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Neutral Impulsive Stochastic Differential Equations Driven by Fractional Brownian Motion with Poisson Jumps and Nonlocal Conditions

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Abstract

In this paper, we present existence and uniqueness of mild solutions for neutral impulsive stochastic differential equations driven by fractional Brownian motion with the Hurst index $H > \frac{1}{2}$ with Poisson jumps. The results are obtained by using Banach fixed point principle in a Hillbert space.
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1. INTRODUCTION

In this paper, we establish the existence, uniqueness and asymptotic behaviour of mild solution to neutral impulsive stochastic differential equations with finite delay and Poisson jumps of the following form driven by fractional Brownian motion in a Hilbert space

$$\begin{cases} d[u(t) + g(t, u_t)] = [Au(t) + f(t, u_t)]dt + \sigma(t)dW^H(t) + \\ \int\limits_Z h(t, u_t, y)\tilde{N}(dt, dy), \quad t \geq 0, t \neq t_k, \\ \Delta u(t_k) := u(t_k^+) - u(t_k) = I_k(x(t_k)), \quad k \in \mathbb{N}, \quad t \in (-\tau, 0] \quad (0 < \tau \leq \infty) \\ u(0) + G(u) = u_0, \end{cases} \tag{1.1}$$

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where $Z \in \mathscr{L}_2^0(U - \{0\})$, A is the infinitesimal generator of an analytic semigroup of bounded linear operators, $(T(t))_{t\geq 0}$ in a Hilbert space X with norm $\|.\|$, W^H is a fractional Brownian motion with $H > \frac{1}{2}$ on a real and seperable Hilbert space Y, $\mathbb{N}$ denotes the set of positive integers, the impulsive moments satisfy $0 < t_1 < t_2 < \cdots$, $\lim_{k\to\infty} t_k = \infty$, and $f, g : [0,\infty)X \to X$, $, G : X \to \mathscr{L}_2^0(Y, X)$, $\sigma : [0,\infty) \to \mathscr{L}_2^0(Y, X)$, $h : [0,\infty)X \times U \to X$, $I_k : X \to X$ are defined later, the initial data $\phi \in C((-\tau, 0], X)$ the space of all continuous functions from $(-\tau, 0]$ to X and has finite second moments. The space $\mathscr{L}_2^0(Y, X)$ will be defined later. We have used Banach fixed point theorem and semigroup theory as a major tool.

As for the stochastic functional differential equations driven by a fractional Brownian motion and Poisson jumps, even much less has been done, as far as we know, there exists only few papers published in this field. In Ferrante and Rovira,[9] the authors studied the existence and regularity of the density by using Skorohod integral based on the Malliavin calculus. Neuenkirch et al. [18] studied the problem by using rough path analysis. In [11]Ferrante and Rovira, studied the existence and convergence when the delay goes to zero by using the Riemann-Stieltjes integral. Using the Riemann-Stieltjes integral, Boufoussi and Hajji(2011) [3] and Boufoussi et al. (2011) [4] proved the existence and uniqueness of a mild solution and studied the dependence of the solution on the initial condition in finite and infinite dimensional spaces. Caraballo et al. (2011) [6] have discussed the existence, uniqueness and exponential asymptotic behaviour of mild solutions by using Wiener integral.

By contrary, there has not been very much study of stochastic differential equations driven by fractional Brownian motion and Poisson jumps with nonlocal conditions, while these have begun to gain attention recently. P. Balasubramaniam et. al [2] showed there exists solutions for semilinear neutral stochastic functional differential equation with nonlocal conditions. Jingyun v and Xiaoyuan Yang [14] studied nonlocal fractional stochastic differential equations driven by fractional Brownian motion. Sayooj Aby Jose and Venkatesh Usha [20] studied existence of Solutions for Random Impulsive Differential Equation with Nonlocal conditions.

On the other hand, to the best of our knowledge, there is no paper which investigates the study of neutral stochastic impulsive differential equations driven by fractional Brownian motion with finite delay, Poisson jumps and nonlocal conditions. Thus, by motivation of above mentioned works we will make the first attempt to study such system in this paper.

