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On a New Subclass of Multivalent Functions Defined by using Generalized Raducanu-Orhan Differential Operator

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Abstract

In recent times, the study of analytic functions has been useful in solving problems in Different field like: Computer Science, Electrical engineering electrical electronics, etc. An analytic function is said to be univalent in a domain if it does not take the same value twice in that domain while an analytic function is said to be p-valent in a domain if it does not take the same value more than p times in that domain. The main object of this paper is to introduce and investigate a new class of normalized analytic functions in the open unit disc ,U which is defined by a generalized Raducanu-Orhan differential operator. We obtain the coefficient inequality, extreme points, integral means inequalities for fractional derivatives of functions in this class.

Keywords: Multivalent functions, Raducanu-Orhan differential operator, extreme points, coefficient inequality, fractional derivatives.

1. INTRODUCTION AND PRELIMINARIES

In the study of Geometric functions theory , the univalent functions and the p-valent functions are very fascinating as we see in recent years; many new articles are written in these areas. Now, operators of normalized analytic functions become very popular, namely for Differential and Integral. Many articles discuss on operators and new generalizations of various authors. Ruscheweyh [11] led the way in the theory of differential operators in 1975.It was followed by Salagean [3] in 1983 giving another version of differential and Integral operator. Many properties have been studied and discussed by many researchers for these operators. In 2004, Al-Oboudi [2] generalized Salagean operator. In 2010, Raducanu and Orhan [1] generalized Al-Oboudi

Differential operator. In this study, we use this operator in [1] to obtain another type of differential operator.

Definition 1.1

Let A denote the class of functions, f(z) normalized by

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \tag{1}$$

which are analytic in the open unit disc, $U = \{z: |z| < 1\}$.

For $f(z) \in A$, Raducanu and Orhan [1] introduced the following operator:

$$D_{\alpha\mu}^{0} f(z) = f(z)$$

$$D_{\alpha\mu}^{1} f(z) = \alpha\mu z^2 f''(z) + (\alpha - \mu) z f'(z) + (1 - \alpha + \mu) f(z)$$

$$D_{\alpha\mu}^{n} f(z) = D_{\alpha\mu}\left(D_{\alpha\mu}^{n-1} f(z)\right) \quad (0 \le \mu \le \alpha \le 1, n \in N) \tag{2}$$

If f is given by (1), then from the definition of the operator $D_{\alpha\mu}^{n} f(z)$ in (2), it is clear to see that

$$D_{\alpha\mu}^{n} f(z) = z + \sum_{j=2}^{\infty} [1 + (\alpha\mu j + \alpha - \mu)(j-1)]^n a_j z^j \tag{3}$$

$$(n \in N_0 = N \cup \{0\})$$

Note:

when $\alpha = 1, \mu = 0$, we get the Salagean differential Operator [3].

when $\mu = 0$, we get the Al-Oboudi differential Operator [2].

Let A_P denote the class of functions of the form

$$f(z) = z^p + \sum_{j=p+1}^{\infty} a_j z^j \quad (p = 1,2,3, \dots) \tag{4}$$

Which are analytic and p-valent in the open unit disc, $U = \{z: |z| < 1\}$.

We can write the following equalities for the functions $f \in A_p$

$$D_{\alpha\mu}^{0,p} f(z) = f(z)$$

$$D_{\alpha\mu}^{1,p} f(z) = \frac{\alpha\mu}{p} z^2 f''(z) + \frac{1}{p}(\alpha-\mu) z f'(z) + \frac{1}{p}(1-\alpha+\mu) f(z) + \frac{(p-1)}{p}[1-p\alpha\mu-\alpha+\mu] z^p \quad (5)$$

$$D_{\alpha\mu}^{n,p} f(z) = D_{\alpha\mu}\left(D_{\alpha\mu}^{n-1} f(z)\right) \qquad (n \in N = 1,2,3\ldots) \quad (6)$$

