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Boundedness of solutions in a chemotaxis-haptotaxis model with logistic source

Long Lei *

College of Mathematic and Information, China West Normal University, Nanchong 637009, P.R. China

Abstract

In this paper, we consider the boundedness of solutions to the following chemotaxis-haptotaxis model:

$$\begin{cases} u_t = \Delta u - \nabla \cdot (\chi(u)\nabla v) - \xi\nabla \cdot (u\nabla w) + \mu u(1 - u^{r-1} - w), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u^{\eta}, & x \in \Omega,\ t > 0, \\ w_t = -vw, & x \in \Omega,\ t > 0, \end{cases}$$

under zero-flux boundary conditions in a smooth bounded domain $\Omega \subset \mathbb{R}^n (n \geq 2)$, with parameters $r \geq 2, \eta \in (0, 1]$ and the parameters $\xi > 0, \mu > 0$. $\chi(u)$ is assumed to satisfy $\chi(u) \leq \rho u^{\beta}, \chi(0) > 0$ for all $u > 0$ with some $\beta \in \mathbb{R}$ and $\rho > 0$. It is proved that if $\beta < \frac{3}{2} - \frac{n\eta}{n+2}$, then for sufficiently smooth initial data (u_0, v_0, w_0), the corresponding initial-boundary problem possesses a unique classical solution which is global in time and bounded.

Keywords: Chemotaxis, Haptotaxis, Boundedness, Logistic source, Nonlinear production
MSC: 35B65 ; 35K55 ; 35Q92 ; 92C17

1. INTRODUCTION

The chemotaxis-haptotaxis model was first introduced by Chaplain and Lolas [3], it describe processes of cancer cell invasion of surrounding healthy tissue. In addition to random motion,cancer cells bias their movement toward increasing concentrations of a diffusible enzyme as well as according to gradients of non-diffusible tissue by detecting matrix molecules

*Corresponding author. E-mail: 253893585@qq.com

such as vitronectin adhered therein. The directed cell motion in response to concentration gradients of some chemical signal is commonly referred to as chemotaxis and the directed migration toward immovable cues is commonly referred to as haptotaxis. Apart from that, in this modeling context the cancer cells are usually also assumed to follow a logistic growth competing for space with healthy tissue. The enzyme is produced by cancer cells and it is supposed to be influenced by diffusion and degradation. The tissue, also named extracellular matrix, can be degraded by enzyme upon contact; on the other hand, the tissue might possess the ability to remodel the healthy level. In [7, 8], authors studied the following chemotaxis-haptotaxis system:

$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u) - \nabla\cdot(S_1(u)\nabla v) - \nabla\cdot(S_2(u)\nabla w) + uf(u,w), & x\in\Omega,\ t>0, \\ v_t = \Delta v - v + u, & x\in\Omega,\ t>0, \\ w_t = -vw, & x\in\Omega,\ t>0, \\ -D(u)\dfrac{\partial u}{\partial \nu} + S_1(u)\dfrac{\partial v}{\partial \nu} + S_2(u)\dfrac{\partial w}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x\in\partial\Omega,\ t>0, \\ u(x,0)=u_0(x),\quad v(x,0)=v_0(x),\quad w(x,0)=w_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^n(n\ge 1)$ is a bounded domain with smooth boundary $\partial\Omega$, $\frac{\partial}{\partial\nu}$ denotes the outward normal derivative on $\partial\Omega$, the unknown functions u, v and w represent the cancer cell density, the enzyme concentration and the extracellular matrix density, respectively, $D(u), S_1(u)$ and $S_2(u)$ stand for the density-dependent motility of cancer cells through the extracellular matrix, the density-dependent chemotactic sensitivity and the density- dependent haptotactic sensitivity, respectively, and $f(u,w)$ denotes the proliferation rate of the cells. For the sake of simplicity, they don't take the remodeling of the extracellular matrix into account here. For D, S_1, S_2 and f are the following assumptions:

$$D, S_1, S_2 \in C^2([0,\infty)) \quad \text{and} \quad f\in C^1([0,+\infty)\times[0,+\infty))$$

and there exist $m, q_1, q_2\in\mathbb{R}, C_D>0, C_{S_1}>0, C_{S_2}>0, \mu>0$ and $b>0$ such that

$$C_D(u+1)^{m-1}\le D(u), S_1\le C_{S_1}u(u+1)^{q_1-1}, S_2\le C_{S_2}u(u+1)^{q_2-1} \qquad \text{for all} \quad u\ge 0$$

and

$$f(u,w)\le \mu - bu \qquad \text{for all} \quad u\ge 0 \quad \text{and} \quad w\ge 0.$$