This paper is constructed as follows. In section 2 we present some basic results and estimates. In section 3 we mentioned hypotheses to establish the main result. In section

4 we studied the existence, uniqueness and asymptotic behaviour of mild solution.

2. PRELIMINARIES

We first introduce some definitions, notations and basic preliminary facts which are used throughout this paper. Let $(\Omega, \mathscr{F}, P)$ be a complete probability space and $T > 0$ be an arbitrary fixed horizon. An one-dimensional fractional Brownian motion(fbm) with Hurst parameter $H \in (0,1)$ is a centered Gaussian process $\beta^H = \{\beta^H(t), 0 \leq t \leq T\}$ with the covariance function $R(t,s) = E[\beta^H(t)\beta^H(s)]$

$$R(t,s) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}).$$

It is known that $\beta^H(t)$ with $H > \frac{1}{2}$ admits the following Volterra representation

$$\beta^H(t) = \int_0^t K(t,s)d\beta(s), \tag{2.1}$$

where β is a standard Brownian motion and the Volterra kernal $K(t,s)$ is given by

$$K(t,s) = c_H \int_s^t (u-s)^{H-\frac{3}{2}} \left(\frac{u}{s}\right)^{H-\frac{1}{2}} du, \quad t \geq s.$$

for the deterministic function $\varphi \in L^2([0,T])$, the fractional Wiener integral of φ with respect to β^H is defined by

$$\int_0^T \varphi(s)d\beta^H(s) = \int_0^T K_H^*\varphi(s)d\beta(s),$$

where $K_H^*\varphi(s) = \displaystyle\int_s^T \varphi(r)\frac{\partial K}{\partial r}(r,s)dr.$

Let $(\mathscr{U}, \epsilon, \nu(du))$ be a $\sigma-$ finite measurable space. Given a stationary Poisson point process $(p_t)_{t>0}$, which is defined on $(\Omega, \mathscr{F}, P)$ with values in $\mathscr{U}$ and with characteristic measure ν. We will denote by $N(t,du)$ be the counting measure of p_t such that $\widehat{N}(t,A) := \mathbb{E}(N(t,A)) = t\nu(A)$ for $A \in \epsilon$. Define $\widehat{N}(t,du) := N(t,du) - t\nu(du)$, the Poisson martingale measure generated by p_t.

Let X and Y be two real, seperable Hilbert spaces and let $\mathscr{L}(Y,X)$ be the space of bounded linear operators from Y to X. For the sake of convenience, we shall use the same notation to denote the norms in X, Y and $\mathscr{L}(Y,X)$. Let $\{e_n, n = 1,2,\ldots\}$ be a complete orthonormal basis in Y and $Q \in \mathscr{L}(Y,X)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite trace $trQ = \displaystyle\sum_{n=1}^{\infty} \lambda_n < \infty$, where $\lambda_n, n = 1,2,\ldots$ are non-negative real numbers. We define the infinite dimensional fbm on Y with covariance Q

as

$$W^H(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n \beta_n^H(t),$$

where $\beta_n^H(t)$ are real, independent fbm's. This process is a $Y-$valued Gaussian, it starts from 0, has zero mean and covariance:

$$E\left\langle W^H(t), x\right\rangle \left\langle W^H(s), y\right\rangle = R(t,s)\left\langle Q(x), y\right\rangle \quad \text{for all} \quad x, y \in Y \quad \text{and} \quad t, s \in [0,T].$$

