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Vertex Polynomial of Graphs with New Results

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Abstract:

The Vertex Polynomial of the graph G is defined as $\mathrm{V(G,x)} = \sum_{\mathrm{k=0}}^{\Delta(\mathrm{G})} \mathrm{v_k x^k}$, where $\Delta(\mathrm{G}) = \max\{\mathrm{d(v)/v\epsilon V}\}$ and $\mathrm{v_k}$ is the number of vertices of degree k. In this paper I found some results on Vertex Polynomial.

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Keywords: Vertex Polynomial, Splitting graph, Degree splitting graph, corona.

1. Introduction:

Here, I consider simple undirected graphs. The terms not defined here we refer Frank Harary [2]. The vertex set is denoted by V and the edge set by E. For v□V, $\mathrm{d(v)}$ is the number of edges incident with v, the maximum degree of G is defined as $\Delta(\mathrm{G}) = \max\{\mathrm{d(v)/v\epsilon V}\}$. Let $\mathrm{G_1} = (\mathrm{V_1}, \mathrm{E_1})$ and $\mathrm{G_2} = (\mathrm{V_2}, \mathrm{E_2})$ be two graphs, the union $\mathrm{G_1} \cup \mathrm{G_2}$ is defined to be $\mathrm{G} = (\mathrm{V}, \mathrm{E})$ where $V = \mathrm{V_1} \cup \mathrm{V_2}$ and $E = \mathrm{E_1} \cup \mathrm{E_2}$, the sum $\mathrm{G_1} + \mathrm{G_2}$ is defined as $\mathrm{G_1} \cup \mathrm{G_2}$ together with all the lines joining points of $\mathrm{V_1}$ to $\mathrm{V_2}$. If G is of order n, the corona of G with H, $\mathrm{G} \odot \mathrm{H}$ is the graph obtained by taking one copy of G and n copies of H and joining the $\mathrm{i^{th}}$ vertex of G with an every vertex in the $\mathrm{i^{th}}$ copy of H. The graph G with $\mathrm{V} = \mathrm{S_1} \cup \mathrm{S_2} \cup \ldots \cup \mathrm{S_i} \cup \mathrm{T}$, where each $\mathrm{S_i}$ is a set of vertices having at least two vertices and having the same degree and $\mathrm{T} = \mathrm{V} \backslash \cup \mathrm{S_i}$. The degree splitting graph of G denoted by $\mathrm{DS(G)}$ and is obtained from G by adding the vertices $\mathrm{w_1}, \mathrm{w_2}, \ldots, \mathrm{w_t}$ and joining $\mathrm{w_i}$ to each vertex of $\mathrm{S_i}$, $1 \leq \mathrm{i} \leq \mathrm{t}$ [5]. For each vertex v of a graph G, take a new vertex $\mathrm{v'}$, join $\mathrm{v'}$ to all the vertices of G which are adjacent to v. The graph $\mathrm{S(G)}$ thus obtained is called splitting graph of G [1]. The graph $G = (V, E)$ is simply denoted by G.

2. MAIN RESULTS:

Theorem 2.1: If H is a sub graph of a simple graph G, then the degree of the vertex polynomial of H is less than or equal to the degree of the vertex polynomial of G.

Proof:

Let H be a sub graph of a simple graph G. Then the order of H is less than or equal to the order of G and degree of each vertex of H is less than or equal to degree of each vertex of G. This gives, the degree of the vertex polynomial of H is less than or equal to the degree of the vertex polynomial of G.

Theorem 2.2: If G and H are two isomorphic graphs, then the vertex polynomial of G and the vertex polynomial of H are equal.

Proof:

Let G and H be two isomorphic graphs. Then G and H have the same degree sequence. Therefore, the graphs G and H have the same vertex polynomial.

Remark 2.3: If the vertex polynomial of two graphs G and H are equal, then G and H need not be isomorphic.

For example, we consider two graphs G and H as given in figure 2.1(a) and figure 2.1(b).