Liu et al. [7, 8] show that besides the impact of the nonlinear diffusion, the dampening effect of the source of logistic type in tumor cells can also contribute to the boundedness of the solutions to (1.1) and proved the global existence, uniqueness and the boundedness of the solutions to (1.1) for space dimension $n=1$ if $q_1<\frac{m}{2}+1, q_2<\min\{\frac{m}{2}+1, 2\}$ or for $n=2$ if $q_1<\frac{m+1}{2}, q_2<\min\{\frac{m}{2}+1,\frac{3}{2}\}$ or for $n\ge 3$ if $q_1<\frac{m}{2}+\frac{2}{n+2}, q_2<\min\{\frac{m}{2}+1, 2-\frac{n-2}{n+2}\}$.

With $D(u)=1, S_1(u)=\chi u, S_2(u)=\xi u, f(u,w)=(a-\mu u^{r-1}-\lambda w)$, (1.1) transforms into the following chemotaxis-haptotaxis system:

$$
\begin{cases}
u_t = \Delta u - \chi \nabla \cdot (u \nabla v) - \xi \nabla \cdot (u \nabla w) + u(a - \mu u^{r-1} - \lambda w), & x \in \Omega,\ t > 0, \\
v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \\
w_t = -vw, & x \in \Omega,\ t > 0, \\
\dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = \dfrac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\
u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad w(x,0) = w_0(x), & x \in \Omega,
\end{cases}
\tag{1.2}
$$

in smoothly bounded domain $\Omega \subset \mathbb{R}^n, n \geq 1, a \in \mathbb{R}$, where $\chi > 0, \xi > 0, \mu > 0, \lambda > 0$ are parameters. Zheng and Ke [17] shown that when $r > 2$, or $r = 2$, with $\mu > \mu^* = \frac{(n-2)_+}{n}(\chi + C_\beta) C_{\frac{n}{2}+1}^{\frac{1}{\frac{n}{2}+1}}$ the problem (1.2) possesses a global classical solution which is bounded, where C_β and $C_{\frac{n}{2}+1}$ are a positive constants.

For the special case $a = \mu, \lambda = \mu, r = 2$ in (1.2), Tao and Wang[12] proved that model (1.2) possesses a unique global-in-time classical solution for any $\chi > 0$ in one space dimension, or for small $\frac{\chi}{\mu} > 0$ in two and three space dimensions. Later, Tao [11] improved the result of [12] for any $\mu > 0$ in two space dimensions. Hillen, Painter and Winkler [5] studied the global boundedness and asymptotic behavior of the solution to (1.2) in one space dimension. Tao [10] proved that the model has a unique classical solution which is global-in-time and bounded in two space dimensions. Cao [2] proved that the model has a unique classical solution which is global-in-time and bounded in three space dimensions. Wang and Ke [16] proved that the model possesses a unique global-in-time classical solution that is bounded in the case $3 \leq n \leq 8$ and μ is appropriately large.

Chen and Tao [4] considered the following chemotaxis-haptotaxis model with generalized logistic source

$$
\begin{cases}
u_t = \Delta u - \chi \nabla \cdot (u \nabla v) - \xi \nabla \cdot (u \nabla w) + \mu u(1 - u - w), & x \in \Omega,\ t > 0, \\
v_t = \Delta v - v + g(u)h(w), & x \in \Omega,\ t > 0, \\
w_t = -vw + \varsigma w(1 - u - w), & x \in \Omega,\ t > 0, \\
\dfrac{\partial u}{\partial \nu} - \chi u \dfrac{\partial v}{\partial \nu} - \xi u \dfrac{\partial w}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\
u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad w(x,0) = w_0(x), & x \in \Omega,
\end{cases}
\tag{1.3}
$$

in a bounded convex domain $\Omega \subset \mathbb{R}^3$ with smooth boundary, where χ, ξ, μ and ς are positive parameters, g and h are prescribed nonnegative and C^1-smooth functions and $g(u)$ is assumed to satisfy $g(u) \leq K u^\eta$ for all $u > 0$ with some $\eta \in (0, 1]$ and $K > 0$. Chen and Tao [4] proved that if $0 < \eta < \frac{5}{6}$, then for any given suitably regular initial data the corresponding Neumann initial-boundary problem possesses a unique global-in-time classical solution that is uniformly bounded.