In order to define Wiener integrals with respect to the $Q-$fbm $W^H(t)$, we introduce the space $\mathscr{L}_2^0 := \mathscr{L}_2^0(Y,X)$ of all $Q-$Hillbert-Schmidt operators $\psi : Y \to X$. and $\psi \in \mathscr{L}(Y,X)$ is called a $Q-$Hilbert-Schmidt operator if

$$\|\psi\|_{\mathscr{L}_2^0} := \sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\psi e_n\|^2 < \infty$$

and that the space $\mathscr{L}_2^0$ equipped with the inner product $\langle \varphi, \psi\rangle_{\mathscr{L}_2^0} := \sum_{n=1}^{\infty} \langle \varphi e_n, \psi e_n\rangle$ is a seperable Hilbert space. The fractional Wiener integral of the function $\psi : [0,T] \to \mathscr{L}_2^0(Y,X)$ with respect to $Q-$fbm is defined by

$$\int_0^t \psi(s)dW^H(s) = \sum_{n=1}^{\infty}\int_0^t \sqrt{\lambda_n}\psi(s)e_n d\beta_n^H(s) = \sum_{n=1}^{\infty}\int_0^t \sqrt{\lambda_n}K_H^*(\psi e_n)(s)d\beta_n(s), \tag{2.2}$$

where β_n is the standard Brownian motion used to present β_n^H as in equation 2.1.

Suppose that $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A, and the semigroup $T(t)$ is uniformly bounded, $\|T(t)\| \leq M$ for some constant $M \geq 1$ and every $t \geq 0$. Then, for $0 < \alpha \leq 1$, it is possible to define the fractional power operator $(-A)^\alpha$ as a closed inear operator on its domain $\mathscr{D}(-A)^\alpha$. Furthermore, the subspace $\mathscr{D}(-A)^\alpha$ is dense in X and the expression

$$\|x\|_\alpha = \|(-A)^\alpha x\|, \quad x \in \mathscr{D}(-A)^\alpha$$

defines a norm on $X_\alpha := \mathscr{D}(-A)^\alpha$.

Lemma 2.1. *If $\psi : [0,T] \to \mathscr{L}_2^0(Y,X)$ satisfies $\int_0^t \|\psi(s)\|_{\mathscr{L}_2^0}^2 ds < \infty$ then the above sum in equation 2.2 is well defined as an $X-$valued random variable and we have*

$$E\left\| \int_0^t \psi(s)dW^H(s)\right\|^2 \leq 2Ht^{2H-1}\int_0^t \|\psi(s)\|_{\mathscr{L}_2^0}^2 ds. \tag{2.3}$$

Lemma 2.2. *[19] Under the above conditions the following properties hold.*

- X_α *is a Banach space for* $0 < \alpha \leq 1$.
- *If the resolvent operator of* A *is compact, then the embedding* $X_\beta \subset X_\alpha$ *is continuous and compact for* $0 < \alpha \leq \beta$.
- *For every* $0 < \alpha \leq 1$, *there exists* M_α *such that*

$$\|(-A)^\alpha T(t)\| \leq M_\alpha t^{-\alpha} e^{-\lambda t}, \ \ \lambda > 0, \ \ t \geq 0. \tag{2.4}$$

Definition 2.1. *An* $X-$*valued stochastic process* $\{u(t), t \in (-\tau, \infty)\}$ *is called a mild solution of equation 1.1 if* $u(t) = \phi(t)$ *on* $(-\tau, 0]$, *and the following conditions hold:*

- $u(.)$ *is continuous on* $(0, t_1]$ *and each interval* $(t_k, t_{k+1}], k \in \mathbb{N}$,
- *for each* t_k, $u(t_k^+) = \lim\limits_{t \to t_k^+} u(t)$ *exists,*
- *for each* $t \geq 0$, *we have a.s.*

$$\begin{aligned} u(t) = {} & T(t)[u_0 - G(u) + g(0,\phi)] - g(t, u_t) - \int_0^t AT(t-s) g(s, u_s) ds \\ & + \int_0^t T(t-s) f(s, u_s) ds + \int_0^t T(t-s)\sigma(s) dW^H(s) \\ & + \int_0^t T(t-s) \int_Z h(s, u_s, y) \tilde{N}(ds, dy) + \sum_{0 < t_k < t} T(t - t_k) I_k(u(t_k)). \end{aligned} \tag{2.5}$$

3. HYPOTHESES

In order to prove the required results, we assume the following conditions:

