

## Fixed Point Results in $S_b$ -Metric Spaces via $(\alpha, \psi, \phi)$ - Generalized Weakly Contractive Maps

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### Abstract

In this paper we introduce the notion of  $(\alpha, \psi, \phi)$  - generalized weakly contractive maps in  $S_b$ -metric spaces and obtain existence of fixed points for such maps. These results extend and improve the results of Babu and Leta[1]. Further, in support of our results examples are also given.

**Key words and phrases:**  $(\alpha, \psi, \phi)$ - generalized weakly contractive maps,  $S_b$ - metric spaces, fixed points.

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### 1. INTRODUCTION AND PRELIMINARIES

Now a days, generalization of metric spaces in various structures have drawn the attention of scientists due to the development and generalization of many results in fixed point theory. Recently, the concept of an  $S_b$ -metric space as a generalization of b-metric spaces and S-metric spaces have been introduced in [11] and proved fixed point theorems in complete  $S_b$ -metric space. But, very recently Tas and Ozur[15] studied some relations between  $S_b$ -metric spaces and some other metric spaces. Pioneering results dealing with fixed point theorems for mappings satisfying certain contractive conditions on  $S_b$ -metric spaces can be referred in [6,8,11,12,13,15,16].

We start by recalling some notations, definitions and lemmas and well known results which are useful in what follows.

In 2012, Sedghi, Shobe and Aliouche[10] introduced  $S$ -metric on a nonempty set  $X$  as follows.

**Definition 1.1[10].** *Let  $X$  be a nonempty set. A  $S$ -metric on  $X$  is a function  $S : X^3 \rightarrow [0, \infty)$  that satisfies the following conditions:*

$$(1.1.1) \quad 0 < S(x, y, z) \text{ for all } x, y, z \in X \text{ with } x \neq y \neq z.$$

$$(1.1.2) \quad S(x, y, z) = 0 \text{ if } x = y = z.$$

$$(1.1.3) \quad S(x, y, z) \leq [S(x, x, a) + S(y, y, a) + S(z, z, a)] \\ \text{for all } x, y, z, a \in X. \text{ The pair } (X, S) \text{ is called } S\text{-metric space.}$$

For more literature on  $S$ -metric spaces we refer[1,3,4,5,7,9,14].

**Definition 1.2[2].** *Let  $X$  be a non-empty set and  $s \geq 1$  be a real number. A function  $d : X \times X \rightarrow [0, \infty)$  is called a  $b$ -metric on  $X$  if it satisfies the following conditions:*

$$(1.2.1) \quad d(x, y) = 0 \text{ if and only if } x = y.$$

$$(1.2.2) \quad d(x, y) = d(y, x) \text{ for all } x, y \in X.$$

$$(1.2.3) \quad d(x, y) \leq s[d(x, z) + d(z, y)], \text{ for all } x, y, z \in X.$$

*Then the order pair  $(X, d)$  is said to be a  $b$ - metric space with  $s \geq 1$ .*

Here we note that the class of  $b$ -metric spaces is larger class than the class of metric spaces, since  $(X, d)$  is a metric space when  $s = 1$ .

Inspired by the works of Bakhtin [2] and Sedghi *et al.*, [10], Souayah and Mlaiki [11] introduced the concept of  $S_b$ -metric space. Tas and Ozur[15] modified the definition of  $S_b$  -metric spaces.

**Definition 1.3[15].** *Let  $X$  be a nonempty set with  $s \geq 1$ . A  $S_b$ -metric on  $X$  is a function  $S_b : X^3 \rightarrow [0, \infty)$  that satisfies the following conditions:*

$$(1.3.1) \quad S_b(x, y, z) = 0 \text{ if } x = y = z.$$

$$(1.3.2) \quad S_b(x, y, z) \leq s[S_b(x, x, a) + S_b(y, y, a) + S_b(z, z, a)]$$

for all  $x, y, z, a \in X$ .

The pair  $(X, S_b)$  is called a  $S_b$ -metric space with index  $s$ .

We note that  $S_b$ -metric spaces are generalizations of  $S$ -metric spaces since every  $S$ -metric is an  $S_b$  metric with  $s = 1$  and its converse need not be true.

**Example 1.4[15].** Let  $X = R$  and the function  $S_b$  be defined as

$$S_b(x, y, z) = \frac{1}{16}(|x - y| + |y - z| + |x - z|)^2.$$

Then  $(X, S_b)$  is an  $S_b$  metric space with  $s = 4$ , but it is not an  $S$ -metric space. Indeed, for  $x = 4, y = 6, z = 8$  and  $a = 5$ , we get

$$S_b(4, 6, 8) = 4 > S_b(4, 4, 5) + S_b(6, 6, 5) + S_b(8, 8, 5).$$

Thus,  $S_b$ - metric spaces are more general than  $S$ -metric spaces.

**Definition 1.5[15].** Let  $(X, S_b)$  be an  $S_b$ -metric space and  $s \geq 1$ . An  $S_b$ -metric is called symmetric if  $S_b(x, x, y) = S_b(y, y, x)$ , for all  $x, y \in X$ .

**Remarks 1.6[15].**

(i) There exists an  $S_b$ - metric space which is not generated by any

$b$ -metric.(Example 2.6[15]).

(ii) There exist non-symmetric  $S_b$ -metric spaces.

(iii) Symmetry condition is automatically satisfied by an  $S$ -metric

(Lemma 2.5 on page 260 in [7]) (Example 2.3[7]).

**Lemma 1.7[8].** In an  $S_b$ - metric space, we have

(i)  $S_b(x, x, y) \leq sS_b(y, y, x)$  and  $S_b(y, y, x) \leq sS_b(x, x, y)$

(ii)  $S_b(x, x, z) \leq 2sS_b(x, x, y) + s^2S_b(y, y, z)$ .