Inspired by the above papers, in the present paper, we consider the boundedness of solutions to the following chemotaxis-haptotaxis model:

$$\begin{cases} u_t = \Delta u - \nabla \cdot (\chi(u)\nabla v) - \xi \nabla \cdot (u \nabla w) + \mu u(1 - u^{r-1} - w), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u^{\eta}, & x \in \Omega,\ t > 0, \\ w_t = -vw, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu} - \chi(u)\dfrac{\partial v}{\partial \nu} - \xi u \dfrac{\partial w}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad w(x,0) = w_0(x), & x \in \Omega, \end{cases} \tag{1.4}$$

under zero-flux boundary conditions in a smooth bounded domain $\Omega \subset \mathbb{R}^n (n \geq 2)$, with parameters $r \geq 2, \eta \in (0,1]$ and the parameters $\xi > 0, \mu > 0$. This paper mainly aims to understand the competition among the haptotaxis, the nonlinear chemotaxis, the nonlinear logistic source and the nonlinear production.

The functions u_0, v_0, w_0 are supposed to satisfy the smoothness assumptions

$$\begin{cases} u_0 \in C(\bar{\Omega}) \ \text{ with } u_0 \geq 0 \text{ in } \Omega \text{ and } u_0 \not\equiv 0, \\ v_0 \in W^{1,\infty}(\Omega) \ \text{ with } v_0 \geq 0 \text{ in } \Omega, \\ w_0 \in C^{2+\vartheta}(\bar{\Omega}) \ \text{ for some } \vartheta \in (0,1) \text{ with } w_0 \geq 0 \text{ in } \bar{\Omega} \text{ and } \dfrac{\partial w_0}{\partial \nu} = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.5}$$

We furthermore assume that

$$\chi \in C^2([0,\infty)), \ \ \chi(0) > 0 \tag{1.6}$$

and

$$\chi(u) \leq \rho u^{\beta} \quad \text{for all} \quad u \geq 0 \tag{1.7}$$

with some $\beta \in \mathbb{R}$ and $\rho > 0$.

We give the main result of this paper reads as follow.

Theorem 1.1. *Let $n \geq 2, \xi > 0, \mu > 0, r \geq 2$ and $\eta \in (0,1]$, and let χ be a function satisfying* (1.6) *and* (1.7) *with $\beta < \frac{3}{2} - \frac{n\eta}{n+2}$. Then for any initial data fulfilling* (1.5)*, the problem* (1.4) *admits a unique nonnegative classical solution which is global and bounded in $\Omega \times (0,\infty)$.*

Remark 1.1. *From our results, it is worth to point out that the nonlinear production affect the nonlinear chemotaxis to guarantee the global boundedness of the solution to* (1.4).

This paper is structured as follows. In section 2, we collect basic facts which will be used later. Section 3 we prove global existence and boundedness by use some L^p-estimate techniques and Moser-Alikakos iteration (see e.g.[1] and Lemma A.1 in [13]).

2. PRELIMINARIES

We first state one result concerning local-in-time existence of classical solution to the model (1.4).

Lemma 2.1. *Let $\xi > 0, \mu > 0$ and assume that u_0, v_0 and w_0 satisfy* (1.5)*. Then the problem* (1.4) *admits a unique classical solution*

$$\begin{cases} u \in C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max})), \\ v \in C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max})), \\ w \in C^{2,1}(\bar{\Omega} \times (0, T_{max})) \end{cases} \tag{2.1}$$

with $u \geq 0, v \geq 0$ and $0 \leq w \leq \|w_0\|_{L^\infty(\Omega)}$ for all $(x,t) \in \Omega \times [0, T_{max})$, where T_{max} denotes the maximal existence time. In addition, if $T_{max} < +\infty$, then

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \to \infty \quad as \quad t \nearrow T_{max}. \tag{2.2}$$

Proof. The proof method can referred to [7, 18]. □

The following lemma provides the basic estimates of solutions of (1.4).

