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A Note on Hausdorff Domination

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Abstract

A set $D \subseteq V$ of a graph $G(V, E)$ is called a dominating set if every vertex in G is either in D or is adjacent to an element of D. A simple graph G is said to be Hausdorff, if for any two distinct vertices u and v of G, either one of u and v is isolated or there exists two nonadjacent edges e_1 and e_2 of G such that e_1 is incident with u and e_2 is incident with v. A dominating set D such that the subgraph $\langle D\rangle$ induced by D is Hausdorff is called a Hausdorff dominating set. If $\langle D\rangle$ is connected and Hausdorff, then it is called a connected Hausdorff dominating set. The minimum cardinality of all Hausdorff dominating sets and connected Hausdorff dominating sets are respectively called Hausdorff domination number $\gamma_H(G)$ and connected Hausdorff domination number $\gamma_{cH}(G)$. In this paper Hausdorff domination number and connected Hausdorff domination number are introduced and some results on these new parameters are established.

Keywords: Domination number, Hausdorff domination number, connected Hausdorff domination number.

MSC(2010): Primary: 05C69

1. INTRODUCTION

Graphs $G = (V(G), E(G))$ discussed in this paper are finite, simple and undirected. Any undefined term in this paper may be found in [2,8]. The *degree* [2] of a vertex v in graph G is denoted by $d_G(v)$, which is the number of edges incident with v. The maximum and minimum degrees of G are denoted respectively by $\triangle(G)$ and $\delta(G)$.

The *complement* $\overline{G}$ [9] of graph G has $V(\overline{G}) = V(G)$ and $uv \in E(\overline{G})$ if and only if uv is not in $E(G)$. For a graph G, the number of vertices is called the the *order* [9] of G and is denoted by $O(G)$. An *empty graph* [2] is a graph with no edges. An *isolated vertex* [8] is one whose degree is zero. A vertex in a graph is called a *pendant vertex* [14] if its degree is one. Any vertex adjacent to a pendant vertex is called a *support vertex*. A simple graph in which each pair of distinct vertices is joined by an edge is called a *complete graph* [2]. A complete graph on n vertices is denoted by K_n. A *bipartite graph* G is one whose vertex set can be partitioned into two subsets X and Y so that each edge has its ends in X and Y respectively. Such a partition (X, Y) is called a bipartition of G. A *complete bipartite graph* [2] is a simple bipartite graph with bipartition (X, Y) in which every vertex of X is joined to every vertex of Y. The complete bipartite graph with $|X| = m$ and $|Y| = n$ is denoted by $K_{m,n}$. The graph H is said to be an *induced subgraph* [3] of the graph G if $V(H) \subseteq V(G)$ and two vertices in H are adjacent if and only if they are adjacent in G. If two vertices u and v are connected in G, the length of the shortest u-v path in G is called the *distance* [2] between u and v and is denoted by $d(u, v)$. The *diameter* [2] of G is the maximum distance between two vertices of G and is denoted by $diam(G)$. A *tree* [2] is a connected acyclic graph. A *cut edge* [2] of a graph G is an edge such that whose removal makes the graph disconnected. The *open neighborhood* [9] of v in $V(G)$ consists of those vertices adjacent to v in G and it is denoted by $N(v)$. The *closed neighborhood* [9] of v is $N[v]$ = $N(v) \cup \{v\}$. The *corona* [7] of two graphs G_1 and G_2 is the graph $G = G_1 \; o \; G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2, where the i^{th} vertex of G_1 is adjacent to every vertex in the i^{th} copy of G_2. The *girth* $g(G)$ [9] of a graph G is length of the shortest cycle in G.

Let $G = (V, E)$ be a graph. A set $D \subseteq V$ is called a *dominating set* [9] if every vertex in G is either in D or is adjacent to an element of D. The minimum cardinality of all dominating sets in G is called the *domination number* and is denoted by $\gamma(G)$. Different types of dominating sets have been studied by imposing conditions on the dominating sets. A detailed survey can be found in [9,10,11]. A dominating set D is called an *independent dominating set* [5] if $\langle D \rangle$ is the empty graph. A dominating set D is called a *connected dominating set* [15] if $\langle D \rangle$ is connected. D is called *total dominating* [4] if $\langle D \rangle$ has no isolated vertices. D is *global dominating* [16] if it is a dominating set of $\overline{G}$, the complement of G. D is *cycle dominating* [12] if $\langle D \rangle$ is a cycle and D is a *dominating clique* [13], if $\langle D \rangle$ is a complete graph. The corresponding minimum cardinality of independent dominating set, connected dominating set, total dominating set, global dominating set, cycle dominating set and clique dominating set are respectively called independent domination number, connected domination number, total domination number, global domination number, cycle domination number and clique domination number and are denoted respectively by $i(G)$, $\gamma_c(G)$, $\gamma_t(G)$, $\gamma_g(G)$, $\gamma_{cy}(G)$ and $\gamma_{cl}(G)$. The maximum size of an independent set of vertices in a graph G is called *independence number* and is denoted by $\alpha(G)$ [6]. In [17], V Seena and Raji Pilakkat defined the *Hausdorff Graph* as follows

A simple graph G is said to be *Hausdorff*, if for any two distinct vertices u and v of G, one of the following hold

1. At least one of u and v is isolated.
2. There exists two nonadjacent edges e_1 and e_2 of G such that e_1 is incident with u and e_2 is incident with v.