**Definition 1.8[8].** If  $(X, S_b)$  is an  $S_b$ -metric space, a sequence  $\{x_n\}$  in  $X$  is said to be :

(i)  $S_b$ - Cauchy sequence, if for each  $\epsilon > 0$ ,

there exists  $n_o \in N$  such that  $S_b(x_n, x_n, x_m) \leq \epsilon$  for all

$$n, m \geq n_o.$$

(ii)  $S_b$ -convergent to a point  $x \in X$ , if for each  $\epsilon > 0$ , there exist

a positive integer  $n_o$  such that  $S_b(x_n, x_n, x) < \epsilon$  and

$S_b(x, x, x_n) < \epsilon$  for all  $n \geq n_o$ , and we denote  $\lim_{n \rightarrow \infty} x_n = x$ .

**Definition 1.9[8].** An  $S_b$ -metric space  $(X, S_b)$  is called complete if every  $S_b$ -Cauchy sequence is  $S_b$ -convergent in  $X$ .

**Lemma 1.10[8].** If  $(X, S_b)$  is an  $S_b$ -metric space with  $s \geq 1$  and  $\{x_n\}$  is  $S_b$ -convergent to  $x$  then we have

$$(i) \frac{1}{2s}S_b(y, y, x) \leq \liminf_{n \rightarrow \infty} S_b(y, y, x_n) \leq \limsup_{n \rightarrow \infty} S_b(y, y, x_n) \\ \leq 2sS_b(y, y, x) \text{ and}$$

$$(ii) \frac{1}{s^2}S_b(x, x, y) \leq \liminf_{n \rightarrow \infty} S_b(x_n, x_n, y) \leq \limsup_{n \rightarrow \infty} S_b(x_n, x_n, y) \\ \leq s^2S_b(x, x, y).$$

Tas and Ozgur[15] proved Banach contraction principle in  $S_b$ -metric spaces.

**Theorem 1.11[15].** Let  $(X, S_b)$  be a complete  $S_b$ -metric space with  $s \geq 1$  and  $T : X \rightarrow X$  be a self-mapping satisfying

$$S_b(Tx, Tx, Ty) \leq hS_b(x, x, y)$$

for all  $x, y, z \in X$ , where  $0 < h < \frac{1}{s^2}$ . Then  $T$  has a fixed point  $x$  in  $X$ .

In particular, if  $x \in X$  and  $x_{n+1} = T^n x_0$  for  $n = 0, 1, 2, \dots$   $\{x_n\}$  converges, say, to  $y$  and  $y$  is a fixed point of  $T$ .

Through this paper we denote:

$$\Psi = \{\psi : [0, \infty) \rightarrow [0, \infty) | \psi \text{ is continuous, nondecreasing and}$$

$$\psi(t) = 0 \text{ iff } t = 0\}$$

$$\Phi = \{\phi : [0, \infty) \rightarrow [0, \infty) | \phi \text{ is continuous, } \phi(t) = 0 \text{ iff } t = 0\}.$$

Recently, in 2017, Babu and Leta[1] introduced  $(\alpha, \psi, \phi)$ -generalized weakly contractive map in  $S$ -metric spaces.

**Definition 1.12** [1]. Let  $(X, S)$  be an  $S$ -metric space. Let  $f : X \rightarrow X$  be a self map of  $X$ . Then  $f$  is said to be  $(\alpha, \psi, \phi)$ - generalized weakly contractive map if it satisfies the following condition:

(1.12.1) if there exist  $\alpha \in (0, 1)$ ,  $\psi \in \Psi$  and  $\phi \in \Phi$  such that

$$\psi(S(fx, fy, fz)) \leq \psi(M_\alpha(x, y, z)) - \phi(M_\alpha(x, y, z))$$

where

$$M_\alpha(x, y, z) = \max\{S(x, y, z), S(x, x, fx), S(y, y, fy), S(z, z, fz),$$

$$\alpha S(fx, fy, fz) + (1 - \alpha)S(fy, fz, z)\}$$

for all  $x, y, z \in X$ .

**Theorem 1.13** [1]. Let  $(X, S)$  be a complete  $S$ -metric space and let  $f$  be an  $(\alpha, \psi, \phi)$ - generalized weakly contractive map. Then  $f$  has unique fixed point  $u$  (say) and  $f$  is  $S$ -continuous at  $u$ .

The aim of the paper is to extend the results of Babu and Leta[1] to more general class ie.,  $S_b$ -metric spaces.

**2. MAIN RESULTS** In this section we introduce the notion of  $(\alpha, \psi, \phi)$ -generalized weakly contractive map in  $S_b$ -metric spaces and we prove fixed point theorems for such maps.

**Definition 2.1** Let  $(X, S_b)$  be an  $S_b$ -metric space with  $s \geq 1$ . Let  $f : X \rightarrow X$  be a self map of  $X$ . Then  $f$  is called  $(\alpha, \psi, \phi)$ - generalized weakly contractive map :

(2.1.1) if there exist  $\alpha \in (0, 1)$ ,  $\psi \in \Psi$  and  $\phi \in \Phi$  such that

$$\psi(4s^4 S_b(fx, fy, fz)) \leq \psi(M_\alpha(x, y, z)) - \phi(M_\alpha(x, y, z))$$

where

$$M_\alpha(x, y, z) = \max\{S_b(x, y, z), S_b(x, x, fx), S_b(y, y, fy), S_b(z, z, fz),$$

$$\frac{1}{2s^2}[\alpha S_b(fx, fy, fz) + (1 - \alpha)S_b(fx, fy, fz)S_b(fy, fz, x)]\}$$

for all  $x, y, z \in X$ .

**Theorem 2.2.** Let  $(X, S_b)$  be a complete  $S_b$ -metric space with  $s \geq 1$  and let  $f$  be an  $(\alpha, \psi, \phi)$ - generalized weakly contractive map. Then  $f$  has a unique fixed point  $z \in X$ .