Lemma 2.2. *Let (u, v, w) be the solution of* (1.4)*. Then there exists $C > 0$ depending on $n, \|v_0\|_{L^1(\Omega)}$ and $\|u_0\|_{L^1(\Omega)}$ such that*

$$\|u(\cdot, t)\|_{L^1(\Omega)} \leq C, \quad \|v(\cdot, t)\|_{L^1(\Omega)} \leq C, \quad \|\nabla v(\cdot, t)\|_{L^2(\Omega)} \leq C \quad for\ all \quad t \in (0, T_{max}). \tag{2.3}$$

Proof. The proof method can referred to [16]. □

Lemma 2.3. *Let (u, v, w) be the classical solution of* (1.4) *in $\Omega \times (0, T_{max})$. Then for any $k > 1$,*

$$-\int_\Omega u^{k-1} \nabla \cdot (u \nabla w) \leq c_1 \left(\int_\Omega u^k + \int_\Omega u^k v + k \int_\Omega u^{k-1} |\nabla u| \right) \tag{2.4}$$

with constant $c_1 > 0$ independent of k.

Proof. The proof method can referred to [14]. □

Lemma 2.4. *Let $n \geq 2, \eta \in (0,1], \beta < \frac{3}{2} - \frac{n\eta}{n+2}, \theta_1 = \frac{2(k+1)}{3-2\beta}, \theta_2 = \frac{2(k+1)(m-1)}{k-2\eta+1}$ and $\kappa_i = \frac{\frac{m}{2} - \frac{m}{\theta_i}}{\frac{m}{2} - \frac{1}{2} + \frac{1}{n}}, i = 1, 2$. Then for all sufficiently large $k > 1$, there exists large $m > 1$ such that the following inequalities are valid*

$$\theta_i > 2, \quad m > \frac{n-2}{2n} \theta_i, \quad 2m > max\{\theta_i \kappa_i, k+1\} \quad for \quad i = 1, 2. \tag{2.5}$$

Proof. Since $2m > \theta_i \kappa_i$ is equivalent to $m > \frac{\theta_i}{2} - \frac{2}{n}$, it is sufficient to show that if $\beta < \frac{3}{2} - \frac{n\eta}{n+2}$, then for all sufficiently large $k > 1$, there exists large $m > 1$ satisfying $m > \frac{\theta_i}{2} - \frac{2}{n}(i = 1, 2)$ and $2m > k + 1$, which can be achieved by the fact that $m > \frac{\theta_i}{2} - \frac{2}{n}(i = 1, 2)$ is equivalent to $\frac{k+1}{3-2\beta} - \frac{2}{n} < m < \frac{(k+1)(n+2)}{2n\eta} - \frac{2}{n}$. □

3. PROOF OF THEOREM 1.1

In this section, we use test function arguments to derive the bound of u in $L^k(\Omega)$ and ∇v in $L^{2m}(\Omega)$ for all sufficiently large $k, m > 1$ and going to establish an iteration step to proof of Theorem 1.1.

Lemma 3.1. *Let* $n \geq 2, r \geq 2, \xi > 0, \mu > 0$*, and assume that* χ *satisfies* (1.6) *and* (1.7) *with* $\beta < \frac{3}{2} - \frac{n\eta}{n+2}$. *Then for all large numbers* $m > 2, k > 1$ *as provided by Lemma 2.4, there exists* $C > 0$ *such that the solution of* (1.4) *enjoys the property*

$$\|u(\cdot, t)\|_{L^k(\Omega)} \leq C, \quad \|\nabla v(\cdot, t)\|_{L^{2m}} \leq C \quad \textit{for all} \quad t \in (0, T_{max}). \tag{3.1}$$

Proof. Multiplying the first equation in (1.4) by ku^{k-1} and integrating over Ω, we get

$$\begin{aligned} &\frac{d}{dt}\|u\|^k_{L^k(\Omega)} + k(k-1)\int_\Omega u^{k-2}|\nabla u|^2 + k\mu\int_\Omega u^{k+r-1} \\ &\leq k(k-1)\int_\Omega \chi(u)u^{k-2}\nabla u \cdot \nabla v - k\xi\int_\Omega \nabla\cdot(u\nabla w)u^{k-1} + k\mu\int_\Omega u^k. \end{aligned} \tag{3.2}$$