In this paper, a new domination parameter, *Hausdorff domination number* is introduced using this concept. A *Hausdorff dominating set* is any dominating set $D \subseteq V$ such that $\langle D \rangle$ is Hausdorff. In this paper it is proved that every independent dominating set in a graph is Hausdorff dominating. So that every graph has a Hausdorff dominating set. Hence the property of Hausdorff domination is applicable to all simple graphs. Also the domination chain can be extended using Hausdorff domination number and an upper bound for $\gamma(G)$ is obtained in terms of $\gamma_H(G)$.

An example of a real life situation where the notion of a non independent Hausdorff dominating set can be used is given below. In a battlefield, sometimes it is needed to locate the places where additional weapons, energy sources, medicines etc are to be located so that it can be accessed easily from more than one or interconnected sources in case of emergency. These resources are represented by a non independent Hausdorff dominating set and the clients are represented by the non isolated vertices outside this set. Self sufficient units are denoted by isolated vertices of the Hausdorff dominating set.

2. HAUSDORFF DOMINATION

Hausdorff domination is defined as follows.

DEFINITION 2.1. A dominating set $D \subseteq V$ is said to be Hausdorff dominating, if $\langle D \rangle$ is Hausdorff. Minimum cardinality of all Hausdorff dominating sets is called the Hausdorff domination number and is denoted by $\gamma_H(G)$. Such a Hausdorff dominating set with cardinality $\gamma_H(G)$ is referred to as a γ_H -set.

For any graph G, $\gamma(G) \leq \gamma_H(G)$

Theorem 2.1 gives a characterization result for a dominating set to be Hausdorff dominating.

Theorem 2.1. Let $G = (V, E)$ be any graph. A dominating set $D \subseteq V$ is a Hausdorff dominating set if and only if one of the following statements hold.

1. $\langle D \rangle$ is an empty graph
2. If $\langle D \rangle$ is triangle free and if $v \in D$ is not an isolated vertex in $\langle D \rangle$, then the degree of v, $d_{\langle D \rangle}(v) \geq 2$
3. If $\langle D \rangle$ contains K_3 as an induced subgraph, then $d_{\langle D \rangle}(v) \geq 3$ for at least two vertices of K_3 and for all other vertices which are non isolated in $\langle D \rangle$ have degree ≥ 2.

Proof. Assume that $D \subseteq V$ is a Hausdorff dominating set of G. If for any two distinct vertices u and v of $\langle D \rangle$, both u and v are isolated, then $\langle D \rangle$ is an empty graph hence there is nothing to prove.

Suppose that $\langle D \rangle$ contains at least one nontrivial connected component. Such components cannot have a vertex of degree one, since then by definition, $\langle D \rangle$ cannot be Hausdorff. Hence for every vertex v in any nontrivial connected component of $\langle D \rangle$, $d_{\langle D \rangle}(v) \geq 2$.

If $\langle D \rangle$ contains K_3 and $d_{\langle D \rangle}(v) < 3$ for at least two vertices of K_3 then those vertices in pairs will not have two non adjacent edges incident with them. Hence $\langle D \rangle$ cannot be Hausdorff. On the other hand if $d_{\langle D \rangle}(v) = 2$ only for one vertex or $d_{\langle D \rangle}(v) \geq 3$ for all vertex in K_3 then there are nonadjacent edges incident with every pair of vertices in K_3. Hence in $\langle D \rangle$, for every non isolated vertex v, $d_{\langle D \rangle}(v) \geq 2$ and $d_{\langle D \rangle}(v) \geq 3$ for at least two vertices in every K_3 which is an induced subgraph of $\langle D \rangle$.

Conversely, assume that D is a dominating set for which one of the three stated conditions hold. Then it is proved that D is a Hausdorff dominating set. If $\langle D \rangle$ is the empty graph, then clearly D is a Hausdorff dominating set.

Suppose that (2) holds. Let (u, v) be a pair of distinct vertices in $\langle D \rangle$. If one of them is an isolated vertex or if u and v belong to different components of $\langle D \rangle$, then there is nothing to prove. If both of them are non isolated and belongs to the same component of $\langle D \rangle$, then there arise the following cases.

Case (i) u and v are adjacent. Then since $d_{\langle D \rangle}(u)$ and $d_{\langle D \rangle}(v)$ are greater than or equal 2, there exists u_1, v_1 in $\langle D \rangle$, such that u_1 is adjacent to u, v_1 is adjacent to v and the edges uu_1 and vv_1 are non adjacent. Here $u_1 \neq v_1$, otherwise $\{u, u_1(= v_1), v\}$ will form the vertices of K_3 in the triangle free graph $\langle D \rangle$.

Case (ii) u and v are non adjacent. Then they are joined by at least one path of length two or greater than two. If the u-v path is of length 2, there exists a vertex w such that uwv is a u-v path and since, $d_{\langle D \rangle}(u) \geq 2$, there exists a vertex $x \neq w$ adjacent to u. So that xu and wv are non adjacent edges incident with u and v respectively. If the length of the u-v path is greater than 2, then there exists at least two vertices $u_1 \neq v_1$ such that, $uu_1...v_1v$ is is a u-v path in $\langle D \rangle$ and u_1u and v_1v are non adjacent edges incident with u and v respectively.