**Proof.** Let  $x_0 \in X$ . We define a sequence  $\{x_n\}$  in  $X$  by  $x_{n+1} = fx_n$  for some  $n$ , then  $x_n$  is fixed point of  $f$  and we are through.

Now we assume that  $x_n \neq x_{n+1}$  for all  $n$ . On choosing  $x = y = x_{n-1}, z = x_n$  in (2.1.1), we have

$$(2.2.1) \quad \psi(S_b(x_n, x_n, x_{n+1})) \leq \psi(4s^4 S_b(fx_{n-1}, fx_{n-1}, fx_n))$$

$$\leq \psi(M_\alpha(x_{n-1}, x_{n-1}, x_n)) - \phi(M_\alpha(x_{n-1}, x_{n-1}, x_n))$$

where

$$(2.2.2) \quad M_\alpha(x_{n-1}, x_{n-1}, x_n) = \max\{S_b(x_{n-1}, x_{n-1}, x_n), S_b(x_n, x_n, x_{n+1}),$$

$$\frac{\alpha}{2s^2} S_b(x_n, x_n, x_{n+1})\}$$

$$= \max\{S_b(x_{n-1}, x_{n-1}, x_n), S_b(x_n, x_n, x_{n+1})\}.$$

If  $S_b(x_n, x_n, x_{n+1})$  is the maximum in (2.2.2), then from (2.2.1), we have

$$\psi(S_b(x_n, x_n, x_{n+1})) \leq \psi(S_b(x_n, x_n, x_{n+1})) - \phi(S_b(x_n, x_n, x_{n+1})).$$

This implies  $\phi(S_b(x_n, x_n, x_{n+1})) = 0$ . Hence  $x_n = x_{n+1}$ , a contradiction to our assumption. Thus

$$(2.2.3) \quad \psi(S_b(x_n, x_n, x_{n+1})) \leq \psi(S_b(x_{n-1}, x_{n-1}, x_n)) - \phi(S_b(x_{n-1}, x_{n-1}, x_n))$$

$$< \psi(S_b(x_{n-1}, x_{n-1}, x_n)).$$

By the property of  $\psi$ , we have

$$S_b(x_n, x_n, x_{n+1}) < S_b(x_{n-1}, x_{n-1}, x_n).$$

Hence  $\{S_b(x_n, x_n, x_{n+1})\}$  is a strictly decreasing sequence of positive real numbers.

Hence there exists  $r \geq 0$  such that

$$\lim_{n \rightarrow \infty} S_b(x_n, x_n, x_{n+1}) = r.$$

Making  $n \rightarrow \infty$  in (2.2.3), we have  $\psi(r) \leq \psi(r) - \phi(r)$ . This implies  $\phi(r) = 0$ . Hence  $r = 0$ . Thus

$$(2.2.4) \quad \lim_{n \rightarrow \infty} S_b(x_n, x_n, x_{n+1}) = 0.$$

On choosing  $x = y = x_{n+1}, z = x_n$  in (2.1.1), we have

$$(2.2.5) \quad \psi(S_b(x_{n+1}, x_{n+1}, x_n)) \leq \psi(4s^4 S_b(fx_n, fx_n, fx_{n-1}))$$

$$\leq \psi(M_\alpha(x_n, x_n, x_{n-1})) - \phi(M_\alpha(x_n, x_n, x_{n-1}))$$

where

$$(2.2.6) \quad M_\alpha(x_n, x_n, x_{n-1}) = \max\{S_b(x_n, x_n, x_{n-1}), S_b(x_n, x_n, x_{n+1}),$$

$$S_b(x_{n-1}, x_{n-1}, x_n), \frac{1}{2s^2}[\alpha S_b(x_{n+1}, x_{n+1}, x_n)$$

$$+(1-\alpha)S_b(x_{n+1}, x_{n+1}, x_n).S_b(x_{n+1}, x_{n+1}, x_{n-1}).S_b(x_n, x_n, x_n)]\}$$

$$= \max\{S_b(x_n, x_n, x_{n-1}), \frac{1}{2s^2}\alpha S_b(x_{n+1}, x_{n+1}, x_n)\}.$$

If  $\frac{1}{2s^2}\alpha S_b(x_{n+1}, x_{n+1}, x_n)$  is the maximum in (2.2.6), then from (2.2.5), we have

$$\psi(S_b(x_{n+1}, x_{n+1}, x_n)) \leq \psi(\frac{\alpha}{2s^2}S_b(x_{n+1}, x_{n+1}, x_n)) - \phi(\frac{\alpha}{2s^2}S_b(x_{n+1}, x_{n+1}, x_n)).$$

Therefore

$$S_b(x_{n+1}, x_{n+1}, x_n) < \frac{\alpha}{2s^2}S_b(x_{n+1}, x_{n+1}, x_n),$$

a contradiction. Thus from (2.2.5), we have

$$(2.2.7) \quad \psi(S_b(x_{n+1}, x_{n+1}, x_n)) \leq \psi(S_b(x_n, x_n, x_{n-1})) - \phi(S_b(x_n, x_n, x_{n-1}))$$

$$< \psi(S_b(x_n, x_n, x_{n-1})).$$

By the property of  $\psi$ , we have

$$S_b(x_{n+1}, x_{n+1}, x_n) < S_b(x_n, x_n, x_{n-1}).$$

Hence  $\{S_b(x_{n+1}, x_{n+1}, x_n)\}$  is a strictly decreasing sequence of positive real numbers.