If f is given by (4), then from (5) and (6) , we see that

$$D_{\alpha\mu}^{n,p} f(z) = z^p + \sum_{j=p+1}^{\infty} \left[\frac{1+(\alpha\mu j+\alpha-\mu)(j-1)}{p}\right]^n a_j z^j \quad (7)$$

$$(n \in N_0 = N \cup \{0\}, p \in N = 1,2,3\ldots)$$

Remarks:

I. If $p = 1$, $D_{\alpha\mu}^{n,p} f(z) = D_{\alpha\mu}^{n} f(z)$ introduced by Raducanu and Orhan [1]

II. If $p = 1, \alpha = 1, \mu = 0, D_{\alpha\mu}^{n,p} f(z) = D^n f(z)$ defined by Salagean [3]

III. If $p = 1, \mu = 0,$ $D_{\alpha\mu}^{n,p} f(z) = D_{\alpha}^{n} f(z)$ defined by Al-Oboudi in [2]

Definition 1.2

Let $T_{\alpha\mu}^{m,n}(b,\lambda,p)$ denote the subclass of A_P consisting of functions f satisfying the inequalities

$$R\left\{1+\frac{1}{b}\left(\frac{D_{\alpha\mu}^{m,p} f(z)}{D_{\alpha\mu}^{n,p} f(z)}-1\right)\right\} > \lambda \quad (8)$$

For some $0 \le \mu \le \alpha \le 1$, $\lambda(0 \le \lambda < 1), b \in C-\{0\}$, $m \in N, n \in N_0$, $D_{\alpha\mu}^{n,p} f(z)$ as in (7) and all $z \in U$.

Remarks:

I. If $\mu = 0, T_{\alpha\mu}^{m,n}(b,\lambda,p)$ becomes the class $N_P(m,n,\alpha,\delta,b)$ studied by Thirucheran and Stalin [7]

II. If p=1, $\mu = 0, T_{\alpha\mu}^{m,n}(b,\lambda,p)$ becomes the class $S_{b,m,n,\delta}(\alpha)$ studied by Thirucheran and Stalin [6]

III. If p=1,b=1, $\mu = 0, T_{\alpha\mu}^{m,n}(b,\lambda,p)$ reduces to the class $S_{m,n,\delta}(\alpha)$ investigated by Eker and Guney [9]

2. *Coefficient Inequalities*

We discuss coefficient inequality of the functions $f(z)$ in the class $T_{\alpha\mu}^{m,n}(b,\lambda,p)$

Theorem 2.1

Let $f(z) \in A_p$ satisfy

$$\sum_{j=p+1}^{\infty} \Psi_p(m,n,b,,j,\alpha,\mu,\lambda)\left|a_j\right| \leq 2(1-\lambda)b \qquad \textbf{(9)}$$

Where

$$\Psi_p(m,n,b,,j,\alpha,\mu,\lambda) = \left|\left[\frac{1+(\alpha\mu j+\alpha-\mu)(j-1)}{p}\right]^m - (1+\lambda b)\left[\frac{1+(\alpha\mu j+\alpha-\mu)(j-1)}{p}\right]^n\right| + \left[\frac{1+(\alpha\mu j+\alpha-\mu)(j-1)}{p}\right]^m + ((2-\lambda)b-1)\left[\frac{1+(\alpha\mu j+\alpha-\mu)(j-1)}{p}\right]^n \qquad \textbf{(10)}$$

For some $0 \leq \mu \leq \alpha \leq 1, \lambda(0 \leq \lambda < 1), b \in C - \{0\}$, $m \in N, n \in N_0$, then $f(z) \in T_{\alpha\mu}^{m,n}(b,\lambda,p)$

Proof: Suppose that (9) hold true for some $0 \leq \mu \leq \alpha \leq 1, \lambda(0 \leq \lambda < 1), b \in C - \{0\}, m \in N, n \in N_0$.For $f(z) \in A_{p,}$

Let's define $\left\{1 + \frac{1}{b}\left(\frac{D_{\alpha\mu}^{m,p} f(z)}{D_{\alpha\mu}^{n,p} f(z)} - 1\right)\right\} - \lambda = F(z)$