Now let (3) hold. Consider two adjacent vertices u, v in $\langle D \rangle$. If $\{u, v\}$ does not belongs to the vertex set of any K_3 in $\langle D \rangle$ then by the above reasoning, non adjacent edges incident with u and v can be found. Otherwise, there exists $w \in D$ such that $\langle \{u, v, w\} \rangle$ is K_3. Then either $d_{\langle D \rangle}(u) \geq 3$ or $d_{\langle D \rangle}(v) \geq 3$ or both $d_{\langle D \rangle}(u)$ and $d_{\langle D \rangle}(v) \geq 3$. Without loss of generality assume that $d_{\langle D \rangle}(u) = 2$ or 3 and $d_{\langle D \rangle}(v) \geq 3$ then $\exists$ a vertex x different from u and w in D adjacent to v in $\langle D \rangle$. Thus in this case the edges e_1 and e_2 are non adjacent, where $e_1 = wu$ is incident with u and $e_2 = xv$ is incident with v. Hence $\langle D \rangle$ is Hausdorff. □

Corollary 2.2 follows directly from Theorem 2.1.

Corollary 2.2. If $D \subseteq V$ is a Hausdorff dominating set of a graph $G(V, E)$ then $\langle D \rangle$ has no vertices of degree one. In other words, $\langle D \rangle$ is free of pendant vertices.

3. HAUSDORFF DOMINATION AND INDEPENDENT DOMINATION

Theorem 3.1. Every independent dominating set is Hausdorff dominating

Proof. Let $G = (V, E)$ be any graph, let $D \subseteq V$ be an independent dominating set of G. Then $\langle D \rangle$ is the empty graph. Hence by Theorem 2.1, it is Hausdorff. □

Corollary 3.2. For any graph G, $\gamma_H(G) \leq i(G)$

Corollary 3.3. The domination chain [7] $\gamma(G) \leq i(G) \leq \alpha(G) \leq \Gamma(G)$ can be extended as $\gamma(G) \leq \gamma_H(G) \leq i(G) \leq \alpha(G) \leq \Gamma(G)$.

Remark 3.4. The converse of Theorem 3.1 need not be true. For example in figure 1, $\{a, b, c, d\}$ is both independent and Hausdorff dominating while $\{e, f, g, h\}$ is Hausdorff dominating but not independent

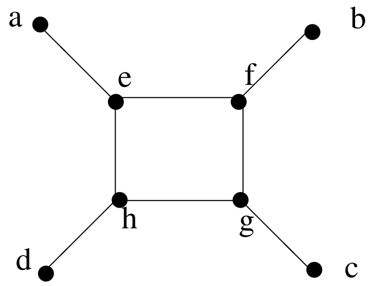


Figure 1

Theorem 3.5. 1. $\gamma_H(P_n) = \lceil \frac{n}{3} \rceil$, for any path P_n on n vertices.

2. $\gamma_H(C_n) = \lceil \frac{n}{3} \rceil$, for any cycle C_n on n vertices.

Proof. (1) For any path P_n on n vertices, a dominating set D can be Hausdorff dominating if and only if $\langle D \rangle$ is an empty graph. Otherwise, $\langle D \rangle$ will contain two or more pendant vertices and hence by corollary 2.2, it cannot be Hausdorff. Therefore $\lceil \frac{n}{3} \rceil = \gamma(P_n) \leq \gamma_H(P_n) \leq i(P_n) = \lceil \frac{n}{3} \rceil$

(2) For any cycle C_n on $n \geq 4$ vertices the set of all vertices constitute a Hausdorff dominating set. By the same reasoning as in the case of paths P_n, any dominating set D of C_n of cardinality $< n$, will be Hausdorff dominating if and only if $\langle D \rangle$ is an empty graph. Hence $\lceil \frac{n}{3} \rceil = \gamma(C_n) \leq \gamma_H(C_n) \leq i(C_n) = \lceil \frac{n}{3} \rceil$ □

Theorem 3.6. Let $G = (V, E)$ be any graph on n vertices then,

1. $\gamma_H(G) = 1$ if and only if $\triangle(G) = n - 1$
2. $\gamma_H(G) = 2$ if and only if $i(G) = 2$
3. $\gamma_H(G) = 3$ if and only if $i(G) = 3$
4. $\gamma_H(G) = n$ if and only if $G = \overline{K_n}$

Proof. (1) If $\gamma_H(G) = 1$, then there exists a vertex v of G which is adjacent to all other vertices of G. Therefore $d(v)$ and hence $\triangle(G)$ = $n - 1$

Conversely, if $\triangle(G)$ = $n - 1$, then G has a vertex v which dominate every vertex of G and $\langle v \rangle$ is Hausdorff. Hence $\gamma_H(G) = 1$

(2) Suppose $\gamma_H(G) = 2$. Let $D \subseteq V(G)$ be a γ_H-set. Since the only Hausdorff graph on two vertices is $\overline{K_2}$, $\langle D \rangle$ has no edges. Therefore, D is independent dominating. Also since $\gamma_H(G) \leq i(G)$, it follows that $i(G) = 2$

Conversely, let $i(G) = 2$. Then $\gamma_H(G) = 2$. Otherwise $\gamma_H(G) = 1$. In this case, a singleton subset of V dominates all the vertices of G. Therefore, $i(G) = 1$, a contradiction.