(H_1) A is the infinitesimal generator of an analytic semigroup, $(T(t))_{t \geq 0}$, of bounded linear operators on X. Moreover, $T(t)$ satisfies the condition that there exists positive constants M, λ such that

$$\|T(t)\| \leq M e^{-\lambda t}, \ \ t \geq 0.$$

(H_2) There exists $L_1 > 0$ such that, for all $t \geq 0, x, y \in X$.

$$\|f(t,u) - f(t,v)\|^2 \leq L_1 \|u - v\|^2.$$

(H_3) There exist constants $0 < \beta < 1, L_2 > 0$ such that the function g is $X_\beta-$ valued and satisfies for all $t \geq 0, \ \ x, y \in X$

$$||(-A)^\beta g(t,u) - (-A)^\beta g(t,v)||^2 \leq L_2 ||u - v||^2.$$

(H_4) The function $(-A)^{\beta}g$ is continuous in the quadratic mean square:
For all functions x,

$$\lim_{t \to s} E\|(-A)^{\beta}g(t, u(t)) - (-A)^{\beta}g(s, u(s))\|^2 = 0.$$

(H_5) There exists some positive numbers $q_k, k \in N$ such that

$$\|I_k(u) - I_k(v)\| \leq q_k\|u - v\|$$

for all $u, v \in X$ and $\sum_{k=1}^{\infty} q_k < \infty$.

(H_6) The function $\sigma : [0, \infty) \to \mathscr{L}_2^0(Y, X)$ satisfies $\int_0^{\infty} e^{2\gamma s}\|\sigma(s)\|^2_{\mathscr{L}_2^0} ds < \infty$ for some $\gamma > 0$.

(H_7) The measurable mappings $f(.), \sigma(.)$ and $h(.)$ satisfy the following conditions:

$(7a)$ for all $t \in (-\tau, 0], \phi_1, \phi_2 \in C((-\tau, 0], X)$,

$$|f(t, \phi_1) - f(t, \phi_2)|^2 \vee |\sigma(t, \phi_1) - \sigma(t, \phi_2)|^2_{\mathscr{L}_2^0}$$
$$\leq \mathcal{K}(\|\phi_1 - \phi_2\|^2_{\mathscr{L}_2^0}).$$

$(7b)$ for any $H-$ valued processes $u(t), v(t), t \in (-\tau, 0]$,

$$\int_0^t \int_Z |h(s, u_s, z)$$
$$- h(s, v_s, z)|^2 v(dz)ds \vee \left(\int_0^t \int_Z |h(s, u_s, z) - h(s, v_s, z)|^4 v(dz)ds\right)^{\frac{1}{2}}$$
$$\leq \int_0^t \mathcal{K}(|u(s) - v(s)|^2)ds,$$
$$\left(\int_0^t \int_Z |h(s, u_s, z)|^4 v(dz)ds\right)^{\frac{1}{2}} \leq \int_0^t \mathcal{K}(|u(s)|^2)ds,$$

where $\mathcal{K}(.)$ is a concave nondecreasing function from $\mathcal{R}_+$ to $\mathcal{R}_+$ such that $\mathcal{K}(0) = 0, \mathcal{K}(u) > 0$ for $u > 0$ and $\int_{0+} \frac{du}{\mathcal{K}(u)} = +\infty$.

(H_8) For all $t \in (-\tau, 0]$, there exists a constant $L_3 > 0$ such that

$$|f(t, 0)|^2 \vee |\sigma(t, 0)|^2 \vee \int_Z |h(t, 0, z)|^2 v(dz) \leq L_3$$

(H_9) There exists a constant $L_4 > 0$ such that $G : \mathbb{S} \to X$ satisfies

$$\|G(x_1) - G(x_2)\|^2 \le L_4 \|x_1 - x_2\|.$$

(H_{10}) There exists a constant c_1 such that

$$\|G(x)\| \le c_1(1 + \|x\|), \quad \forall x \in X.$$

4. EXISTENCE AND UNIQUENESS RESULTS

Theorem 1. *Assume that* $f(t, 0) = g(t, 0) = I_k(0) = 0, \quad \forall t \ge 0,\ k \in \mathbb{N}$. *The assumptions* $(H_1) - (H_8)$ *hold and that*