By the definition of the class,

$$\Rightarrow \quad F(z) \prec \frac{1+z}{1-z}$$

There exist a schwarz function $w(z)$, with $w(0) = 0$ and $|w(z)| < 1$, such that

$$F(z) = \frac{1+w(z)}{1-w(z)} \qquad \textbf{(11)}$$

This implies that

$$w(z)=\frac{F(z)-1}{F(z)+1}$$

$$|w(z)|=\left|\frac{F(z)-1}{F(z)+1}\right|<1$$

We note that

$$\left|\frac{F(z)-1}{F(z)+1}\right|=\left|\frac{D_{\alpha\mu}^{m,p}f(z)-(1+\lambda b)D_{\alpha\mu}^{n,p}f(z)}{D_{\alpha\mu}^{m,p}f(z)-[1+(\lambda-2)b]D_{\alpha\mu}^{n,p}f(z)}\right|$$

$$=\left|\frac{z^p+\sum_{j=p+1}^{\infty}\left[\frac{1+(\alpha\mu j+\alpha-\mu)(j-1)}{p}\right]^m a_jz^j-(1+\lambda b)\left\{z^p+\sum_{j=p+1}^{\infty}\left[\frac{1+(\alpha\mu j+\alpha-\mu)(j-1)}{p}\right]^n a_jz^j\right\}}{z^p+\sum_{j=p+1}^{\infty}\left[\frac{1+(\alpha\mu j+\alpha-\mu)(j-1)}{p}\right]^m a_jz^j-[1+(\lambda-2)b]\left\{z^p+\sum_{j=p+1}^{\infty}\left[\frac{1+(\alpha\mu j+\alpha-\mu)(j-1)}{p}\right]^n a_jz^j\right\}}\right| \tag{12}$$

$$=\left|\frac{-\lambda bz^p+\sum_{j=p+1}^{\infty}\left\{\left[\frac{1+(\alpha\mu j+\alpha-\mu)(j-1)}{p}\right]^m-(1+\lambda b)\left[\frac{1+(\alpha\mu j+\alpha-\mu)(j-1)}{p}\right]^n\right\}a_jz^j}{(2-\lambda)bz^p+\sum_{j=p+1}^{\infty}\left\{\left[\frac{1+(\alpha\mu j+\alpha-\mu)(j-1)}{p}\right]^m+((2-\lambda)b-1)\left[\frac{1+(\alpha\mu j+\alpha-\mu)(j-1)}{p}\right]^n\right\}a_jz^j}\right|$$

$$=\left|\frac{\lambda b-\sum_{j=p+1}^{\infty}\left\{\left[\frac{1+(\alpha\mu j+\alpha-\mu)(j-1)}{p}\right]^m-(1+\lambda b)\left[\frac{1+(\alpha\mu j+\alpha-\mu)(j-1)}{p}\right]^n\right\}a_jz^{j-p}}{(2-\lambda)b+\sum_{j=p+1}^{\infty}\left\{\left[\frac{1+(\alpha\mu j+\alpha-\mu)(j-1)}{p}\right]^m+((2-\lambda)b-1)\left[\frac{1+(\alpha\mu j+\alpha-\mu)(j-1)}{p}\right]^n\right\}a_jz^{j-p}}\right|$$

$$\leq\frac{\lambda b+\sum_{j=p+1}^{\infty}\left\{\left|\left[\frac{1+(\alpha\mu j+\alpha-\mu)(j-1)}{p}\right]^m-(1+\lambda b)\left[\frac{1+(\alpha\mu j+\alpha-\mu)(j-1)}{p}\right]^n\right|\right\}|a_j|\,|z|^{j-p}}{(2-\lambda)b-\sum_{j=p+1}^{\infty}\left\{\left[\frac{1+(\alpha\mu j+\alpha-\mu)(j-1)}{p}\right]^m+((2-\lambda)b-1)\left[\frac{1+(\alpha\mu j+\alpha-\mu)(j-1)}{p}\right]^n|a_j||z|^{j-p}\right\}}$$

$$<\frac{\lambda b+\sum_{j=p+1}^{\infty}\left\{\left|\left[\frac{1+(\alpha\mu j+\alpha-\mu)(j-1)}{p}\right]^m-(1+\lambda b)\left[\frac{1+(\alpha\mu j+\alpha-\mu)(j-1)}{p}\right]^n\right|\right\}|a_j|}{(2-\lambda)b-\sum_{j=p+1}^{\infty}\left\{\left[\frac{1+(\alpha\mu j+\alpha-\mu)(j-1)}{p}\right]^m+((2-\lambda)b-1)\left[\frac{1+(\alpha\mu j+\alpha-\mu)(j-1)}{p}\right]^n|a_j|\right\}}$$