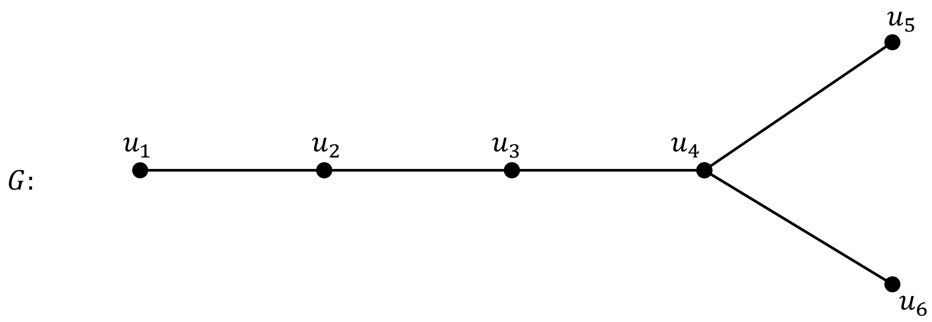


Figure 2.1(a)

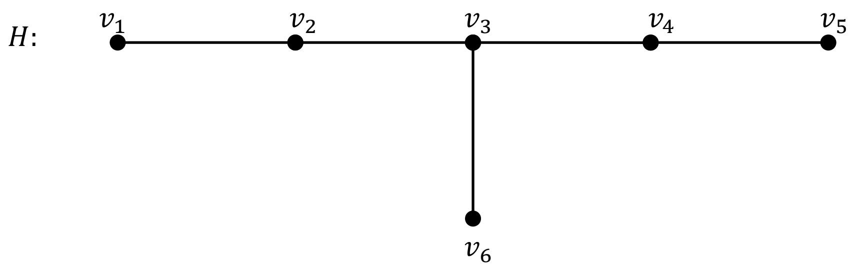


Figure 2.1(b)

Here, the vertex polynomial of the graph G is $\mathrm{V}(G, \mathrm{x}) = \mathrm{x}^3 + 2\mathrm{x}^2 + 3\mathrm{x}$.

And the vertex polynomial of the graph H is $\mathrm{V}(H, \mathrm{x}) = \mathrm{x}^3 + 2\mathrm{x}^2 + 3\mathrm{x}$.

The vertex u_4 in G corresponds to the vertex v_3 in H as both have the same degree 3. But, there are two pendant vertices adjacent to the vertex $u_4$and there is only one pendant vertex adjacent to v_3. So, adjacency relation not preserved and hence G and H are not isomorphic.

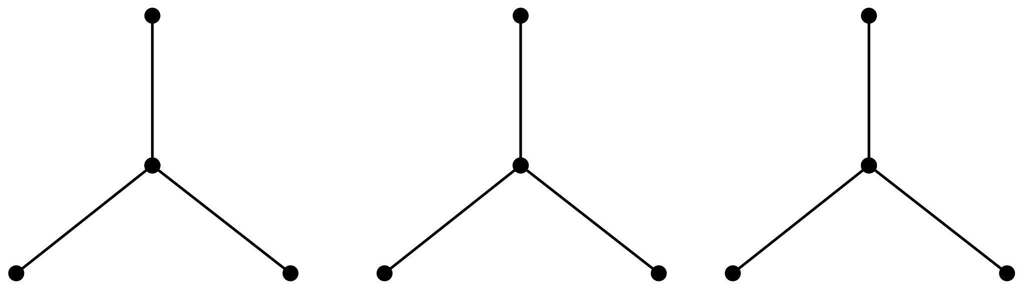
Theorem 2.4: Let G be a graph and $\zeta = G \cup G \cup \dots \cup G$ $(m\ times)$, then $\mathrm{V}(\zeta, \mathrm{x}) = \mathrm{mV}(G, \mathrm{x})$.

Proof:

Let G be a graph and take m copies of G. From the definition of union of graphs, the number of vertices of the graph G increased by m copies but degree of each vertex remains unchanged in ζ. Therefore, each coefficient of the vertex polynomial of G is multiplied by m gives the result.

Example 2.5: Consider the graph $K_{1,3}$ with 3 copies. That is, $\zeta = K_{1,3} \cup K_{1,3} \cup K_{1,3}$.