$$\begin{aligned} 4\bigg(& L_2\|(-A)^{-\beta}\|^2 + M_{1-\beta}^2 L_2 \Gamma^2(\beta)\lambda^{-2\beta} + M^2 L_1 \lambda^{-2} \\ & + M^2\lambda^{-2} + L_4 M^2 e^{-\lambda t} + M^2 \bigg(\sum_{k=1}^{\infty} q_k \bigg)^2 \bigg) < 1, \end{aligned} \tag{4.1}$$

where $\Gamma(.)$ *is the Gamma function,* $M_{1-\beta}$ *is the corresponding constant in Lemma 2.2. Then the mild solution to equation 1.1 exists uniquely and is exponential decay to zero in mean square, i.e., there exists a pair of positive constants* $a > 0$ *and* $M^* = M^*(\phi, a) > 0$ *such that*

$$E\|u(t)\|^2 \le M^* e^{-at}, \ \ t \ge 0. \tag{4.2}$$

Proof: Denote by $\mathbb{S}$ the space of all stochastic processes $u(t, \omega) : (-\tau, \infty) \times \Omega \to X$ satisfying $u(t) = \phi(t), t \in (-\tau, 0]$ and the conditions (i), (ii) in Definition 2.1 and there exist some constants $a > 0$ and $M^* = M^*(\phi, a) > 0$ such that

$$E\|u(t)\|^2 \le M^* e^{-at}, \ \ t \ge 0. \tag{4.3}$$

Now we check that $\mathbb{S}$ is a banach space endowed with a norm $|x|^2_{\mathbb{S}} = \sup_{t \ge 0} E|u(t)|^2$. Without loss of generality, we may assume that $a < \lambda$. We define the operator Ψ on $\mathbb{S}$ by
$(\Psi x)(t) = \phi(t), t \in (-\tau, 0]$ and

$$\begin{aligned} (\Psi x)(t) &= T(t)[x_0 - G(x) + g(0, \phi)] - g(t, x_t) \\ &\quad - \int_0^t AS(t-s)g(s, x_t)ds + \int_0^t S(t-s)f(s, x_s)ds \\ &\quad + \int_0^t S(t-s)\sigma(s)dW^H(s) + \int_0^t S(t-s)\int_Z h(s, x_s, y)\tilde{N}(ds, dy) \\ &\quad + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k)) := \sum_{i=1}^{6} P_i(t), \ \ t \ge 0. \end{aligned}$$

It is enough to show that the operator Ψ has a unique fixed point in $\mathbb{S}$. To prove this we use the contraction mapping principle.

Step 1: Lets check that $\Psi(\mathbb{S}) \subset \mathbb{S}$. We denote by $M_i^*, i = 1, 2, \ldots$ the finite positive constants depending on ϕ, a. By the assumption (H_1) we have

$$\begin{aligned} E\|P_1(t)\|^2 &\leq M^2 E\|x_0 - G(x) + g(0,\phi)\|^2 e^{-\lambda t} \\ &\leq 3M^2[E\|x_0\|^2 + E\|G(x)\|^2 + E\|g(0,\phi)\|^2]e^{-\lambda t} \\ &\leq 3M^2[E\|x_0\|^2 + 2c_1^2(1 + \|x\|^2)]e^{-\lambda t} \\ &\leq 3M^2[E\|x_0\|^2 + 2c_1^2]e^{-\lambda t} + 6M^2 c_1^2 e^{-\lambda t} E\|x\|^2 \\ &\leq M_1^* e^{-\lambda t} + M_2^* e^{-(\lambda + a)t} \end{aligned} \tag{4.4}$$