The last expression is bounded above by 1 if

$$=\lambda b+\sum_{j=p+1}^{\infty}\left\{\left|\left[\frac{1+(\alpha\mu j+\alpha-\mu)(j-1)}{p}\right]^m-(1+\lambda b)\left[\frac{1+(\alpha\mu j+\alpha-\mu)(j-1)}{p}\right]^n\right|\right\}|a_j|$$

$$\leq(2-\lambda)b-\sum_{j=p+1}^{\infty}\left\{\left[\frac{1+(\alpha\mu j+\alpha-\mu)(j-1)}{p}\right]^m+((2-\lambda)b-1)\left[\frac{1+(\alpha\mu j+\alpha-\mu)(j-1)}{p}\right]^n|a_j|\right\}$$

$$=\sum_{j=p+1}^{\infty}\left\{\left|\left[\frac{1+(\alpha\mu j+\alpha-\mu)(j-1)}{p}\right]^m-(1+\lambda b)\left[\frac{1+(\alpha\mu j+\alpha-\mu)(j-1)}{p}\right]^n\right|+\left[\frac{1+(\alpha\mu j+\alpha-\mu)(j-1)}{p}\right]^m+((2-\lambda b)-1)\left[\frac{1+(\alpha\mu j+\alpha-\mu)(j-1)}{p}\right]^n\right\}|a_j|$$

$$\leq 2(1-\lambda)b$$

Which is equivalent to the condition in (9).This completes the proof of Theorem 2.1.

3 *Extreme points*

In view of Theorem2.1, we now introduce the subclass

$$\bar{T}_{\alpha\mu}^{m,n}(b,\lambda,p)\subset T_{\alpha\mu}^{m,n}(b,\lambda,p)$$

Which consist of functions

$$f(z)=z^p+\sum_{j=p+1}^{\infty}a_jz^j\quad(a_j\geq 0)$$

Where Taylor-Maclaurin coefficients satisfy inequality (9).Now, let us determine the extreme points of the class $\bar{T}_{\alpha\mu}^{m,n}(b,\lambda,p)$.

Theorem 3.1

Let $f_p(z)=z^p$, $f_j(z)=z^p+\frac{2(1-\lambda)b}{\Psi_p(m,n,b,,j,\alpha,\mu,\lambda)}z^j$

$(j\geq p+1)$ Where $\Psi_p(m,n,b,,j,\alpha,\mu,\lambda)$ is as given in (10), then $f\in\bar{T}_{\alpha\mu}^{m,n}(b,\lambda,p)$ if and only if it can be expressed in the form

$$f(z)=\eta_pf_p(z)+\sum_{j=p+1}^{\infty}\eta_jf_j(z)\tag{13}$$

Where $\eta_j\geq 0$ and $\eta_p=1-\sum_{j=p+1}^{\infty}\eta_j$

Proof: Assume that

$$f(z) = \eta_p f_p(z) + \sum_{j=p+1}^{\infty} \eta_j f_j(z)$$

Then

$$f(z) = (1 - \sum_{j=p+1}^{\infty} \eta_j) z^p + \sum_{j=p+1}^{\infty} \eta_j \left\{ z^p + \frac{2(1-\lambda)b}{\Psi_p(m,n,b,,j,\alpha,\mu,\lambda)} z^j \right\} \quad \textbf{(14)}$$

$$f(z) = z^p + \sum_{j=p+1}^{\infty} \eta_j (\frac{2(1-\lambda)b}{\Psi_p(m,n,b,,j,\alpha,\mu,\lambda)} z^j)$$