Then$\mathrm{V}(\zeta, \mathrm{x}) = 3\mathrm{V}(K_{1,3}, \mathrm{x})$.



$K_{1,3} \cup K_{1,3} \cup K_{1,3}$:

Figure 2.2

Here,$\mathrm{V}(K_{1,3}, \mathrm{x}) = \mathrm{x}^3 + 3\mathrm{x}$.

$$\begin{aligned} \text{Now, } \mathrm{V}(K_{1,3} \cup K_{1,3} \cup K_{1,3}, \mathrm{x}) &= 3\mathrm{x}^3 + 9\mathrm{x}. \\ &= 3(\mathrm{x}^3 + 3\mathrm{x}). \\ &= 3\mathrm{V}(K_{1,3}, \mathrm{x}). \end{aligned}$$

Theorem 2.6: Let G be a graph with order n.

Then $\mathrm{V}(mG, \mathrm{x}) = \mathrm{mx}^{(\mathrm{m}-1)\mathrm{n}}\mathrm{V}(G, \mathrm{x})$.

Proof:

Let G be a graph with order n and take mG. Using the definition of sum of the graphs, each vertex of the graph G is increased by m times in mG and each vertex of mG is adjacent to all vertices of $(\mathrm{m}-1)$ copies of G. Since G has order n, when we multiply $\mathrm{mx}^{(\mathrm{m}-1)\mathrm{n}}$ to vertex polynomial of G we get vertex polynomial of $\mathrm{V}(mG,\mathrm{x})$. That is, $\mathrm{V}(mG,\mathrm{x}) = \mathrm{mx}^{(\mathrm{m}-1)\mathrm{n}}\mathrm{V}(G,\mathrm{x})$.

Example 2.7: Consider the graph $K_{1,3}$ and take $K_{1,3} + K_{1,3}$.

Then $\mathrm{V}\left(K_{1,3} + K_{1,3},\mathrm{x}\right) = 2\mathrm{x}^4\mathrm{V}\left(K_{1,3},\mathrm{x}\right)$.

$K_{1,3} + K_{1,3}$:

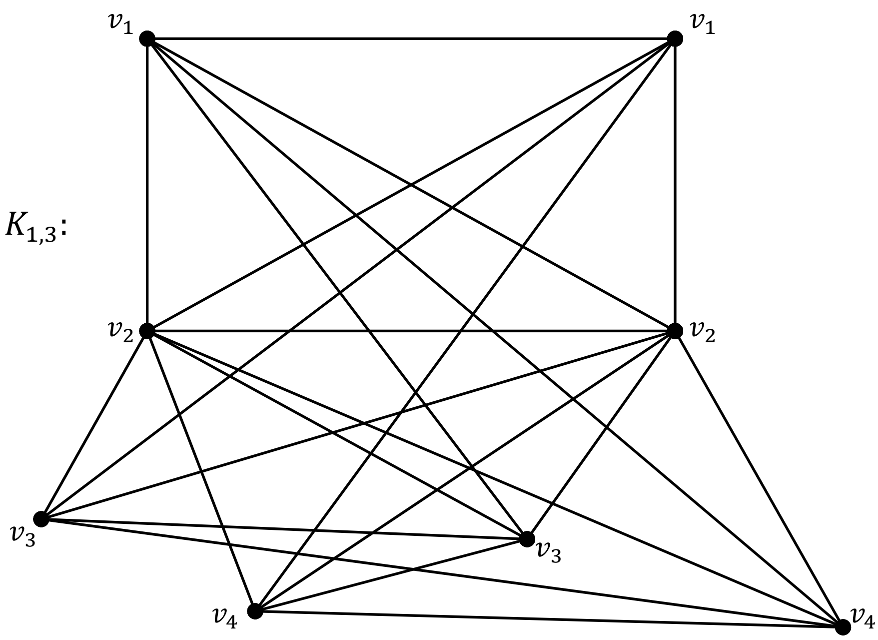


Figure 2.3

Here, $\mathrm{V}\left(K_{1,3},\mathrm{x}\right) = \mathrm{x}^3 + 3\mathrm{x}$.