Hence there exists  $r \geq 0$  such that

$$\lim_{n \rightarrow \infty} S_b(x_{n+1}, x_{n+1}, x_n) = r.$$

Making  $n \rightarrow \infty$  in (2.2.7), we have  $\psi(r) \leq \psi(r) - \phi(r)$ . This implies  $\phi(r) = 0$ . Hence  $r = 0$ . Thus

$$(2.2.8) \quad \lim_{n \rightarrow \infty} S_b(x_{n+1}, x_{n+1}, x_n) = 0.$$

We now show that  $\{x_n\}$  is a  $S_b$ -Cauchy sequence in  $X$ . Suppose that  $\{x_n\}$  is not a  $S_b$ -Cauchy sequence. Then there exists  $\varepsilon > 0$  and monotone increasing sequences of

real numbers  $m(k)$  and  $n(k)$  with  $n(k) > m(k) > k$  such that

$$(2.2.9) \quad S_b(x_{m(k)}, x_{m(k)}, x_{n(k)}) \geq \epsilon \text{ and } S_b(x_{m(k)}, x_{m(k)}, x_{n(k)-1}) < \epsilon.$$

Now from (2.1.1), (2.2.5) and (2.2.9), we have

$$\begin{aligned} (2.2.10) \quad \psi(4s^4\epsilon) &\leq \psi(4s^4S_b(x_{m(k)}, x_{m(k)}, x_{n(k)})) \\ &\leq \psi(M_\alpha(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1})) - \phi(M_\alpha(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1})) \end{aligned}$$

where

$$\begin{aligned} M_\alpha(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1}) &= \max\{S_b(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1}), \\ &\quad S_b(x_{m(k)-1}, x_{m(k)-1}, fx_{m(k)-1}), S_b(x_{m(k)-1}, x_{m(k)-1}, fx_{m(k)-1}), \\ &\quad S_b(x_{n(k)-1}, x_{n(k)-1}, fx_{n(k)-1}), \frac{1}{2s^2}[\alpha S_b(fx_{m(k)-1}, fx_{m(k)-1}, fx_{n(k)-1}) \\ &\quad + (1-\alpha)S_b(fx_{m(k)-1}, fx_{m(k)-1}, x_{m(k)-1}) \\ &\quad S_b(fx_{m(k)-1}, fx_{m(k)-1}, x_{n(k)-1})S_b(fx_{n(k)-1}, fx_{n(k)-1}, x_{m(k)-1})]\} \\ &= \max\{S_b(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1}), S_b(x_{m(k)-1}, x_{m(k)-1}, x_{m(k)}), \\ &\quad S_b(x_{m(k)-1}, x_{m(k)-1}, x_{m(k)}), S_b(x_{n(k)-1}, x_{n(k)-1}, x_{n(k)}), \\ &\quad \frac{1}{2s^2}[\alpha S_b(x_{m(k)}, x_{m(k)}, x_{n(k)}) + (1-\alpha)S_b(x_{m(k)}, x_{m(k)}, x_{m(k)-1}) \\ &\quad S_b(x_{m(k)}, x_{m(k)}, x_{n(k)-1})S_b(x_{n(k)}, x_{n(k)}, x_{m(k)-1})]\}. \end{aligned}$$

As  $k \rightarrow \infty$ ,

$$\begin{aligned} (2.1.11) \quad \lim_{n \rightarrow \infty} M_\alpha(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1}) \\ &= \max\{S_b(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1}), \frac{\alpha}{2s^2}S_b(x_{m(k)}, x_{m(k)}, x_{n(k)})\}. \end{aligned}$$

If  $\frac{\alpha}{2s^2}S_b(x_{m(k)}, x_{m(k)}, x_{n(k)})$  is maximum,

$$\begin{aligned} \psi(4s^4S_b(x_{m(k)}, x_{m(k)}, x_{n(k)})) &\leq \psi(\frac{\alpha}{2s^2}S_b(x_{m(k)}, x_{m(k)}, x_{n(k)})) \\ &\quad - \phi(\frac{\alpha}{2s^2}S_b(x_{m(k)}, x_{m(k)}, x_{n(k)})). \end{aligned}$$

This implies

$$\psi(4s^4 S_b(x_{m(k)}, x_{m(k)}, x_{n(k)})) < \psi\left(\frac{\alpha}{2s^2} S_b(x_{m(k)}, x_{m(k)}, x_{n(k)})\right).$$

By the property of  $\psi$ , we have

$$4s^4 S_b(x_{m(k)}, x_{m(k)}, x_{n(k)}) < \frac{\alpha}{2s^2} S_b(x_{m(k)}, x_{m(k)}, x_{n(k)}).$$

This gives rise to

$4s^4 < \frac{\alpha}{2s^2} \Rightarrow 8s^6 < \alpha$ , a contradiction since  $\alpha \in (0, 1)$  and  $s \geq 1$ . Therefore

$$\begin{aligned} (2.1.12) \quad \psi(4s^4 S_b(x_{m(k)}, x_{m(k)}, x_{n(k)})) &\leq \psi\left(\frac{\alpha}{2s^2} S_b(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1})\right) \\ &\quad - \phi\left(\frac{\alpha}{2s^2} S_b(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1})\right) \\ &< \psi\left(\frac{\alpha}{2s^2} S_b(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1})\right). \end{aligned}$$

Now from Lemma 1.7, we have

$$\begin{aligned} 4s^4 S_b(x_{m(k)}, x_{m(k)}, x_{n(k)}) &\leq \\ 2s S_b(x_{m(k)-1}, x_{m(k)-1}, x_{m(k)}) + s^2 S_b(x_{m(k)}, x_{m(k)}, x_{n(k)-1}) &. \end{aligned}$$

Letting  $k \rightarrow \infty$ , we get  $4s^4 \epsilon \leq s^2 \epsilon$ ,

a contraction. Hence  $\{x_n\}$  is  $S_b$ -Cauchy sequence.