Thus,

$$\sum_{j=p+1}^{\infty} \Psi_p(m,n,b,,j,\alpha,\mu,\lambda)\eta_j \frac{2(1-\lambda)b}{\Psi_p(m,n,b,,j,\alpha,\mu,\lambda)}$$

$$= 2(1-\lambda)b \sum_{j=p+1}^{\infty} \eta_j$$

$$= 2(1-\lambda)b(1-\eta_p) \leq 2(1-\lambda)b$$

Which shows that f satisfy the condition (9) and therefore, $f \in \bar{T}_{\alpha\mu}^{m,n}(b,\lambda,p)$.
Conversely, suppose that $f \in \bar{T}_{\alpha\mu}^{m,n}(b,\lambda,p)$, since

$$a_j \leq \frac{2(1-\lambda)b}{\Psi_p(m,n,b,,j,\alpha,\mu,\lambda)}, \qquad j \geq p+1$$

We may set

$$\eta_j = \frac{\Psi_p(m,n,b,,j,\alpha,\mu,\lambda)}{2(1-\lambda)b} a_j$$

$$\eta_p = 1 - \sum_{j=p+1}^{\infty} \eta_j$$

Then we obtain from

$$f(z) = z^p + \sum_{j=p+1}^{\infty} a_j z^j$$

$$f(z) = (\eta_p + \sum_{j=p+1}^{\infty} \eta_j) z^p + \sum_{j=p+1}^{\infty} \eta_j \frac{2(1-\lambda)b}{\Psi_p(m,n,b,,j,\alpha,\mu,\lambda)} z^j$$

$$\Rightarrow f(z) = \eta_p z^p + \sum_{j=p+1}^{\infty} \eta_j \left\{ z^p + \frac{2(1-\lambda)b}{\Psi_p(m,n,b,,j,\alpha,\mu,\lambda)} z^j \right\}$$

$$= \eta_p z^p + \sum_{j=p+1}^{\infty} \eta_j f_j(z)$$

This completes the proof of Theorem 3.1

Corollary 3.2

The extreme points of the class $\bar{T}_{\alpha\mu}^{m,n}(b,\lambda,p)$are given by

$f_p(z) = z^p$, $f_j(z) = z^p + \frac{2(1-\lambda)b}{\Psi_p(m,n,b,,j,\alpha,\mu,\lambda)} z^j$

$(j \geq p+1)$ Where $\Psi_p(m,n,b,,j,\alpha,\mu,\lambda)$ is as given(10).

Theorem 3.3

Let the functions

$$f(z) = z^p + \sum_{j=p+1}^{\infty} a_j z^j \quad (a_j \geq 0)$$

$$g(z) = z^p + \sum_{j=p+1}^{\infty} b_j z^j \quad (b_j \geq 0)$$

Be in the class $\bar{T}_{\alpha\mu}^{m,n}(b,\lambda,p)$ for $0 \leq t \leq 1$, then the function $h(z)$ defined by

$$h(z) = (1-t)f(z) + tg(z) = z^p + \sum_{j=p+1}^{\infty} c_j z^j \tag{15}$$

$$c_j = (1-t)a_j + tb_j \geq 0$$

Is also in the class $\bar{T}_{\alpha\mu}^{m,n}(b,\lambda,p)$

Proof: suppose that each of the functions f, g is in the class $\bar{T}_{\alpha\mu}^{m,n}(b,\lambda,p)$, then making use of (9),

We see that

$$\sum_{j=p+1}^{\infty} \Psi_p(m,n,b,,j,\alpha,\mu,\lambda)c_j \tag{16}$$

$$= (1-t)\sum_{j=p+1}^{\infty} \Psi_p(m,n,b,,j,\alpha,\mu,\lambda)a_j$$

$$+ t\sum_{j=p+1}^{\infty} \Psi_p(m,n,b,,j,\alpha,\mu,\lambda)b_j$$

$$\leq (1-t)2(1-\lambda)b + t2(1-\lambda)b$$

$$= 2(1-\lambda)b$$

This completes the proof of Theorem 3.3.