By (1.7) and using the Young's inequality, the first item on the right side of the inequality (3.2) becomes

$$\begin{aligned} k(k-1)\int_\Omega \chi(u)u^{k-2}\nabla u\cdot\nabla v &\leq k(k-1)\rho\int_\Omega u^{k+\beta-2}\nabla u\cdot\nabla v \\ &\leq \frac{k(k-1)}{4}\int_\Omega u^{k-2}|\nabla u|^2 + k(k-1)\rho^2\int_\Omega u^{k+2\beta-2}|\nabla v|^2. \end{aligned} \tag{3.3}$$

The second item of the right side of the inequality (3.2), combining with (2.4), yields

$$\begin{aligned} -k\xi\int_\Omega u^{k-1}\nabla\cdot(u\nabla w) &\leq c_1k\xi\int_\Omega u^k + c_1k\xi\int_\Omega u^k v + c_1k^2\xi\int_\Omega u^{k-1}|\nabla u| \\ &\leq c_1k\xi\int_\Omega u^k + c_1k\xi\int_\Omega u^k v + \frac{k(k-1)}{4}\int_\Omega u^{k-2}|\nabla u|^2 + \frac{c_1^2\xi^2k^3}{k-1}\int_\Omega u^k. \end{aligned} \tag{3.4}$$

Hence, inserting (3.3) and (3.4) into (3.2) yields

$$\begin{aligned}
&\frac{d}{dt}\|u\|^k_{L^k(\Omega)} + \frac{\delta k(k-1)}{2}\int_\Omega u^{k-2}|\nabla u|^2 + k\mu\int_\Omega u^{k+r-1}\\
&\quad\le k(k-1)\rho^2\int_\Omega u^{k+2\beta-2}|\nabla v|^2 + c_1k\xi\int_\Omega u^k + c_1k\xi\int_\Omega u^k v\\
&\qquad+\frac{c_1^2\xi^2k^3}{k-1}\int_\Omega u^k + k\mu\int_\Omega u^k.
\end{aligned}\tag{3.5}$$

Furthermore, using the Young's inequality, we can find

$$\frac{d}{dt}\|u\|^k_{L^k(\Omega)} + c_2\int_\Omega u^{k+1} \le k(k-1)\rho^2\int_\Omega u^{k+2\beta-2}|\nabla v|^2 + c_2\int_\Omega v^{k+1} + c_2,\tag{3.6}$$

where $c_2 > 0$, as all subsequently appearing constants $c_3, c_4, ...c_{16}$ possibly depend on $k, m, \mu, \xi, r, \eta, |\Omega|$ and ρ.

Differentiating the second equation in (1.4), we obtain

$$\frac{d}{dt}|\nabla v|^2 = 2\nabla v\cdot\nabla\Delta v - 2|\nabla v|^2 + 2\nabla u^\eta\cdot\nabla v,$$

and hence, according to identity

$$\Delta|\nabla v|^2 = 2\nabla v\cdot\nabla\Delta v + 2|D^2v|^2,$$

we obtain

$$\frac{d}{dt}|\nabla v|^2 = \Delta|\nabla v|^2 - 2|D^2v|^2 + 2\nabla u^\eta\cdot\nabla v.$$

Testing this by $m|\nabla v|^{2m-2}$ yields

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega|\nabla v|^{2m} + m(m-1)\int_\Omega|\nabla v|^{2m-4}\left|\nabla|\nabla v|^2\right|^2 + 2m\int_\Omega|\nabla v|^{2m-2}|D^2v|^2 + 2m\int_\Omega|\nabla v|^{2m}\\
&\quad\le 2m\int_\Omega|\nabla v|^{2m-2}\nabla u^\eta\cdot\nabla v + m\int_{\partial\Omega}\frac{\partial|\nabla v|^2}{\partial\nu}|\nabla v|^{2m-2}.
\end{aligned}\tag{3.7}$$

Based on the estimate of Mizoguchi-Souplet [9], the Gagliardo-Nirenberg inequality and boundedness of ∇v in $L^2(\Omega)$, we can conclude that

$$m\int_{\partial\Omega}\frac{\partial|\nabla v|^2}{\partial\nu}|\nabla v|^{2m-2} \le c_3\left(\int_\Omega|\nabla|\nabla v|^m|^2\right)^b + c_3\tag{3.8}$$

with some $b\in(0,1)$. Therefore, combining (3.7) with (3.8) and applying the Young's inequality, we have

$$\frac{d}{dt}\int_\Omega |\nabla v|^{2m} + \frac{m(m-1)}{2}\int_\Omega |\nabla v|^{2m-4}\left|\nabla|\nabla v|^2\right|^2 + 2m\int_\Omega |\nabla v|^{2m-2}|D^2 v|^2 + 2m\int_\Omega |\nabla v|^{2m} \\ \leq 2m\int_\Omega |\nabla v|^{2m-2}\nabla u^\eta \cdot \nabla v + c_4, \tag{3.9}$$

due to $\int_\Omega |\nabla v|^{2m-4}|\nabla|\nabla v|^2|^2 = \frac{4}{m}\int_\Omega |\nabla|\nabla v|^m|^2$.