Now, $\mathrm{V}\left(K_{1,3} + K_{1,3},\mathrm{x}\right) = 2\mathrm{x}^7 + 6\mathrm{x}^5$.

$= 2\mathrm{x}^4(\mathrm{x}^3 + 3\mathrm{x})$.

$= 2\mathrm{x}^4\mathrm{V}\left(K_{1,3},\mathrm{x}\right)$.

Theorem 2.8: Let G be a graph with order n and H be a graph of order m, then $\mathrm{V}(G \odot H,\mathrm{x}) = \mathrm{x}^{\mathrm{m}}\mathrm{V}(G,\mathrm{x}) + \mathrm{nxV}(H,\mathrm{x})$.

Proof:

Let G be a graph with order n and H be a graph of order m. From the definition of Corona, we have n copies of H, order of H has been increased by n times and each degree of the vertices of the n copies H has been increased by one. This gives the term $nxV(H, x)$. Also, each degree of the vertices of G has been increased by m. This gives the term $x^m V(G, x)$. Adding these two terms, we get the vertex polynomial of $V(G \odot H, x) = x^m V(G, x) + nxV(H, x)$.

Example 2.9: Consider the graphs C_4 and P_3, then

$C_4 \odot P_3$:

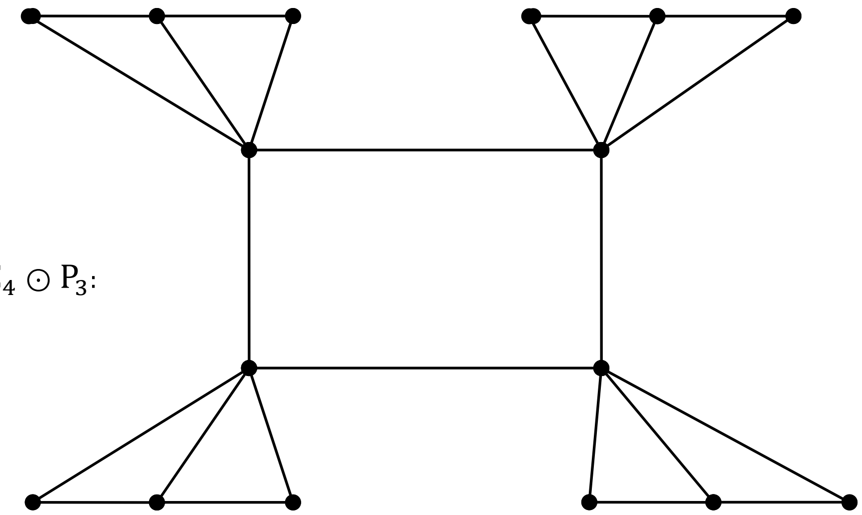


Figure 2.4

Here, $V(C_4, x) = 4x^2$.

$V(P_3, x) = x^2 + 2x$.

Now, $V(C_4 \odot P_3, x) = 4x^5 + 4x^3 + 8x^2$.

$= x^3(4x^2) + 4x(x^2 + 2x)$.

$= x^3 V(C_4, x) + 4xV(P_3, x)$.

Theorem 2.10: If G is an n-regular graph with order m, then $V(S(G), x) = x^n V(G, x) + V(G, x)$.

Proof:

Let G be an n-regular graph with order m. From the definition of splitting graph, each new vertex corresponding to every vertices of G has same degree as in G and every existing vertices of G has twice the degree. This gives the result $x^n V(G, x) + V(G, x)$ which is equal to $V(S(G), x)$.