4 Integral Mean Inequalities for fractional derivatives

We will make use of the definitions of fractional derivatives by Owa [10], Srivastava and Owa [4].

Definition 4.1

The fractional derivative of order l is defined, for functions f by

$$D_z^l = \frac{1}{\Gamma(1-l)}\frac{d}{dz}\int_0^z \frac{f(t)}{(z-t)^l}d(t), \quad (0 \leq l < 1) \tag{17}$$

Where f is an analytic function in a simply connected region of the z-plane containing the origin, and the multiplicity of $(z-t)^{-l}$ is removed by requiring $\log(z-t)$ to be real when $(z-t) > 0$

Definition 4.2

Under the hypothesis of definition 4.1, the fractional derivative of order $q+l$

Is defined, for function f, by

$$D_Z^{q+l}f(z) = \frac{d^q}{d_{z^q}}D_z^l(0 \leq l < 1, q \in N_0) \tag{18}$$

It readily follows from (18)

$$D_Z^l z^k = \frac{\Gamma(k+1)}{\Gamma(k-l+1)} z^{k-l} (0 \leq l < 1, k \in N) \quad (19)$$

Further, we need the concept of subordination between analytic functions and a subordination theorem of Littlewoods in our investigation.

Definition 4.3

For two functions f and g, analytic in U, and write

$$f \prec g, (z \in U)$$

If there exist a schwarz function $w(z)$ analytic in U with $w(0) = 0$ and $|w(z)| < 1$, such that

$$f(z) = g(w(z), (z \in U)$$

In particular, if the function $g(z)$ is univalent in U, the above subordination is equivalent to

$$f(0) = g(0), f(U) = g(U), \ \ in \ [8]$$

In 1925, Littlewoods [5], proved the following subordination theorem.

Lemma 4.4

If $f(z)$ and $g(z)$ are analytic in U with $f(z) \prec g(z)$, then for $\sigma > 0$ and $z = re^{i\theta} (0 < r < 1)$

$$\int_0^{2\pi} |f(z)|^\sigma d\theta \leq \int_0^{2\pi} |g(z)|^\sigma d\theta \quad (20)$$

Theorem 4.5

Let $f \in \bar{T}_{\alpha\mu}^{m,n}(b, \lambda, p)$ and suppose that

$$\sum_{j=p+1}^{\infty} (j-q)_{q+1} a_j \leq \frac{2(1-\lambda)b\Gamma(k+1)\Gamma(2+p-l-q)}{\Psi_p(m,n,b,,j,\alpha,\mu,\lambda)\Gamma(k+1-l-q)\Gamma(p+1-q)} \quad (21)$$

For some $j \geq q, 0 \leq l < 1, (j-q)_{q+1}$ denotes the pochhammer symbol defined by $(j-q)_{q+1} = (j-q)(j-q+1) \dots j$. Also, let the function

$$f_k(z) = z^p + \frac{2(1-\lambda)b}{\Psi_p(m,n,b,,j,\alpha,\mu,\lambda)} z^k \; (k \geq p+1) \qquad \textbf{(22)}$$

If there exist an analytic function $w(z)$ given by

$$(w(z))^{k-p} = \frac{\Psi_p(m,n,b,,j,\alpha,\mu,\lambda)}{2(1-\lambda)b} \frac{\Gamma(k+1-l-q)}{\Gamma(k+1)} \sum_{j=p+1}^{\infty} (j-q)_{q+1}\Phi(j)a_j z^{j-p} \qquad \textbf{(23)}$$

$$k \geq q$$

$$\Phi(j) = \frac{\Gamma(j-q)}{\Gamma(k+1-l-q)} \; (0 \leq l < 1, j \geq p+1) \qquad \textbf{(24)}$$

then for $\sigma > 0$ and $z = re^{i\theta}(0 < r < 1)$

$$\int_0^{2\pi} \left|D_Z^{q+l} f(z)\right|^{\sigma} d\theta \leq \int_0^{2\pi} \left|D_Z^{q+l} f_k(z)\right|^{\sigma} d\theta \qquad \textbf{(25)}$$