Hence, due to the pointwise identities $\nabla|\nabla v|^{2m-2} = (m-1)|\nabla v|^{2m-4}\nabla|\nabla v|^2$ and $|\Delta v|^2 \leq n|D^2 v|^2$, and together with an integration by the right part in (3.9) and using Young's inequality, we have

$$\begin{aligned} &2m\int_\Omega |\nabla v|^{2m-2}\nabla u^\eta \cdot \nabla v \\ &= -2m(m-1)\int_\Omega u^\eta |\nabla v|^{2m-4}\nabla v\cdot\nabla|\nabla v|^2 - 2m\int_\Omega u^\eta|\nabla v|^{2m-2}\Delta v \\ &\leq \frac{m(m-1)}{4}\int_\Omega |\nabla v|^{2m-4}\left|\nabla|\nabla v|^2\right|^2 + 4m(m-1)\int_\Omega u^{2\eta}|\nabla v|^{2m-2} \\ &\quad + \frac{m}{n}\int_\Omega |\nabla v|^{2m-2}|\Delta v|^2 + mn\int_\Omega u^{2\eta}|\nabla v|^{2m-2} \\ &\leq \frac{m(m-1)}{4}\int_\Omega |\nabla v|^{2m-4}\left|\nabla|\nabla v|^2\right|^2 + (4m(m-1)+mn)\int_\Omega u^{2\eta}|\nabla v|^{2m-2} \\ &\quad + m\int_\Omega |\nabla v|^{2m-2}|D^2 v|^2. \end{aligned} \tag{3.10}$$

Hence, inserting (3.10) into (3.9) yields

$$\frac{d}{dt}\int_\Omega |\nabla v|^{2m} + (m-1)\int_\Omega |\nabla|\nabla v|^m|^2 + 2m\int_\Omega |\nabla v|^{2m} \\ \leq (4m(m-1)+mn)\int_\Omega u^{2\eta}|\nabla v|^{2m-2} + c_4. \tag{3.11}$$

Hence combining (3.6) with (3.11) and using the Young's inequality, we can find

$$\begin{aligned} &\frac{d}{dt}\int_\Omega (u^k + |\nabla v|^{2m}) + c_5\int_\Omega \left(\left|\nabla|\nabla v|^m\right|^2 + |\nabla v|^{2m}\right) + c_5\int_\Omega u^{k+1} \\ &\leq c_6\int_\Omega u^{k+2\beta-2}|\nabla v|^2 + c_6\int_\Omega u^{2\eta}|\nabla v|^{2m-2} + c_6\int_\Omega v^{k+1} + c_6 \\ &\leq \frac{c_5}{2}\int_\Omega u^{k+1} + c_7\int_\Omega (|\nabla v|^{\theta_1} + |\nabla v|^{\theta_2}) + c_6\int_\Omega v^{k+1} + c_6 \end{aligned} \tag{3.12}$$

with $\theta_i (i = 1, 2)$ as shown in Lemma 2.4. According to the Gagliardo-Nirenberg inequality, (2.3) and Lemma 2.4, we have

$$\begin{aligned} c_7\int_\Omega |\nabla v|^{\theta_i} &= c_7 \| |\nabla v|^m \|_{L^{\frac{\theta_i}{m}}}^{\frac{\theta_i}{m}} \\ &\le c_8\left(\|\nabla|\nabla v|^m\|_{L^2(\Omega)}^{\kappa_i} \| |\nabla v|^m\|_{L^{\frac{2}{m}}(\Omega)}^{1-\kappa_i} + \||\nabla v|^m\|_{L^{\frac{2}{m}}(\Omega)} \right)^{\frac{\theta_i}{m}} \\ &\le c_9 \|\nabla|\nabla v|^m\|_{L^2(\Omega)}^{\frac{\theta_i\kappa_i}{m}} + c_9 \\ &\le \frac{c_5}{2}\|\nabla|\nabla v|^m\|_{L^2(\Omega)}^2 + c_{10}. \end{aligned} \tag{3.13}$$