Since  $X$  is  $S_b$ complete, there exists  $u \in X$  such that

$$(2.2.13) \quad \lim_{n \rightarrow \infty} x_n = u.$$

We now show that  $fu = u$ . Suppose that  $fu \neq u$ . Then by lemma 1.10, we have

$$\frac{1}{2s} S_b(fu, fu, u) \leq \liminf_{n \rightarrow \infty} S_b(fu, fu, fx_n)$$

This implies

$$\begin{aligned} \frac{4s^4}{2s} S_b(fu, fu, u) &\leq 4s^4 \liminf_{n \rightarrow \infty} S_b(fu, fu, fx_n) \\ &\leq 4s^4 \limsup_{n \rightarrow \infty} S_b(fu, fu, fx_n). \end{aligned}$$

Thus

$$\begin{aligned} 2s^3 S_b(fu, fu, u) &\leq 4s^4 \liminf_{n \rightarrow \infty} S_b(fu, fu, fx_n) \\ &\leq 4s^4 \limsup_{n \rightarrow \infty} S_b(fu, fu, fx_n). \end{aligned}$$

By the property of  $\psi$ , we have

$$\begin{aligned}
 (2.2.14) \quad & \psi(2s^3 S_b(fu, fu, u)) \leq \psi(4s^4 \limsup_{n \rightarrow \infty} S_b(fu, fu, fx_n)) \\
 & \leq \psi(\limsup_{n \rightarrow \infty} M_\alpha(u, u, x_n)) \\
 & \quad - \phi(\limsup_{n \rightarrow \infty} M_\alpha(u, u, x_n)).
 \end{aligned}$$

Now,

$$\begin{aligned}
 M_\alpha(u, u, x_n) &= \max\{S_b(u, u, x_n), S_b(u, u, fu), S_b(u, u, fu), S_b(x_n, x_n, x_{n+1}), \\
 &\quad \frac{1}{2s^2}[\alpha S_b(fu, fu, u) + (1 - \alpha)S_b(fu, fu, u)S_b(fu, fu, x_n)S_b(x_{n+1}, x_{n+1}, u)]\}. \\
 &= \max\{S_b(u, u, fu), \frac{1}{2s^2}\alpha S_b(fu, fu, u)\}.
 \end{aligned}$$

If  $\frac{1}{2s^2}\alpha S_b(fu, fu, u)$  is maximum,

$$\begin{aligned}
 \psi(2s^3 S_b(fu, fu, u)) &\leq \psi\left(\frac{\alpha}{2s^2} S_b(fu, fu, u)\right) - \phi\left(\frac{\alpha}{2s^2} S_b(fu, fu, u)\right) \\
 &< \psi\left(\frac{\alpha}{2s^2} S_b(fu, fu, u)\right).
 \end{aligned}$$

By the property of  $\psi$ , we have

$$2s^3 S_b(fu, fu, u) < \frac{\alpha}{2s^2} S_b(fu, fu, u),$$

this implies

$4s^5 < \alpha$ , a contradiction. Therefore

$$\begin{aligned}
 (2.1.15) \quad & \psi(2s^3 S_b(fu, fu, u)) \leq \psi(S_b(u, u, fu)) - \phi(S_b(u, u, fu)) \\
 & \Rightarrow \psi(2s^3 S_b(fu, fu, u)) < \psi(S_b(u, u, fu)).
 \end{aligned}$$

If  $u \neq fu$ , in (2.2.15), we have

$$2s^3 S_b(fu, fu, u) < S_b(u, u, fu) \leq s S_b(fu, fu, u),$$

which implies

$2s^2 < 1$ , a contradiction. Therefore  $fu = u$

To prove uniqueness, let  $u$  and  $v$  be two fixed points of  $f$  then

$$(2.2.17) \quad \psi(4s^4 S_b(u, v, v)) \leq \psi(M_\alpha(u, v, v)) - \phi(M_\alpha(u, v, v))$$

where

$$\begin{aligned} (2.2.18) \quad M_\alpha(u, v, v) &= \max\{S_b(u, v, v), S_b(u, u, u), S_b(v, v, v), S_b(v, v, v) \\ &\quad \frac{1}{2s^2}[\alpha S_b(u, v, v) + (1 - \alpha)S_b(u, u, v)S_b(v, v, v)S_b(v, v, u)]\} \\ &= \max\{S_b(u, v, v), \frac{\alpha}{2s^2}S_b(u, v, v)\} = S_b(u, v, v). \end{aligned}$$

Therefore from (2.1.18), we have

$$\psi(4s^4 S_b(u, v, v)) \leq \psi(S_b(u, v, v)) - \phi(S_b(u, v, v)) < \psi(S_b(u, v, v)).$$

By the property of  $\psi$ , we have  $4s^4 < 1$ , a contradiction. Therefore,  $S_b(u, v, v) = 0$ , hence it follows that  $u = v$ .

### 3. COROLLARIES AND EXAMPLES

**Corollary 3.1.** *Let  $(X, S_b)$  be a complete  $S_b$ -metric space with  $s \geq 1$ . Let  $f : X \rightarrow X$  be a self map of  $X$ . Suppose there exist  $\alpha \in (0, 1)$ ,  $\psi \in \Psi$  and  $\phi \in \Phi$  such that*

$$(3.1.1) \quad \psi(4s^4 S_b(fx, fy, fz)) \leq \psi(N_\alpha(x, y, z)) - \phi(N_\alpha(x, y, z))$$

where

$$N_\alpha(x, y, z) = \max\{S_b(x, y, z), S_b(x, x, fx), S_b(y, y, fy), S_b(z, z, fz),$$

$$\frac{S_b(x, x, fx)S_b(y, y, fy)}{1+S_b(x, x, fx)+S_b(x, y, z)}, \frac{S_b(x, x, fx)S_b(z, z, fz)}{1+S_b(z, z, fz)+S_b(x, y, z)}$$

$$\frac{1}{2s^2}[\alpha S_b(fx, fy, fz) + (1 - \alpha)S_b(fx, fx, y)S_b(fy, fy, z)S_b(fz, fz, x)]\}$$

for all  $x, y, z \in X$ . Then  $f$  has a unique fixed point  $z \in X$ .