Proof: Let

$$f(z) = z^p + \sum_{j=p+1}^{\infty} a_j z^j$$

By means of (19) and definition 4.1, we have

$$D_Z^{q+l} f(z) = \frac{\Gamma(p+1)z^{p-l-q}}{\Gamma(p+1-l-q)} \left[1 + \sum_{j=p+1}^{\infty} \frac{\Gamma(j+1)\Gamma(p+1-l-q)}{\Gamma(p+1)\Gamma(j+1-l-q)} a_j z^{j-p}\right] \qquad (22)$$

$$= \frac{\Gamma(p+1)z^{p-l-q}}{\Gamma(p+1-l-q)} \left[1 + \sum_{j=p+1}^{\infty} \frac{\Gamma(p+1-l-q)}{\Gamma(p+1)} (j-q)_{q+1}\Phi(j)a_j z^{j-p}\right]$$

Where

$$\Phi(j) = \frac{\Gamma(j-q)}{\Gamma(j+1-l-q)}, 0 \leq l < 1, j \geq p+1$$

Since Φ is decreasing function of j,we get

$$0 < \Phi(j) \leq \Phi(p+1) = \frac{\Gamma(p+1-q)}{\Gamma(2+p-l-q)}$$

Similarly, from (9), (19) and definition 4.1

$$D_Z^{q+l} f_k(z) = \frac{\Gamma(p+1) z^{k-l-q}}{\Gamma(p+1-l-q)} \left[1 + \frac{2(1-\lambda)b\Gamma(k+1)\Gamma(p+1-l-q)}{\Psi_p(m,n,b,,j,\alpha,\mu,\lambda)\Gamma(p+1)\Gamma(k+1-l-q)} z^{k-p}\right] \quad \textbf{(27)}$$

for $\sigma > 0$ and $z = re^{i\theta}$ $(0 < r < 1)$,we show that

$$\int_0^{2\pi} \left|1 + \sum_{j=p+1}^{\infty} \frac{\Gamma(j+1)\Gamma(p+1-l-q)}{\Gamma(p+1)\Gamma(j+1-l-q)} a_j z^{j-p}\right|^{\sigma} d\theta \leq \int_0^{2\pi} \left|1 + \frac{2(1-\lambda)b\Gamma(k+1)\Gamma(p+1-l-q)}{\Psi_p(m,n,b,,j,\alpha,\mu,\lambda)\Gamma(p+1)\Gamma(k+1-l-q)} z^{k-p}\right|^{\sigma} d\theta \quad \textbf{(28)}$$

So,by applying lemma 4.4,it is enough to show that

$$1 + \sum_{j=p+1}^{\infty} \frac{\Gamma(j+1)\Gamma(p+1-l-q)}{\Gamma(p+1)\Gamma(j+1-l-q)} a_j z^{j-p} \prec 1 + \frac{2(1-\lambda)b\Gamma(k+1)\Gamma(p+1-l-q)}{\Psi_p(m,n,b,,j,\alpha,\mu,\lambda)\Gamma(p+1)\Gamma(k+1-l-q)} z^{k-p}$$

$$j \geq p+1$$

If the above subordination holds true, then we have analytic function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$, such that

$$1 + \sum_{j=p+1}^{\infty} \frac{\Gamma(j+1)\Gamma(p+1-l-q)}{\Gamma(p+1)\Gamma(j+1-l-q)} a_j z^{j-p} = 1 + \frac{2(1-\lambda)b\Gamma(k+1)\Gamma(p+1-l-q)}{\Psi_p(m,n,b,,j,\alpha,\mu,\lambda)\Gamma(p+1)\Gamma(k+1-l-q)} w(z)^{k-p}$$

By the condition of the theorem, we define the function $w(z)$ by

$$w(Z)^{k-p} = \frac{\Psi_p(m,n,b,,j,\alpha,\mu,\lambda)}{2(1-\lambda)b} \frac{\Gamma(k+1-l-q)}{\Gamma(k+1)} \sum_{j=p+1}^{\infty} (j-q)_{q+1} \Phi(j) a_j z^{j-p} \quad \textbf{(29)}$$