Due to the boundedness of $\|v\|_{W^{1,2}(\Omega)}$ (see Lemma 2.2) and Lemma 2.4, and by the Sobolev inequality and Young's inequality, we can find

$$c_6\int_\Omega v^{k+1} \le c_{11}\|v\|_{L^\infty(\Omega)}^{k+1} \le c_{12}\|v\|_{L^{n+1}(\Omega)}^{k+1} + c_{12} \le c_{13}\|v\|_{L^{2m}(\Omega)}^{k+1} + c_{12} \le \frac{c_5}{2}\int_\Omega |\nabla v|^{2m} + c_{14}. \tag{3.14}$$

Hence substituting (3.13) and (3.14) into (3.12) yields

$$\frac{d}{dt}\int_\Omega (u^k + |\nabla v|^{2m}) + \frac{c_5}{2}\int_\Omega (u^{k+1} + |\nabla v|^{2m}) \le c_{15},$$

by the Young's inequality, we can find

$$\frac{d}{dt}\int_\Omega (u^k + |\nabla v|^{2m}) + \frac{c_5}{2}\int_\Omega (u^{k} + |\nabla v|^{2m}) \le c_{16},$$

for sufficiently large $k > 1, m > 1$. Consequently, $y(t) := \int_\Omega (u^k + |\nabla v|^{2m})$ satisfies $y'(t) + \frac{c_5}{2}y(t) \le c_{16}$.

Upon an ODE comparison argument, we have $y(t) \le max\{y(0), \frac{2c_{16}}{c_5}\}$ for all $t \in (0, T_{max})$. The proof of Lemma 3.1 is complete. □

Due to $\|u(\cdot,t)\|_{L^k(\Omega)} \le C$ is bounded for any large k, by the fundamental estimates for Neumann semigroup (see[6, Lemma 2.1]) or the standard regularity theory of parabolic equation, we immediately have the following Corollary.

Corollary 3.1. *Let $T \in (0, T_{max}), \xi > 0$ and $\mu > 0$, and assume that (u_0, v_0, w_0) satisfy* (1.5). *Then there exists $C > 0$ independent of T such that the solution (u, v, w) of* (1.4) *satisfies*

$$\|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C \quad \textit{for all} \quad t \in (0,T). \tag{3.15}$$

Next we can prove our main result. Use the standard Moser-Alikakos iteration and choose (3.5) as a starting point for our proof.

Lemma 3.2. *Let $n \ge 2, T \in (0, T_{max}), \xi > 0, \mu > 0, \varrho > 0$, and assume Lemma 3.1. Then there* *exists* $C > 0$ *independent of* T *such that the solution* (u, v, w) *of* (1.4) *satisfies*

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C \quad \textit{for all} \quad t \in (0, T). \tag{3.16}$$

Proof. We begin with (3.5)

$$\begin{aligned}
&\frac{d}{dt}\|u\|^k_{L^k(\Omega)} + \frac{\delta k(k-1)}{2}\int_\Omega u^{k-2}|\nabla u|^2 + k\mu\int_\Omega u^{k+r-1} \\
&\leq k(k-1)\rho^2\int_\Omega u^{k+2\beta-2}|\nabla v|^2 + c_1k\xi\int_\Omega u^k + c_1k\xi\int_\Omega u^k v \\
&\quad + \frac{c_1^2\xi^2k^3}{k-1}\int_\Omega u^k + k\mu\int_\Omega u^k
\end{aligned}$$

which, along with (3.15), implies that

$$\begin{aligned}
&\frac{d}{dt}\|u\|^k_{L^k(\Omega)} + \frac{\delta k(k-1)}{2}\int_\Omega u^{k-2}|\nabla u|^2 + k\mu\int_\Omega u^{k+r-1} \\
&\leq c_{17}k(k-1)\int_\Omega u^{k+2\beta-2} + c_{17}k\int_\Omega u^k + \left(c_1k\xi + \frac{c_1^2\xi^2k^3}{k-1} + k\mu\right)\int_\Omega u^k,
\end{aligned}$$

where $c_{17} > 0$, as all subsequently appearing constants $c_{18}, c_{19}, \ldots$ are independent of k as well as T.