(3) Suppose $\gamma_H(G) = 3$. Let $D \subseteq V(G)$ be a γ_H-set. Then as in the proof of part (2), the only Hausdorff graph on three vertices is $\overline{K_3}$. Therefore, D is independent dominating. Hence by Corollary 3.2, it follows that $i(G) = 3$.

Conversely, let $i(G) = 3$. Then $\gamma_H(G)$ cannot be 1 or 2, as in these cases, every γ_H-set is also an independent dominating set of cardinality less than $i(G)$. Therefore, $\gamma_H(G) = 3$.

(4) Let $\gamma_H(G) = n$. Then for any γ_H-set D, $|D| = n$. ie., every vertex of G belongs to every γ_H-set. Hence $\langle D \rangle = G$. By Corollary 3.2, $\gamma_H(G) \leq i(G)$. Also since $O(G) = n$, $i(G) = n$. Hence $G = \overline{K_n}$, an empty graph on n vertices.

The converse is obvious. □

Since all the graphs G mentioned in Corollary 3.7 have $\triangle(G) = n - 1$, it follows immediately from Theorem 3.6.

Corollary 3.7.

1. For any complete graph K_n, $\gamma_H(K_n) = 1, \forall n \geqslant 1$
2. For any star graph $K_{1,n}$, $\gamma_H(K_{1,n}) = 1, n \geqslant 1$
3. For any wheel graph $W_{n+1} = C_n + K_1, \gamma_H(W_{n+1}) = 1, n \geqslant 3$

Corollary 3.8. If $i(G) = 4$, then $\gamma_H(G) = 4$

Proof. Let $i(G) = 4$. By Corollary 3.2, $\gamma_H(G) \leq i(G)$. Therefore $\gamma_H(G) < 4$ will imply that the γ_H-set is independent dominating, which contradicts $i(G) = 4$. Hence $\gamma_H(G) = 4$ □

Corollary 3.9. If $i(G) \leq 4$, then $\gamma_H(G) = i(G)$

Remark 3.10. By Theorem 3.6, whenever $i(G) \leq 3$ the i-set and γ_H-set are the same. But when $i(G) = 4$ even though $\gamma_H(G) = 4$, there may exist γ_H-set which is different from an i-set. For example, the graph $G(V, E)$ in Figure 1 has $A \subset V$, $B \subset V$, where, $A = \{a, b, c, d\}$ forms an i-set which is also Hausdorff dominating. But $B = \{e, f, g, h\}$ is a γ_H-set which is not independent.

Remark 3.11. The conclusion of Corollary 3.9 need not be true for $i(G) \geq 5$. The graph in figure 2 has $\gamma_H(G) = 4 < i(G) = 5$

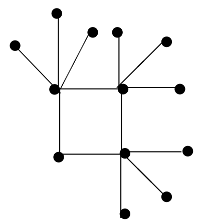


Figure 2

Theorem 3.12. For any complete bipartite graph $K_{m,n}$,

$$\gamma_H(K_{m,n}) = \begin{cases} 1, & \text{if either } m \text{ or } n = 1; \\ 2, & \text{if } m \geq 2, n \geq 2 \text{ and at least one of } m \text{ or } n \text{ is } 2; \\ 3, & \text{if } m \geq 3, \text{n} \geq 3 \text{ and at least one of } m \text{ or } n \text{ is } 3; \\ 4, & \text{if } m \geq 4 \text{ and } n \geq 4. \end{cases}$$

Proof. $\gamma_H(K_{m,n}) = 1$ if either m or n = 1 is a particular case of Corollary 3.7.

Since $\gamma(K_{m,n}) = 2$ for $m \geq 2$ $n \geq 2$ and $i(K_{m,n}) = 2$ if $m \geq 2$, $n \geq 2$ and at least one of m or n is 2 then since $\gamma(K_{m,n}) \leq \gamma_H(K_{m,n}) \leq i(K_{m,n})$, $\gamma_H(K_{m,n})$ = 2 if $m \geq 2$, $n \geq 2$ and at least one of m or n is 2

When $m \geq 3$, $n \geq 3$ and at least one of m or n is 3 then since $K_{m,n}$ does not have a vertex of degree $m + n - 1$, by Theorem 3.6, $\gamma_H(K_{m,n}) \neq 1$. Suppose if possible, D is a γ_H-set of $K_{m,n}$ of cardinality 2 then either $\langle D \rangle$ is K_2 or an empty graph on 2 vertices. In the first case $\langle D \rangle$ is not Hausdorff and in the second case, D is not dominating. Hence $\gamma(K_{m,n}) = 3 = min\{m, n\}$ if $m \geq 3$, $n \geq 3$ and at least one of m or n is 3.