Which readily yields $w(0)=0$. For such a function $w(z)$, we have

$$|w(z)|^{k-p} \le \frac{\Psi_p(m,n,b,,j,\alpha,\mu,\lambda)}{2(1-\lambda)b}\frac{\Gamma(k+1-l-q)}{\Gamma(k+1)}\sum_{j=p+1}^{\infty} (j-q)_{q+1}\Phi(j)a_j|z|^{j-p}$$

$$\le |z|\frac{\Psi_p(m,n,b,,j,\alpha,\mu,\lambda)}{2(1-\lambda)b}\frac{\Gamma(k+1-l-q)}{\Gamma(k+1)}\Phi(p+1)\sum_{j=p+1}^{\infty} (j-q)_{q+1}a_j$$

$$\le |z| < 1$$

Where $\Psi_p(m,n,b,,j,\alpha,\mu,\lambda)$ is as in (10). By means of the hypothesis of the theorem, thus, the Theorem is proved.

As a special case $q=0$**,** we have the following from Theorem 4.5.

Corollary 4.6

Let $f \in \bar{T}_{\alpha\mu}^{m,n}(b,\lambda,p)$ and suppose that

$$\sum_{j=p+1}^{\infty} ja_j \le \frac{2(1-\lambda)b\Gamma(k+1)\Gamma(2+p-l)}{\Psi_p(m,n,b,,j,\alpha,\mu,\lambda)\Gamma(k+1-l)\Gamma(p+1)} \qquad \textbf{(30)}$$

$(j \ge p+1)$ if $t$here exist analytic function $w(z)$ define by

$$(w(z))^{k-p} = \frac{\Psi_p(m,n,b,,j,\alpha,\mu,\lambda)}{2(1-\lambda)b}\frac{\Gamma(k+1-l)}{\Gamma(k+1)}\sum_{j=p+1}^{\infty} j\Phi(j)a_jz^{j-p} \qquad \textbf{(31)}$$

With

$$\Phi(j) = \frac{\Gamma(j)}{\Gamma(j+1-l)}(0 \le l < 1, j \ge p+1)$$

Then $\sigma > 0$ and $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} \left|D_z^l f(z)\right|^{\sigma} d\theta \le \int_0^{2\pi} \left|D_z^l f_k(z)\right|^{\sigma} d\theta \qquad \textbf{(32)}$$

Letting $q=1$, we have the following from Theorem 4.5,

Corollary 4.7

Let $f \in \bar{T}_{\alpha\mu}^{m,n}(b,\lambda,p)$ and suppose that

$$\sum_{j=p+1}^{\infty} j(j-1)a_j \le \frac{2(1-\lambda)b\Gamma(k+1)\Gamma(p+1-l)}{\Psi_p(m,n,b,,j,\alpha,\mu,\lambda)\Gamma(k-l)\Gamma(p)} \qquad \textbf{(33)}$$

$(j \geq p+1)$ if there exist analytic function $w(z)$ define by

$$(w(z))^{k-p} = \frac{\Psi_p(m,n,b,,j,\alpha,\mu,\lambda)}{2(1-\lambda)b} \frac{\Gamma(k-l)}{\Gamma(k+1)} \sum_{j=p+1}^{\infty} j(j-1)\Phi(j)a_j z^{j-p} \quad (34)$$

With

$$\Phi(j) = \frac{\Gamma(j-1)}{\Gamma(j-l)}, (0 \leq l < 1, j \geq p+1) \quad (35)$$

Then $\sigma > 0$ and $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} \left|D_Z^{1+l} f(z)\right|^{\sigma} d\theta \leq \int_0^{2\pi} \left|D_Z^{1+l} f_k(z)\right|^{\sigma} d\theta \quad (36)$$

CONCLUSION

In this paper, using a generalized Raducanu-Orhan differential operator, we defined new subclass of Multivalent functions and established its properties. Results obtained provide new Properties of certain subclasses of Analytic functions.

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