gamma(\frac{a+1}{2})}, \tag{18}$$

$$\left(|y| < 1;\ \frac{2a+5}{6} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right).$$

In the equation (2) put $A = a, B = \frac{3a+1}{3}, C = \frac{4-3a}{3}$ and $x = \frac{9y-1}{8}$, using a result of Lavoie and Trottier [7, p.45, Equation (8)], see also Prudnikov *et al.* [9, p.494, Equation (18)], we get

$$F_3\left[a, \frac{4-6a}{3}; \frac{3a+1}{3}, -2a+1; \frac{4-3a}{3}; \frac{9y-1}{8}, \frac{-y}{1-y}\right] =$$

$$= \left(\frac{2}{3}\right)^{3a} \frac{(1-y)^{-3a+1}\Gamma(\frac{2}{3}-a)\Gamma(\frac{4}{3}-a)}{\Gamma(\frac{2}{3})\Gamma(\frac{4}{3}-2a)}, \tag{19}$$

$$\left(|y| < 1;\ \frac{4-3a}{3} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right).$$

In the equation (2) put $A = \frac{1}{2}, B = 1-b, C = \frac{4b+1}{2}$ and $x = \frac{1+3y}{4}$, use the summation theorem given by the Spiegel [13, p.894] and Luke [8, p.273, Equation 6.8(20)], we get

$$F_3\left[\frac{1}{2}, 2b; 1-b, \frac{6b-1}{2}; \frac{4b+1}{2}; \frac{1+3y}{4}, \frac{-y}{1-y}\right] =$$

$$= \frac{2(1-y)^{3b-1}\Gamma(b)\Gamma(\frac{4b+1}{2})}{3\,\Gamma(2b)\Gamma(\frac{2b+1}{2})}, \tag{20}$$

$$\left(|y| < 1;\ b,\ \frac{4b+1}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right).$$

In the equation (2) put $A = a, B = \frac{2-a}{2}, C = \frac{2a+1}{2}$ and $x = \frac{4y-1}{3}$ and using a result of Prudnikov *et al.* [9, p.494, Equation (14)], we get

$$F_3\left[a, \frac{1}{2}; \frac{2-a}{2}, \frac{3a+1}{2}; \frac{2a+1}{2}; \frac{4y-1}{3}, \frac{-y}{1-y}\right] =$$

$$= (1-y)^{\frac{a+1}{2}} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{2a+1}{2})}{3^{\frac{a}{2}}\Gamma(\frac{a+1}{2})\Gamma(\frac{a+1}{2})}, \tag{21}$$

$$\left(|y|<1;\ \frac{2a+1}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right).$$

In the equation (2) put $A = 2, B = a, C = \frac{5-a}{2}$ and $x = \frac{3y-1}{2}$ and using a result of Prudnikov *et al.* [9, p.494, Equation (13)], we get

$$F_3\left[2, \frac{1-a}{2}; a, \frac{5-3a}{2}; \frac{5-a}{2}; \frac{3y-1}{2}, \frac{-y}{1-y}\right] =$$

$$= (1-y)^{\frac{1-3a}{2}} \frac{3-a}{3}, \tag{22}$$

$$\left(|y|<1;\ \frac{5-a}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right).$$

In the equation (2) put $A = a, B = \frac{2a+1}{4}, C = \frac{2a+1}{2}$ and $x = (2\sqrt{2}-2) + y(3-2\sqrt{2})$ and using a result of Prudnikov *et al.* [9, p.495, Equation (36)], we get
$F_3\left[a, \frac{1}{2}; \frac{2a+1}{4}, \frac{2a+1}{4}; \frac{2a+1}{2}; (2\sqrt{2}-2) + y(3-2\sqrt{2}), \frac{-y}{1-y}\right] =$

$$= (1-y)^{\frac{1-2a}{4}} \left(\frac{2+\sqrt{2}}{2}\right)^a \frac{\Gamma(\frac{1}{2})\Gamma(\frac{3+2a}{4})}{\Gamma(\frac{a+2}{4})\Gamma(\frac{a+3}{4})}, \tag{23}$$

$$\left(|y|<1\,;\ \frac{2a+1}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right).$$

In the equation (2) put $A = a, B = \frac{4a+1}{6}, C = \frac{4a+1}{3}$ and $x = (12\sqrt{2}-16) + y(17+12\sqrt{2})$ and using a result of Prudnikov *et al.* [9, p.496, Equation (43)], see also Brychkov [4, p.587, Equation (172)], we get

$$F_3\left[a, \frac{a+1}{3}; \frac{4a+1}{6}, \frac{4a+1}{6}; \frac{4a+1}{3}; x, \frac{-y}{1-y}\right] =$$

$$= \left(\frac{2+\sqrt{2}}{2}\right)^{2a} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{2a+2}{3})\,(1-y)^{\frac{1-2a}{6}}}{\Gamma(\frac{a+1}{2})\Gamma(\frac{a+4}{6})}, \tag{24}$$

$$\left(|y|<1\,;\ \frac{4a+1}{3} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right).$$