If $m \geq 4$ and $n \geq 4$, then four vertices, two each from the bipartite sets will form a Hausdorff dominating set. So $\gamma_H(K_{m,n}) \leq 4$. Suppose, if possible, $\gamma_H(K_{m,n}) < 4$. Then by the above reasoning, $\gamma_H(K_{m,n})$ cannot be 1 or 2. If D is a γ_H-set of cardinality 3, then either $\langle D \rangle$ is P_3 or $K_2 \cup K_1$ or an empty graph on three vertices. If $\langle D \rangle$ is P_3 or $K_2 \cup K_1$ then it is not Hausdorff and D is not dominating if $\langle D \rangle$ an empty graph on three vertices. Hence $\gamma_H(K_{m,n})$ cannot be 3. □

Corollary 3.13. For any complete bipartite graph $K_{m,n}$, $m \geq 1, n \geq 1, \gamma_H(K_{m,n}) \leq 4$

Theorem 3.14. The graph induced by any Hausdorff dominating set which is not independent, contains a cycle C_m on $m \geqslant 4$ vertices

Proof. Let $D \subseteq V$ be any Hausdorff dominating set. Suppose D is not an independent dominating set. Then $\langle D \rangle$ is not an empty graph. Let $v \in D$. If v is not an isolated vertex in $\langle D \rangle$, then v is a vertex of a connected component G_1 of $\langle D \rangle$. Since $\langle D \rangle$ is Hausdorff, the subgraph G_1 also should be Hausdorff. Then by Theorem 2.1,

$d_{G_1}(v) \geq 2$, $\forall v \in V(G_1)$. So that G_1 cannot be a tree. Hence G_1 is not acyclic and contains a cycle C_m for $m \geq 3$. Now if $d_{G_1}(v) = 2\, \forall v \in V(G_1)$, then G_1 is a cycle C_m with m vertices. Since G_1 is Hausdorff, $m \geq 4$. If G_1 contains K_3, by Theorem 2.1, $d_{G_1}(v) \geq 3$ for at least two vertices of K_3. Let u_1 and u_2 be the vertices adjacent to the vertices of K_3 of degree > 2 in G_1. Consider the following cases.

Case 1: $u_1 = u_2$, then u_1 together with the vertices of K_3 will form a cycle of length 4

Case 2: $u_1 \neq u_2$, and if u_1 and u_2 are adjacent. In this case, two internally disjoint paths can be found from u_1 to u_2, one along the vertices of K_3 and the other along the edge u_1u_2. Adjoining these two paths from u_1 to u_2 a cycle of length 5 will be obtained.

Case 3: $u_1 \neq u_2$, and u_1 and u_2 are not adjacent in G_1. Since $d_{G_1}(u_1)$ and $d_{G_1}(u_2)$ are greater than or equal to 2, if at least one of u_1 or u_2 is adjacent to the third vertex of K_3 under consideration, then there is a cycle of length 4 in G_1. If u_1 and u_2 are joined by a path not along the vertices of K_3 then also a cycle of length greater 4 can be obtained by adjoining these two internally disjoint u_1-u_2 paths.

Case 4: u_1 and u_2 are not connected through any path other than that along the vertices of K_3. In this case, suppose if possible the other end blocks in the direction opposite to that of K_3 from u_1 and u_2 do not contain any cycle of length greater than or equal to 4. Then these blocks are either a triangle or a pendant edge. In both cases G_1 cannot be Hausdorff. Hence both these blocks should contain a cycle of length greater than or equal to 4.

Now let G_1 be triangle free. Let $u, v \in V(G_1)$. As G_1 is a connected Hausdorff graph, the order of G_1 is ≥ 4. Let e be any edge in G_1 with end points v_1 and v_2, which is not a cut edge of G_1. Since G_1 is not a tree such an edge will exist. As G_1 is Hausdorff and $d(v) \geq 2$ for all $v \in G_1$, a path from v_1 to v_2 not through e can be found. Then since G_1 is triangle free this path together with e will form a cycle of length ≥ 4. Hence the proof. □

Since any tree is acyclic, the corollary follows from Theorem 3.14

Corollary 3.15. For any tree T, the Hausdorff dominating set and independent dominating set are the same. Hence $\gamma_H(T) = i(T)$, for any tree T.

Theorem 3.16. For any graph G of order $n \geq 2$, $3 \leq \gamma_H(G) + \gamma_H(\overline{G}) \leq n + 1$

Proof. Let G be any graph of order $n \geq 2$. If $\gamma_H(G) = 1$, then by Theorem 3.6, $\exists$ a vertex v of degree $n - 1$ in G. Hence v is an isolated vertex in $\overline{G}$. Hence $\gamma_H(\overline{G}) \geq 2$. Similarly if $\gamma_H(\overline{G}) = 1$, then $\gamma_H(G) \geq 2$. In this case, $\gamma_H(G) + \gamma_H(\overline{G}) \geq 3$ Also the lower bound is obvious when $\gamma_H(G) \geq 2$.