To analyze $P_i(t), i = 2, \ldots, 6$, we found that for $u \in \mathbb{S}$ the following evaluation holds

$$\begin{aligned} E\|u_t\|^2 &\leq (M^* e^{-at} + E\|\phi_t\|^2) \\ &\leq (M^* e^{-at} + E\|\phi\|_C^2 e^{-at}) \\ &\leq (M^* + E\|\phi\|_C^2)e^{-at}, \end{aligned}$$

where $\|\phi\|_C = \sup_{-\tau < s \leq 0} \|\phi(s)\| < \infty$. Then by assumption (H_3) we have

$$\begin{aligned} E\|P_2(t)\|^2 &\leq \|(-A)^{-\beta}\|^2 E\|(-A)^\beta g(t, u_t) - (-A)^\beta g(t, 0)\|^2 \\ &\leq L_2\|(-A)^{-\beta}\|^2 E\|u_t\|^2 \\ &\leq L_2\|(-A)^{-\beta}\|^2 (M^* + E\|\phi\|_C^2)e^{-at} \\ &\leq M_2^* e^{-at}. \end{aligned} \tag{4.5}$$

Using Lemma 2.2, Holder's inequality and assumption (H_3) we get that

$$\begin{aligned} E\|P_3(t)\|^2 &= E\left\|\int_0^t AT(t-s)g(s,u_s))ds\right\|^2 \\ &\leq \int_0^t \|(-A)^{1-\beta}T(t-s)\|ds \int_0^t \|(-A)^{1-\beta}T(t-s)\|E\|(-A)^\beta g(s,u_s)\|^2 ds \\ &\leq M_{1-\beta}^2 L_2 \int_0^t (t-s)^{\beta-1} e^{-\lambda(t-s)} ds \int_0^t (t-s)^{\beta-1} e^{-\lambda(t-s)} E\|u_s\|^2 ds \\ &\leq M_{1-\beta}^2 L_2 \frac{\Gamma(\beta)}{\lambda^\beta} \int_0^t (t-s)^{\beta-1} e^{-\lambda(t-s)} (M^* + E\|\phi\|_C^2) e^{-as} ds \\ &\leq M_{1-\beta}^2 L_2 \frac{\Gamma(\beta)}{\lambda^\beta} (M^* + E\|\phi\|_C^2) e^{-at} \int_0^t (t-s)^{\beta-1} e^{(a-\lambda)(t-s)} ds \\ &\leq M_{1-\beta}^2 L_2 \frac{\Gamma^2(\beta)}{\lambda^\beta(\lambda-a)^\beta} (M^* + E\|\phi\|_C^2) e^{-at}. \end{aligned}$$

Hence we retrieve that

$$E\|P_3(t)\|^2 \leq M_3^* e^{-at}. \tag{4.6}$$

we acquire by assumption (H_2) that

$$\begin{aligned} E\|P_4(t)\|^2 &= E\left\| \int_0^t T(t-s)f(s,u_s)ds \right\|^2 \\ &\leq M^2 L_1 \int_0^t e^{-\lambda(t-s)}ds \int_0^t e^{-\lambda(t-s)} E\|u_s\|^2 ds \\ &\leq M^2 L_1 \lambda^{-1} \int_0^t e^{-\lambda(t-s)}(M^* + E\|\phi\|_C^2)e^{-as}ds \\ &\leq M^2 L_1 \lambda^{-1}(\lambda-a)^{-1}(M^* + E\|\phi\|_C^2)e^{-at} \\ &\leq M_4^* e^{-at}. \end{aligned} \tag{4.7}$$

By using Lemma 2.1 we get that

$$E\|p_5(t)\|^2 \leq 2M^2 H t^{2H-1} \int_0^t e^{-2\lambda(t-s)} \|\sigma(s)\|^2_{\mathscr{L}_2^0} ds \tag{4.8}$$

From this inequality we can establish that ,

$$E\|P_5(t)\|^2 \leq 2M^2 H t^{2H-1} e^{-2\lambda' t} \int_0^\infty e^{2\gamma s} \|\sigma(s)\|^2_{\mathscr{L}_2^0} ds, \tag{4.9}$$

where $\lambda' = \lambda \wedge \gamma$. Indeed, if $\lambda < \gamma$, then $\lambda' = \lambda$ and we have

$$\begin{aligned} E\|P_5(t)\|^2 &\leq 2M^2 H t^{2H-1} e^{-2\lambda t} \int_0^t e^{2\lambda s} \|\sigma(s)\|^2_{\mathscr{L}_2^0} ds \\ &\leq 2M^2 H t^{2H-1} e^{-2\lambda' t} \int_0^\infty e^{2\gamma s} \|\sigma(s)\|^2_{\mathscr{L}_2^0} ds. \end{aligned}$$