*Proof.* Proof of this corollary follows from Theorem 2.2 since

$$N_\alpha(x, y, z) = M_\alpha(x, y, z).$$

**Corollary 3.2.** *Let  $(X, S_b)$  be a complete  $S_b$ -metric space with  $s \geq 1$ . Let  $f : X \rightarrow X$  be a self map of  $X$ . If there exist  $\alpha \in (0, 1)$  and  $\psi \in \Psi$  such that*

$$(3.2.1) \quad 4s^4 S_b(fx, fy, fz) \leq \psi(N_\alpha(x, y, z))$$

where

$$N_\alpha(x, y, z) = \max\{S_b(x, y, z), S_b(x, x, fx), S_b(y, y, fy), S_b(z, z, fz),$$

$$\frac{S_b(x, x, fx)S_b(y, y, fy)}{1+S_b(x, x, fx)+S_b(x, y, z)}, \frac{S_b(x, x, fx)S_b(z, z, fz)}{1+S_b(z, z, fz)+S_b(x, y, z)}$$

$$\frac{1}{2s^2}[\alpha S_b(fx, fy, fz) + (1 - \alpha)S_b(fx, fx, y)S_b(fy, fy, z)S_b(fz, fz, x)]\}$$

for all  $x, y, z \in X$ . Then  $f$  has a unique fixed point  $z \in X$ .

**Proof.** This theorem follows as Corollary to Corollary 3.1 by choosing  $\psi(t) = \frac{t+\psi(t)}{2}$  and  $\phi(t) = \frac{t-\psi(t)}{2}$ .

**Corollary 3.3.** Let  $(X, S_b)$  be a complete  $S_b$ -metric space with  $s \geq 1$ . Let  $f : X \rightarrow X$  be a self map of  $X$ . If there exist  $\alpha, \lambda \in (0, 1)$  such that

$$(3.3.1) \quad S_b(fx, fy, fz) \leq \frac{\lambda}{4s^4} M_\alpha(x, y, z)$$

where

$$M_\alpha(x, y, z) = \max\{S_b(x, y, z), S_b(x, x, fx), S_b(y, y, fy), S_b(z, z, fz),$$

$$\frac{1}{2s^2}[\alpha S_b(fx, fy, fz) + (1 - \alpha)S_b(fy, fy, z)S_b(fy, fy, z)S_b(fz, fz, x)]\}$$

for all  $x, y, z \in X$ . Then  $f$  has a unique fixed point  $z \in X$ .

**Proof.** Proof follows by choosing  $\psi(t) = \frac{1+\lambda(t)}{2}$  and  $\phi(t) = \frac{1-\lambda(t)}{2}$  in Theorem 2.2.

**Corollary 3.4.** Let  $(X, S)$  be a complete  $S$ -metric space. Let  $f : X \rightarrow X$  be a self map of  $X$ . Suppose that there exist  $\alpha \in (0, 1)$  and  $\psi, \phi \in \Phi$  such that

$$(3.4.1) \quad \psi(S(fx, fy, fz)) \leq \psi(M_\alpha(x, y, z)) - \phi(M_\alpha(x, y, z))$$

where

$$M_\alpha(x, y, z) = \max\{S(x, y, z), S(x, x, fx), S(y, y, fy), S(z, z, fz),$$

$$[\alpha S(fx, fy, fz) + (1 - \alpha)S(fy, fy, z)]\}$$

for all  $x, y, z \in X$ . Then  $f$  has a unique fixed point  $z \in X$ .

**Proof.** Proof follows from Theorem 2.2 by choosing  $s=1$ .

**Example 3.5.** Let  $X = [0, \frac{12}{5}]$ . We define  $S_b : X^3 \rightarrow R$  by  $S_b(x, y, z) = \frac{1}{16}[|x - y| + |y - z| + |z - x|]^2$ . Then  $(X, S_b)$  is a complete  $S_b$ -metric space with  $s = 4$ . We define  $f : X \rightarrow X$  by

$$fx = \begin{cases} \frac{1}{8} & \text{if } x \in [0, 2] \\ \frac{x}{16} - \frac{1}{32} & \text{if } x \in (2, \frac{12}{5}] \end{cases}$$

Also, we define  $\psi, \phi : [0, \infty) \rightarrow [0, \infty)$  by  $\psi(t) = t$  for all  $t \geq 0$  and  $\phi(t) = \frac{t}{4}$  for all  $t \geq 0$ . Now, we verify the inequality (2.1.1).

**Case (i)** When  $x, y, z \in [0, 2]$ , we have  $\psi(4s^4 S_b(fx, fy, fz)) = 0$ . Then inequality (2.1.1) holds good.

**Case (ii)** Let  $x, y, z \in (2, \frac{12}{5}]$ . Without loss of generality suppose that  $x > y > z$ . Then

$$\begin{aligned} \psi(4s^4 S_b(fx, fy, fz)) &= 4^5 \frac{1}{16} [| \frac{x}{16} - \frac{y}{16} | + | \frac{y}{16} - \frac{z}{16} | + | \frac{z}{16} - \frac{x}{16} |]^2 \\ &\leq \frac{4^5}{16} [3 | \frac{x}{16} - \frac{y}{16} |]^2 \\ &\leq \frac{9}{4} |x - z|^2 = \frac{9}{25} \\ &\leq \frac{15987}{16384} = \frac{5329}{4096} - \frac{5329}{16384} \\ &= \frac{3}{4} S_b(x, x, fx) = \frac{3}{4} M_\alpha(x, y, z) \\ &= M_\alpha(x, y, z) - \frac{1}{4} M_\alpha(x, y, z) \\ &= \psi(M_\alpha(x, y, z)) - \phi(M_\alpha(x, y, z)) \end{aligned}$$