Now an upper bound is obtained by proceeding as follows. Since $i(G) \leq n - \Delta(G)$ [1] and since $\gamma_H(G) \leq i(G)$,

$$\begin{aligned}
\gamma_H(G) + \gamma_H(\overline{G}) &\leq i(G) + i(\overline{G}) \\
&\leq n - \Delta(G) + n - \Delta(\overline{G}) \\
&= 2n - [\Delta(G) + \Delta(\overline{G})] \\
&\leq 2n - [\Delta(G) + \delta(\overline{G})] \\
&= 2n - (n-1) \\
&= n + 1
\end{aligned}$$

□

Remark 3.17. The bounds are sharp. By considering $G = K_{1,n-1}$, $n \geq 5$ $\gamma_H(G) = 1$, $\gamma_H(\overline{G}) = 2$, we see that the lower bound is sharp. Also by considering $G = K_n, \gamma_H(G) = 1, \gamma_H(\overline{G}) = n$ we get $\gamma_H(G) + \gamma_H(\overline{G}) = n + 1$

Theorem 3.18. If G is a connected triangle free graph of order ≥ 2, then $\gamma_H(\overline{G}) = 2$

Proof. Since *G* is a connected graph of order ≥ 2, it contains an edge say uv. If $O(G) = 2$ then G is isomorphic to K_2 and $\overline{G}$ is isomorphic to an empty graph on two vertices. Therefore, $\gamma_H(\overline{G}) = 2$. If $O(G) > 2$, then no vertex of G is adjacent to both u and v, because G is triangle free. Therefore every vertex in G which are adjacent to u are dominated by v in $\overline{G}$ and those vertices adjacent to v in G are dominated by u in $\overline{G}$ and all vertices which are non adjacent to both u and v are dominated by both u and v in $\overline{G}$. So $\{u, v\}$ forms an independent dominating set of $\overline{G}$. Therefore it is also a Hausdorff dominating set of $\overline{G}$. Hence $\gamma_H(\overline{G}) \leq 2$. Now let if possible $\gamma_H(\overline{G}) = 1$, then *G* would have an isolated vertex, a contradiction. Which proves $\gamma_H(\overline{G}) = 2$ □

4. CONNECTED HAUSDORFF DOMINATION

DEFINITION 4.1. Let $G = (V, E)$ be any graph. A dominating set $D \subseteq V$ is called a connected Hausdorff dominating set, if $\langle D \rangle$ is both connected and Hausdorff. Any connected Hausdorff dominating set with minimum cardinality $\gamma_{cH}(G)$, is called a γ_{cH} -set and $\gamma_{cH}(G)$ is called the connected Hausdorff domination number of G.

Theorem 4.1. For any graph G, $\gamma_H(G) \leq \gamma_{cH}(G)$

Proof. Since every connected Hausdorff dominating set is Hausdorff dominating, it follows that $\gamma_H(G) \leq \gamma_{cH}(G)$. □

DEFINITION 4.2. [14] A star graph $K_{1,n}$, $n \geq 0$ is a tree on n pendant vertices and one central vertex of degree n.

Theorem 4.2. No tree other than the star graph has a connected Hausdorff dominating set.

Proof. For $K_{1,n}$, $n \geq 1$ the vertex of degree n will form a connected Hausdorff dominating set.

Let T be any tree other than $K_{1,n}$, $n \geq 1$. If possible, let T have a connected Hausdorff dominating set D. Then $\langle D \rangle$, which is a subgraph of T is connected and Hausdorff. $\langle D \rangle$ being a connected subgraph of T, it is acyclic and hence a tree. As every tree of order greater than one contains at least two pendant vertices, and since any graph with pendant vertices is not Hausdorff, $|D| = 1$ ie., $D = \{v\}$ a singleton subset of V. But in this case, the given graph T is a star, a contradiction. Hence T cannot have a connected Hausdorff dominating set. □

Observation 4.3. If a graph G has a spanning cycle $C_n, n \geq 4$, it contains a connected Hausdorff dominating set. In particular, Every Hamiltonian graph with more than four vertices contains a connected Hausdorff dominating set. But the condition is not sufficient, ie, the existence of a connected Hausdorff dominating set in a graph G need not imply that G is Hamiltonian. For example, the graph G in figure 3 has a connected Hausdorff dominating set. But G is not Hamiltonian.

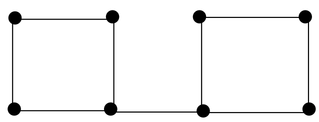


Figure 3

Theorem 4.4. For any nontrivial connected Hausdorff dominating set D of any graph $G \ \exists$ a subset V_1 of D such that $\langle V_1 \rangle = C_m$, for $m \geqslant 4$

Proof. Proof follows directly from Theorem 3.14

□

Corollary 4.5. For any nontrivial connected Hausdorff dominating set $D \subseteq V$ of G, $|D| \geq 4$. In particular $\gamma_{cH}(G) \geq 4$

Corollary 4.6. If G is triangle free and has a nontrivial connected Hausdorff dominating set then the girth $g(G) \geq 4$.