If $\gamma < \lambda$, then $\lambda' = \gamma$ and we have

$$\begin{aligned} E\|P_5(t)\|^2 &\leq 2M^2 H t^{2H-1} e^{-2\gamma t} \int_0^t e^{-2(\lambda-\gamma)(t-s)} e^{2\gamma s} \|\sigma(s)\|^2_{\mathscr{L}_2^0} ds. \\ &\leq 2M^2 H t^{2H-1} e^{-2\lambda' t} \int_0^\infty e^{2\gamma s} \|\sigma(s)\|^2_{\mathscr{L}_2^0} ds. \end{aligned}$$

We know that $\sup_{t \geq 0}(t^{2H-1} e^{-\lambda' t}) < \infty$, and using the inequality 4.9, gives us

$$E\|P_5(t)\|^2 \leq M_5^* e^{-\lambda' t}. \tag{4.10}$$

Using assumptions (H_7), (H_8) and Burholder's inequality, we found that

$$\begin{aligned}
E\|P_6(t)\|^2 &= E\bigg\| \int_0^t \int_Z T(t-s)h(s,u_s,z)\tilde{N}(ds,dz)\bigg\|^2 \\
&\leq c\bigg\{ \int_0^t \int_Z E\|T(t-s)h(s,u_s,z)\|^2 v(dz)ds \\
&\quad + E\bigg(\int_0^t \int_z \|T(t-s)h(s,u_s,z)\|^4 v(dz)ds\bigg)^{\frac{1}{2}}\bigg\} \\
&\leq c\bigg\{ \int_0^t E\|T(t-s)\|ds \int_0^t \int_z T(t-s)\|h(s,u_s,z)\|^2 v(dz)ds \\
&\quad + E\int_0^t \|T(t-s)\|^2 ds\bigg(\int_0^t \int_z \Big(T(t-s))^2\|h(s,u_s,z)\|^4 v(dz)ds\Big)^{\frac{1}{2}}\bigg\} \\
&\leq c\bigg\{ 2M^2\lambda^{-1}\bigg(\int_0^t \int_z e^{-\lambda(t-s)}\|h(s,u_s,z)-h(s,0,z)\|^2 v(dz)ds \\
&\quad + \int_0^t \int_z e^{-\lambda(t-s)}\|h(s,0,z)\|^2 v(dz)ds\bigg) \\
&\quad + M^2\lambda^{-2}\bigg(M^2\int_0^t e^{-2\lambda(t-s)}\bigg)^{\frac{1}{2}} \int_0^t \|\|u(s)\|^2 ds\bigg\} \\
&\leq c\bigg\{ 2M^2\lambda^{-1}\int_0^t e^{-\lambda(t-s)}(M^* + E\|\phi\|^2)e^{-as}ds \\
&\quad + 2M^2L_3\lambda^{-2} + M^3\lambda^{-2}\int_0^t e^{-\lambda(t-s)}(M^* + E\|\phi\|^2)e^{-as}ds\bigg\}
\end{aligned}$$

After reckoning we found the following.

$$\begin{aligned}
E\|P_6(t)\|^2 &\leq c\bigg\{ 2M^2\lambda^{-1}(\lambda-a)^{-1}(M^* + E\|\phi\|^2)e^{-at} + 2M^2L_3\lambda^{-2}e^{-\lambda_1 t} \\
&\quad + M^3\lambda^{-2}\|(M^* + E\|\phi\|^2)(\lambda-a)^{-1}e^{-at}\bigg\} \leq M_6^* e^{-(a+\lambda_1)t}. \qquad (4.11)
\end{aligned}$$