**Case (iii)** When  $x, y \in [0, 2]$ ,  $z \in (2, \frac{12}{5}]$  without loss of generality suppose that  $x > y$ ,

$$4s^4 S_b(fx, fy, fz) = 4^5 S_b(\frac{1}{8}, \frac{1}{8}, \frac{z}{16} - \frac{1}{32})$$

$$\begin{aligned} &= \frac{4^5}{16} [2 | \frac{1}{8} - \frac{z}{16} + \frac{1}{32} |]^2 \\ &= 4^4 [\frac{4-2z+1}{32}]^2 \\ &= \frac{1}{4} [5 - 2z]^2 \\ &\leq \frac{1}{4} = \frac{15987}{16384} = \frac{5329}{4096} - \frac{5329}{16384} \\ &= S_b(z, z, fz) - \frac{1}{4} S_b(z, z, fz) \\ &= M_\alpha(x, y, z) - \frac{1}{4} M_\alpha(x, y, z) \\ &= \psi(M_\alpha(x, y, z)) - \phi(M_\alpha(x, y, z)) \end{aligned}$$

**Case (iv)** When  $y, z \in [0, 2]$ ,  $x \in (2, \frac{12}{5}]$  without loss of generality suppose that  $y > z$ ,

$$\begin{aligned} 4s^4S_b(fx, fy, fz) &= 4^5S_b(\frac{1}{8}, \frac{1}{8}, \frac{x}{16} - \frac{1}{32}) \\ &= \frac{4^5}{16}[|2| - \frac{1}{8} + \frac{x}{16} - \frac{1}{32}]^2 \\ &= 4^4[\frac{2x-5}{32}]^2 \\ &= \frac{1}{4}[5 - 2x]^2 \\ &\leq \frac{1}{4} = \frac{15987}{16348} = \frac{5329}{4096} - \frac{5329}{16384} \\ &= S_b(x, x, fx) - \frac{1}{4}S_b(x, x, fx) \\ &= M_\alpha(x, y, z) - \frac{1}{4}M_\alpha(x, y, z) \\ &= \psi(M_\alpha(x, y, z)) - \phi(M_\alpha(x, y, z)) \end{aligned}$$

**Case (v)** When  $z \in [0, 2]$ ,  $x, y \in (2, \frac{12}{5}]$  without loss of generality suppose that  $x > y$ ,

$$\begin{aligned} 4s^4S_b(fx, fy, fz) &= 4^5S_b(\frac{x}{16} - \frac{1}{32}, \frac{y}{16} - \frac{1}{32}, \frac{1}{8}) \\ &= \frac{4^5}{16}[|\frac{x-y}{16} + \frac{2x-y}{32} + \frac{2y-5}{32}|]^2 \\ &= 4^3[\frac{x-y}{16} + \frac{2x-5}{16}]^2 \\ &= 4^3[\frac{x-y+5-2x}{16}]^2 \\ &\leq \frac{1}{4} = \frac{15987}{16348} = \frac{5329}{4096} - \frac{5329}{16384} \\ &= S_b(z, z, fz) - \frac{1}{4}S_b(z, z, fz) \\ &= M_\alpha(x, y, z) - \frac{1}{4}M_\alpha(x, y, z) \\ &= \psi(M_\alpha(x, y, z)) - \phi(M_\alpha(x, y, z)) \end{aligned}$$

**Case (vi)** When  $x \in [0, 2]$ ,  $z, y \in (2, \frac{12}{5}]$  without loss of generality suppose that  $z > y$ ,

$$\begin{aligned} 4s^4S_b(fx, fy, fz) &= 4^5S_b(\frac{1}{8}, \frac{y}{16} - \frac{1}{32}, \frac{z}{16} - \frac{1}{32}) \\ &= \frac{4^5}{16}[|\frac{1}{8} - \frac{y}{16} + \frac{1}{32}| + |\frac{1}{8} - \frac{z}{16} + \frac{1}{32}| + |\frac{y-z}{16}|]^2 \\ &= 4^3[\frac{5-2y}{32} + \frac{5-2z}{32} + \frac{y-z}{16}]^2 \end{aligned}$$

$$\begin{aligned}
&= \frac{4^3}{(32)^2} [10 - 4z]^2 \\
&\leq \frac{1}{4} = \frac{15987}{16348} = \frac{5329}{4096} - \frac{5329}{16384} \\
&= S_b(z, z, fz) - \frac{1}{4} S_b(z, z, fz) \\
&= M_\alpha(x, y, z) - \frac{1}{4} M_\alpha(x, y, z) \\
&= \psi(M_\alpha(x, y, z)) - \phi(M_\alpha(x, y, z))
\end{aligned}$$

**Case (vii)** When  $x, z \in [0, 2]$ ,  $y \in (2, \frac{12}{5}]$  without loss of generality suppose that  $x > z$ ,

$$\begin{aligned}
4s^4 S_b(fx, fy, fz) &= 4^5 S_b(\frac{1}{8}, \frac{1}{8}, \frac{z}{16} - \frac{1}{32}) \\
&= \frac{4^5}{16} [| \frac{1}{8} - \frac{1}{8} | + 2 | \frac{1}{8} - \frac{z}{16} + \frac{1}{32} |]^2 \\
&\leq \frac{1}{4} [5 - 2z]^2 \leq \frac{1}{4} = \frac{15987}{16348} = \frac{5329}{4096} - \frac{5329}{16384} \\
&= S_b(z, z, fz) - \frac{1}{4} S_b(z, z, fz) \\
&= M_\alpha(x, y, z) - \frac{1}{4} M_\alpha(x, y, z) \\
&= \psi(M_\alpha(x, y, z)) - \phi(M_\alpha(x, y, z))
\end{aligned}$$