In the equation (2) put $A = a, B = \frac{4a-1}{2}, C = 4a-1$ and $x = (12\sqrt{2}-16) + y(17+12\sqrt{2})$ and using a result of Prudnikov *et al.* [9, p.496, Equation (46)], we get

$$F_3\left[a, 3a-1; \frac{4a-1}{2}, \frac{4a-1}{2}; 4a-1; x, \frac{-y}{1-y}\right] =$$

$$= (1-y)^{\frac{2a-1}{2}}\left(\frac{2+\sqrt{2}}{2}\right)^{2a}\frac{\Gamma(a)\Gamma(\frac{1}{2})}{\Gamma(\frac{3a}{2})\Gamma(\frac{a+1}{2})}, \tag{25}$$

$$\left(|y|<1\,;\;\; 4a-1\in\mathbb{C}\backslash\mathbb{Z}_0^-\right).$$

In the equation (2) put $A = 2a, B = \frac{4a+1}{4}, C = \frac{4a+3}{4}$ and $x = \frac{\sqrt{2}-1+2y}{\sqrt{2}+1}$, use the summation theorem given by the Andrews *et al.* [1, p.177, Question 3(b)], we get

$$F_3\left[2a, \frac{3-4a}{4}; \frac{4a+1}{4}, \frac{1}{2}; \frac{4a+3}{4}; \frac{\sqrt{2}-1+2y}{\sqrt{2}+1}, \frac{-y}{1-y}\right] =$$

$$= \frac{\sqrt{(\pi)}\Gamma(\frac{4a+3}{4})(1-y)^{-2a+\frac{1}{2}}}{(4-2\sqrt{2})^{2a}\Gamma(\frac{2a+3}{4})\Gamma(\frac{a+1}{2})}, \tag{26}$$

$$\left(|y|<1;\; a+\frac{3}{4}\in\mathbb{C}\backslash\mathbb{Z}_0^-\right).$$

In the equation (2) put $A = \frac{1}{2}, B = a, C = \frac{2a+3}{4}$ and $x = \frac{1-\sqrt{2}+y(1+\sqrt{2})}{2}$ and using a result of Prudnikov *et al.* [9, p.494, Equation (16)], we get

$$F_3\left[\frac{1}{2}, \frac{2a+1}{4}; a, \frac{3a-2}{4}; \frac{2a+3}{4}, \frac{2a+3}{4}; \frac{1-\sqrt{2}+y(1+\sqrt{2})}{2}, \frac{-y}{1-y}\right] =$$

$$= (1-y)^{\frac{1-2a}{4}}\frac{\Gamma(\frac{1}{2})\Gamma(\frac{2a+3}{4})}{2^{\frac{a}{2}}\Gamma(\frac{a+2}{4})\Gamma(\frac{a+3}{4})}, \tag{27}$$

$$\left(|y|<1;\;\; \frac{2a+3}{4}\in\mathbb{C}\backslash\mathbb{Z}_0^-\right).$$

In the equation (2) put $A = a, B = \frac{2-a}{3}, C = \frac{2a+5}{6}$ and $x = \frac{(4-3\sqrt{2})+y(4+3\sqrt{2})}{8}$ and using a result of Prudnikov *et al.* [9, p.495, Equation (22)], we get

$$F_3\left[a, \frac{5-4a}{6}; \frac{2-a}{3}, \frac{4a+1}{6}; \frac{2a+5}{6}; \frac{(4-3\sqrt{2})+y(4+3\sqrt{2})}{8}, \frac{-y}{1-y}\right] =$$

$$= (1-y)^{\frac{1-2a}{6}}\left(\frac{2}{3}\right)^{\frac{a}{2}}\frac{\Gamma(\frac{1}{2})\Gamma(\frac{2a+5}{6})}{\Gamma(\frac{a+3}{6})\Gamma(\frac{a+5}{6})}, \tag{28}$$

$$\left(|y|<1\,;\;\; \frac{2a+5}{6}\in\mathbb{C}\backslash\mathbb{Z}_0^-\right).$$

In the equation (2) put $A = a, B = 2-3a, C = \frac{3-2a}{2}$ and $x = \frac{(2-\sqrt{3})+y(2+\sqrt{3})}{4}$ and using a result of Prudnikov *et al.* [9, p.495, Equation (25)], we get

$$F_3\left[a, \tfrac{3-4a}{2}; 2-3a, \tfrac{4a-1}{2}; \tfrac{3-2a}{2}; \tfrac{(2-\sqrt{3})+y(2+\sqrt{3})}{4}, \tfrac{-y}{1-y}\right] =$$

$$= (1-y)^{\frac{2a-1}{2}} \frac{3^{\frac{3a}{2}}\Gamma(\frac{4}{3})\Gamma(\frac{3-2a}{2})}{2^{2a-1}\Gamma(\frac{1}{2})\Gamma(\frac{4-3a}{3})}, \tag{29}$$

$$\left(|y| < 1\,;\ \frac{3-2a}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right).$$

4. CONCLUSION

By this technique we can find the exact solutions of numerous problem in engineering, whose solution leads to Appell's function of third kind.

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$$1 + \frac{\alpha.\beta}{1.\gamma}x + \frac{\alpha(\alpha+1).\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2 + \ldots,$$

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