Remark 4.7. For the wheel graph $G = W_{n+1} = K_1 + C_n$, $n \geq 4$, the vertex v of degree n is a Hausdorff dominating set. Also the graph induced by the vertices of C_n is connected Hausdorff, for $n \geq 4$. For this graph, both $D = \{v\}$ and $V - D$ are connected Hausdorff dominating sets and $\gamma_H(G) = \gamma_{cH}(G) = 1$

Theorem 4.8. *Unicyclic graphs can have a connected Hausdorff dominating set if and only if $G \cong C_m$ for $m \geq 4$ or G contains a cycle C_m for $m \geq 4$ with one or more pendant vertices adjacent to all or some of the vertices of C_m*

Proof. If G is any of the two types of graphs as mentioned in the statement, then the vertices of C_m, $m \geq 4$ forms a connected Hausdorff dominating set of G with $\gamma_{cH}(G) = m$

Conversely, let G be any uni-cyclic graph with a connected Hausdorff dominating set D. Then by Theorem 4.4, $\langle D \rangle$ contains at least one cycle C_m where $m \geq 4$. ie., the unique cycle of G is in fact contained in the graph induced by every connected Hausdorff dominating set. If $G = C_m$, then there is nothing to prove. On the other hand, let $v \in V - V(C_m)$. Since G has a connected Hausdorff dominating set, G itself is connected. Therefore there exists a path from v to every vertex of C_m. It is claimed that there do not exist two internally disjoint paths from v to the vertices of C_m. Otherwise G contains more than one cycle. Hence there exists exactly one vertex $u \in V(C_m)$ and exactly one path from v to u such that $d(u, v)$ is minimum.

Now it is claimed that v is a pendant vertex of G, which is adjacent to a vertex of C_m. If v is not a pendant vertex,then a pendant vertex say u in $V(G)$ and a unique path containing v, joining u to the nearest vertex say w in C_m can be found. Since G has a connected Hausdorff dominating set D, in order to dominate all the vertices in this u-w path they should belong to D. Hence either the pendant vertex u or a support vertex of u belong to $D \Rightarrow \langle D \rangle$ is not Hausdorff. Now if v is a pendant vertex, but it is not adjacent to any vertex of C_m, then either v is dominated by a support vertex u, where $u \in D$ or $v \in D$. In both cases $\langle D \rangle$ contains a pendant vertex and hence is not Hausdorff. Hence if $v \in V - V(C_m)$ then it should be adjacent to a vertex of C_m □

Theorem 4.9. *If D is a γ_{cH}-set of a connected graph G, then both endpoints of every cut edge of G belongs to D.*

Proof. Suppose, if possible, any or both endpoints of a cut edge do not belong to the γ_{cH}-set D. Then the graph induced by D is disconnected, a contradiction to D is a γ_{cH}-set of G. □

5. RELATION OF HAUSDORFF DOMINATING SET AND CONNECTED HAUSDORFF DOMINATING SET WITH OTHER DOMINATION PARAMETERS

From the very definition, Every nontrivial connected Hausdorff dominating set is connected dominating and total dominating. Therefore, $\gamma_c(G) \leq \gamma_{cH}(G)$ and $\gamma_t(G) \leq \gamma_{cH}(G)$

Theorem 5.1. Every cycle dominating set D with $|D| \geq 4$ is connected Hausdorff dominating. In particular $\gamma_{cy}(G) = \gamma_{cH}(G)$

Theorem 5.2. If G is the corona $C_m o K_1$, $m \geq 4$ then $i(G) = \gamma_H(G) = \gamma_t(G) = \gamma_c(G) = \gamma_{cy}(G) = \gamma_{cH}(G) = m$

Proof. In $C_m o K_1$, $m \geq 4$ the pendant vertices will form a γ_H-set. This set is also independent dominating. Since, $\gamma_H(G) \leq i(G)$ by Corollary 3.2, $i(G) = \gamma_H(G) = m.$

Clearly vertices of C_m will form a total dominating, connected dominating,cycle dominating and connected Hausdorff dominating set. So $\gamma_t(G) = \gamma_c(G) = \gamma_{cy}(G) = \gamma_{cH}(G) = m$ □

Theorem 5.3. Every clique dominating set of a graph G with clique domination number $\gamma_{cl}(G) \geq 4$ is a connected Hausdorff dominating set.

Proof. Since every complete graph K_n is Hausdorff for $n \geq 4$, every dominating clique is a connected Hausdorff dominating set. □

Corollary 5.4. If a graph G has a dominating clique with $\gamma_{cl}(G) \geq 4$, then $\gamma_{cH}(G) \leq \gamma_{cl}(G)$

Remark 5.5. [7] If G has a dominating clique and if $\gamma(G) \geq 2$, then $\gamma(G) \leq \gamma_t(G) \leq \gamma_c(G) \leq \gamma_{cl}(G)$

Therefore if G has a dominating clique with $\gamma_{cl}(G) \geq 4$ and if $\gamma(G) \geq 2$, then the above domination chain can be extended as $\gamma(G) \leq \gamma_t(G) \leq \gamma_c(G) \leq \gamma_{cH}(G) \leq \gamma_{cl}(G)$. If $\gamma(G) = 1$, Then $\gamma(G) = \gamma_t(G) = \gamma_c(G) = \gamma_H(G) = \gamma_{cH}(G) = \gamma_{cl}(G)$

Corollary 5.4 need not hold if $\gamma_{cl}(G) < 4$ Figures 4 and 5 are examples of graphs for which $\gamma_{cl}(G) = 3 < \gamma_{cH}(G) = 4$ and $\gamma_{cl}(G) = 2 < \gamma_{cH}(G) = 4$ respectively.