By the Young's inequality and an obvious rearrangement implies, we can find

$$\frac{d}{dt}\int_\Omega u^k + c_{18}\int_\Omega |\nabla u^{\frac{k}{2}}|^2 \leq c_{19}k^2\int_\Omega u^k. \tag{3.17}$$

Let $k_i = 2^i, i \in \mathbb{N}$ and $M_i := \sup_{t\in(0,T)}\int_\Omega u^{k_i}, i \in \mathbb{N}$. Since $k_i \geq 1$, it is easy to find $c_{20} > 0$ such that

$$\frac{d}{dt}\int_\Omega u^{k_i} + c_{18}\int_\Omega |\nabla u^{\frac{k_i}{2}}|^2 + \int_\Omega u^{k_i} \leq c_{19}k_i^2\int_\Omega u^{k_i} + \int_\Omega u^{k_i} \leq c_{20}k_i^2\int_\Omega u^{k_i}. \tag{3.18}$$

Using the Gagliardo-Nirenberg inequality, we find $c_{21} > 0$ independent of k, such that

$$\int_\Omega u^{k_i} = \|u^{\frac{k_i}{2}}\|^2_{L^2} \leq c_{21}\|\nabla u^{\frac{k_i}{2}}\|^{2a}_{L^2(\Omega)} \cdot \|u^{\frac{k_i}{2}}\|^{2(1-a)}_{L^1(\Omega)} + c_{21}\|u^{\frac{k_i}{2}}\|^2_{L^1(\Omega)},$$

for all $t \in (0, T)$, with $a = \frac{\frac{n}{2}}{1+\frac{n}{2}} \in (0, 1)$.

By the Young's inequality and ensure that there are $c_{22} > 0$ and $e > 1$ satisfying

$$\begin{aligned} c_{20}k_i^2 \int_\Omega u^{k_i} &\le c_{18} \int_\Omega |\nabla u^{\frac{k_i}{2}}|^2 + c_{22}(k_i^2)^{\frac{1}{1-a}} \left(\int_\Omega u^{\frac{k_i}{2}}\right)^2 + c_{20}c_{21}k_i^2 \left(\int_\Omega u^{\frac{k_i}{2}}\right)^2 \\ &\le c_{18} \int_\Omega |\nabla u^{\frac{k_i}{2}}|^2 + e^i \left(\int_\Omega u^{\frac{k_i}{2}}\right)^2, \end{aligned} \tag{3.19}$$

Combining (3.18) and (3.19) we find that

$$\frac{d}{dt}\int_\Omega u^{k_i} + \int_\Omega u^{k_i} \le e^i \left(\int_\Omega u^{\frac{k_i}{2}}\right)^2 = e^i M_{i-1}^2.$$

Upon an ODE comparison argument, we have

$$M_i \le \max\left\{\|u_0\|_{L^\infty(\Omega)}^{k_i}, e^i M_{i-1}^2\right\}.$$

If $\|u_0\|_{L^\infty(\Omega)}^{k_i} \ge e^i M_{i-1}^2$ for infinitely many $i \ge 1$, which implies that

$$\sup_{t\in(0,T)} \|u(\cdot,t)\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)}$$

and thereby proves the lemma in this case.

Otherwise, a direct induction entails

$$M_i \le e^i M_{i-1}^2 \le e^{i+\sum_{j=1}^{i-1} 2^j(i-j)} M_0^{2^i} \quad \text{for all} \quad i \ge 1. \tag{3.20}$$

Here we observe that

$$i + \sum_{j=1}^{i-1} 2^j(i-j) = 2 + 2^2 + 2^3 + \cdots + 2^i - i \le 2^{i+1} \quad \text{for all} \quad i \ge 1.$$

From this and (3.20) we infer

$$M_i^{\frac{1}{k_i}} \le e^2 M_0 \quad \text{for all} \quad i \ge 1.$$

which implies that

$$\sup_{t\in(0,T)} \|u(\cdot,t)\|_{L^\infty(\Omega)} \le e^2 M_0$$

and thereby yields the assertion in this case. $\square$

Now, we prove Theorem 1.1.

Proof of Theorem 1.1. First we see that boundedness of u and v follows from Lemma 3.2 and Corollary 3.1. Therefore the assertion of Theorem 1.1 is immediately obtained from Lemma 2.1. □

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