**Case (viii)** When  $x \in [0, 2]$ ,  $z, y \in (2, \frac{12}{5}]$  without loss of generality suppose that  $z > y$ ,

$$\begin{aligned}
4s^4 S_b(fx, fy, fz) &= 4^5 S_b(\frac{1}{8}, \frac{y}{16} - \frac{1}{32}, \frac{z}{16} - \frac{1}{32}) \\
&= \frac{4^5}{16} [| \frac{1}{8} - \frac{y}{16} + \frac{1}{32} | + | \frac{1}{8} - \frac{z}{16} + \frac{1}{32} | + | \frac{y-z}{16} |]^2 \\
&= 4^3 [\frac{5-2y}{32} + \frac{5-2z}{32} + \frac{y-z}{16}]^2 \\
&= \frac{4^3}{(32)^2} [10 - 4z]^2 \\
&\leq \frac{1}{4} = \frac{15987}{16348} = \frac{5329}{4096} - \frac{5329}{16384} \\
&= S_b(z, z, fz) - \frac{1}{4} S_b(z, z, fz) \\
&= M_\alpha(x, y, z) - \frac{1}{4} M_\alpha(x, y, z).
\end{aligned}$$

Thus  $f$  satisfies all the conditions of Theorem 2.2, also  $\frac{1}{8}$  is the unique fixed point of  $f$ . Also, we note that Theorem 1.13 is not applicable since  $(X, S_b)$  is not an S-metric space. Thus our theorem generalizes Theorem 1.13.

**Example 3.7.** Let  $X = \{1, 2, 3\}$ . We write

$$S_1 = \{(1, 2, 1), (2, 1, 1), (2, 2, 1), (1, 1, 2), (2, 1, 2)\}$$

$$S_2 = \{(1, 2, 2), (2, 1, 2), (1, 1, 3)\} \text{ and}$$

$$S_3 = \{(3, 3, 1), (2, 3, 1), (1, 3, 1), (3, 1, 1), (3, 1, 3), (3, 2, 2)\}$$

$$S_4 = \{(3, 3, 2), (1, 3, 2)\}$$

$$S_5 = \{(1, 2, 3), (2, 2, 3), (3, 2, 1), (2, 3, 2)\}$$

$$S_6 = \{(2, 1, 3), (3, 2, 3), (2, 3, 3), (1, 3, 3), (3, 1, 2)\}.$$

We define  $S_b : X^3 \rightarrow R$  by

$$S_b(x, y, z) = \begin{cases} 1 & \text{if } (x, y, z) \in S_1 \\ 2 & \text{if } (x, y, z) \in S_2 \\ 35 & \text{if } (x, y, z) \in S_3 \\ 66 & \text{if } (x, y, z) \in S_4 \\ 67 & \text{if } (x, y, z) \in S_5 \\ 130 & \text{if } (x, y, z) \in S_6 \end{cases}$$

Then  $(X, S_b)$  is a complete  $S_b$ -metric space with  $s = 2$ , but it is not an S-metric space. Indeed, at  $x = 1, y = 2, z = 3$  and  $a = 1$ , we have

$$S_b(1, 2, 3) = 67 > [S_b(1, 1, 1) + S_b(2, 2, 1) + S_b(3, 3, 1)] = 36.$$

Also,  $S_b$  is not a symmetric  $S_b$ -metric space.

We define  $f : X \rightarrow X$  by  $f1 = f2 = 1$  and  $f3 = 2$ .

Now we verify the inequality (2.1.1) with  $b = 2, \phi(t) = \frac{t}{1+t}, \psi(t) = t$  and  $\alpha = \frac{1}{2}$ .

*Case (i) :* When  $(x, y, z) \in \{(1, 2, 3), (3, 2, 1), (2, 2, 3), (2, 3, 2)\}$ . Then

$$\psi(4s^4 S_b(fx, fy, fz)) = 64 \leq 67 - \frac{67}{68} = \psi(M_\alpha(x, y, z)) - \phi(M_\alpha(x, y, z)).$$

*Case (ii):* When  $(x, y, z) \in \{(1, 3, 2), (3, 1, 3), (2, 3, 1), (3, 3, 1), (1, 3, 1), (3, 1, 1), (3, 3, 2), (3, 2, 2), (1, 1, 3)\}$ . Then

$$\psi(4s^4 S_b(fx, fy, fz)) = 64 \leq 66 - \frac{66}{67} = \psi(M_\alpha(x, y, z)) - \phi(M_\alpha(x, y, z)).$$

*Case (iii):* When  $(x, y, z) \in \{(1, 1, 1), (2, 2, 2), (3, 3, 3), (1, 1, 2), (1, 2, 2), (2, 2, 1), (1, 2, 1), (2, 1, 2), (2, 1, 1)\}$ . Then

$$\psi(4s^4 S_b(fx, fy, fz)) = 0 \leq \psi(M_\alpha(x, y, z)) - \phi(M_\alpha(x, y, z)).$$

*Case (iv):* When  $(x, y, z) \in \{(1, 3, 3), (2, 1, 3), (3, 2, 3), (2, 3, 3)\}$ . Then

$$\psi(4s^4 S_b(fx, fy, fz)) = 128 \leq 130 - \frac{130}{131} = \psi(M_\alpha(x, y, z)) - \phi(M_\alpha(x, y, z)).$$

*Case (v) :* When  $(x, y, z) \in \{(3, 1, 2)\}$ , Then

$$\psi(4s^4 S_b(fx, fy, fz)) = 64 \leq 130 - \frac{130}{131} = \psi(M_\alpha(x, y, z)) - \phi(M_\alpha(x, y, z)).$$

Thus  $f$  satisfies all the conditions of Theorem 2.2 and 0 is the unique fixed point of  $f$ . Also, we note that Theorem 1.13 is not applicable since  $(X, S_b)$  is not an S-metric space. Thus our theorem generalizes Theorem 1.13.

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