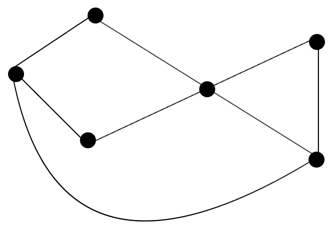


Figure 4

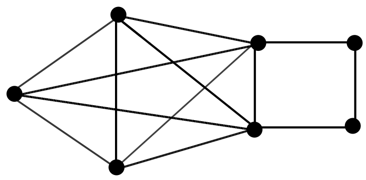


Figure 5

The corona $K_p o K_1$ has $\gamma = \gamma_t = \gamma_c = \gamma_{cH} = \gamma_{cl} = p$ if $p \geq 4$

Theorem 5.6. If D is a γ_H-set of the graph G such that $\langle D \rangle$ contains an isolated vertex, and if $diam(G) \geq 5$, then D is a global dominating set.

Proof. Consider any graph G with $diam(G) \geq 5$. Let D be a γ_H-set of G such that $\langle D \rangle$ contains at least one isolated vertex. It is asserted that that D is a global dominating set of G. As $diam(G) \geq 5$, D must contain more than one vertices otherwise $diam(G) = 2$. Since $\langle D \rangle$ contains an isolated vertex it will dominate all the vertices of D in $\overline{G}$. Now it is claimed that for every vertex $v \in V \setminus D$, $|N[v] \cap D| < |D|$. Otherwise, there exists a vertex $v \in V \setminus D$ such that $|N[v] \cap D| = |D|$. Then for any two vertices u_1, u_2 of G, there arise the following cases.

Case(i) If u_1, u_2 are in D, then $u_1 v u_2$ is a path connecting u_1 and u_2. Hence $d(u_1, u_2) \leq 2$

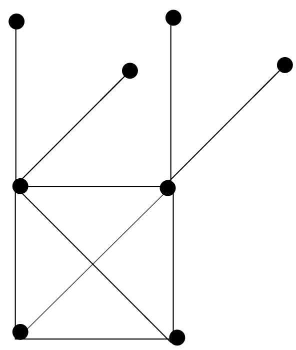
Case(ii) Let u_1, u_2 are in $V \setminus D$, then there exists $u_1^{'}$, $u_2^{'}$ in D such that u_1 is adjacent to $u_1^{'}$ and u_2 is adjacent to $u_2^{'}$. So that $u_1 u_1^{'} v u_2^{'} u_2$ is a path joining u_1 and u_2, when $v \neq u_1$ and $v \neq u_2$. If $v = u_1$ or $v = u_2$ then $u_1 = v u_2^{'} u_2$, $u_1 u_1^{'} v = u_2$ respectively form u_1-u_2 paths and hence $d(u_1, u_2) \leq 4$

Case(iii) If $u_1 \in D$ and $u_2 \in V \setminus D$ and if $u_2^{'} \in D$ dominates u_2 then $u_1 v u_2^{'} u_2$ is a path joining u_1 and u_2. Therefore $d(u_1, u_2) \leq 3$

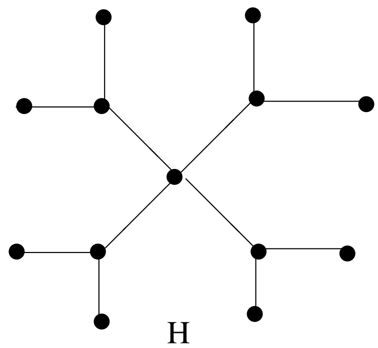
So that the distance between any pair of vertices is at most four, a contradiction to $diam(G) \geq 5$. Hence if $v \in V \setminus D$, then $|N[v] \cap D| < |D|$. So there exists a vertex u in D which is not in $N[v] \cap D$ and dominates v in $\overline{G}$. Thus D is a dominating set of $\overline{G}$. Hence the theorem. □

Corollary 5.7. If D is a γ_H-set of G containing an isolated vertex, and if $diam(G) \geq 5$, then $\gamma_g(G) \leq \gamma_H(G)$

Remark 5.8. Theorem 5.6 need not hold for graphs with diameter ≤ 4. For example, complete graphs $K_n, n \geq 2$ has diameter 1. $\gamma_H(K_n) = 1$ and $\gamma_g(K_n) = n$. For $K_{1,3}$, diameter = 2, $\gamma_H(K_{1,3}) = 1$ and $\gamma_g(K_{1,3}) = 2$ In figure 6, diameter of the graph G is 3, $\gamma_H(G) = 3$ and $\gamma_g(G) = 4$ and in figure 7, diameter H is 4, $\gamma_H(H) = 4$ and $\gamma_g(H) = 5$



G
Figure 6



H
Figure 7

CONCLUSION

In this paper Hausdorff domination number and connected Hausdorff domination number are introduced. A characterization property for a dominating set to be Hausdorff dominating is proved. The relation between Hausdorff domination and independent domination are discussed. Also an attempt is made to compare Hausdorff domination with other domination parameters. Still there are many characterizations which are not dealt with and hence there is a wide scope for future study.

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