Example 2.11: Consider 2-regular graph with order 4, that is C_4. The graph $S(C_4)$ illustrated as follows;

$S(C_4)$:

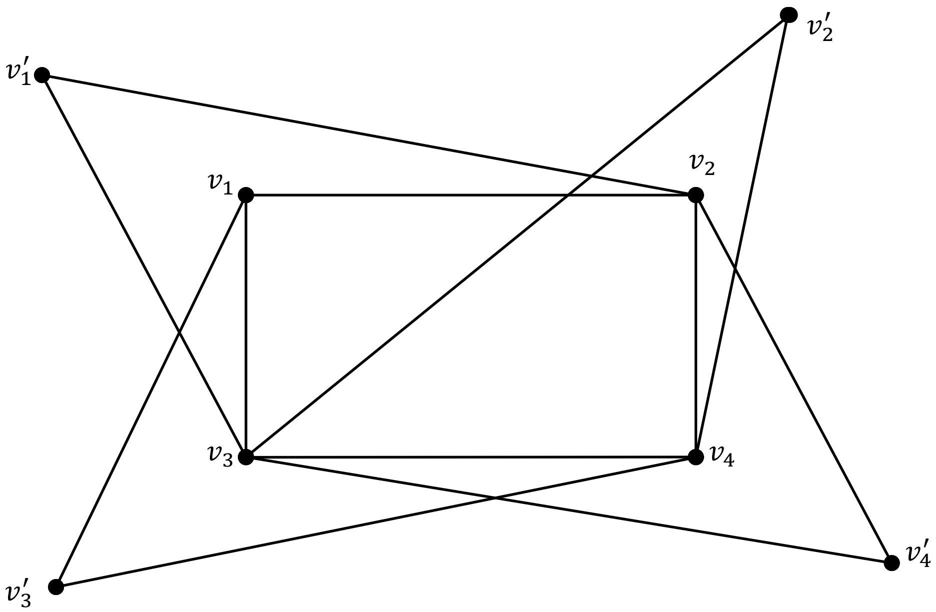


Figure 2.5

Here, $V(C_4, x) = 4x^2$.

$$V(S(C_4), x) = 4x^4 + 4x^2.$$
$$= x^2(4x^2) + 4x^2.$$
$$= x^2 V(C_4, x) + V(C_4, x).$$

Theorem 2.12: If G is an n-regular graph with order m, then

$$V(DS(G), x) = x^m + xV(G, x).$$

Proof:

Let G be an n-regular graph with order m. That is, G has order m and each vertex of G has same degree n. Therefore, from the definition of degree splitting graph, we introduce the new vertex w, the vertex w adjacent to every vertices of G. That is, w is adjacent to m vertices of G. Therefore, w has degree m and degree of each vertices of G has been increased by one. This gives the term $x^m + xV(G, x)$ and is equal to the vertex polynomial for the degree splitting graph of G. Hence, $V(DS(G), x) = x^m + xV(G, x)$.

Example 2.13: Consider 2-regular graph with order 4. that is C_4. The graph $DS(C_4)$ illustrated as follows;

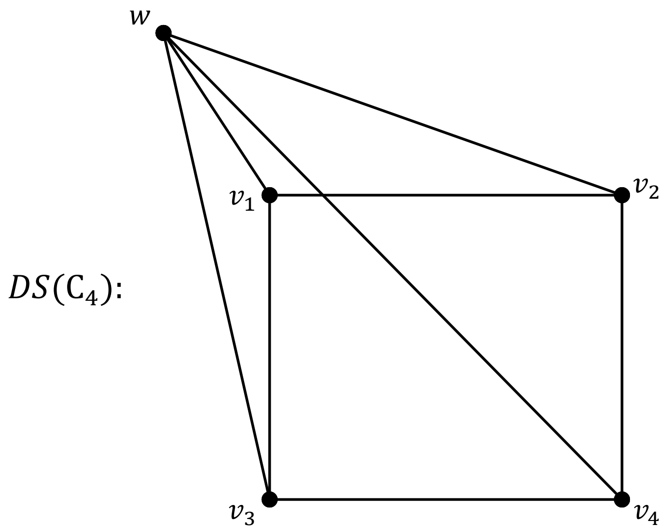


Figure 2.6

Here, $V(C_4, x) = 4x^2$.

$$V(DS(C_4), x) = x^4 + 4x^3.$$
$$= x^4 + x(4x^2).$$
$$= x^4 + xV(C_4, x).$$

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