From (H_5) and Holder's inequality, we get the following estimate for $P_6(t)$

$$\begin{aligned}
E\|P_7(t)\|^2 &= E\bigg\|\sum_{0<t_k<t} T(t-t_k)I_k(u(t_k))\bigg\|^2 \\
&\le E\bigg(\sum_{0<t_k<t}\|T(t-t_k)\|\|I_k(u(t_k))-I_k(0)\|\bigg)^2 \\
&\le M^2E\bigg(\sum_{0<t_k<t} e^{-\lambda(t-t_k)}q_k\|u(t_k)\|\bigg)^2 \\
&\le M^2\sum_{0<t_k<t} q_k \sum_{0<t_k<t} q_k e^{-2\lambda(t-t_k)}E\|u(t_k)\|^2 \\
&\le M^2\sum_{k=1}^{\infty} q_k \sum_{0<t_k<t} q_k e^{-2\lambda(t-t_k)}M^* e^{-at_k} \\
&\le M^2M^*e^{-at}\sum_{k=1}^{\infty} q_k \sum_{0<t_k<t} q_k e^{(a-2\lambda)(t-t_k)} \\
&\le M^2M^*e^{-at}\bigg(\sum_{k=1}^{\infty} q_k\bigg)^2 \le M_6^* e^{-at}. \qquad (4.12)
\end{aligned}$$

Combining 4.4 - 4.7 and 4.10 - 4.12 we found there exist $\overline{M}^* > 0$ and $\overline{a} > 0$ such that

$$E\|(\Psi u)(t)\|^2 \le \overline{M}^* e^{-\overline{a}t}, t \ge 0. \qquad (4.13)$$

It is easy to check that $(\Psi u)(t)$ satisfies the conditions (i), (ii) in definition 3.1. Hence, we can conclude that $\Psi(\mathbb{S}) \subset \mathbb{S}$.

Step 2 We now show that Ψ is a contraction mapping. For any $u, v \in \sim$, we have

$$E\|(\Psi u)(t) - (\Psi v)(t)\|^2 \le 4\sum_{i=1}^{4} Q_i. \qquad (4.14)$$

Since $u(t) = v(t) = \phi(t), t \in (-\tau, 0]$, this implies that

$$E\|u_t - v_t\|^2 \le \sup_{t\ge 0} E\|u(t) - v(t)\|^2. \qquad (4.15)$$

$$\begin{aligned}
Q_1 &= E\|T(t)[G(u_t) - G(v_t)]\|^2 \le \|T(t)\|^2 E\|G(u_t) - G(v_t)\|^2 \qquad (4.16)\\
&\le M^2e^{-\lambda t}E\|G(u_t) - G(v_t)\|^2 \le L_4M^2\sup_{t\ge 0} e^{-\lambda t}E\|G(u_t) - G(v_t)\|. \qquad (4.17)
\end{aligned}$$

Using assumption (H_3), we get the following result.

$$\begin{aligned}
Q_1 &= E\|g(t,u_t) - g(t,v_t)\|^2 \\
&\le L_2\|(-A)^{-\beta}\|^2 E\|u_t - v_t\|^2 \\
&\le L_2\|(-A)^{-\beta}\|^2 \sup_{t\ge 0} E\|u(t) - v(t)\|^2.
\end{aligned}$$

and

$$\begin{aligned} Q_2 &= E\Big\| \int_0^t AT(t-s)[g(s,u_s)-g(s,v_s)]ds\Big\|^2 \\ &\le M_{1-\beta}^2 L_2 \int_0^t (t-s)^{\beta-1}e^{-\lambda(t-s)}ds \int_0^t (t-s)^{\beta-1}e^{-\lambda(t-s)}E\|u_s-v_s\|^2 ds \\ &\le M_{1-\beta}^2 L_2 \frac{\Gamma(\beta)}{\lambda^\beta}\int_0^t (t-s)^{\beta-1}e^{-\lambda(t-s)}E\|u_s-v_s\|^2 ds \\ &\le M_{1-\beta}^2 L_2 \frac{\Gamma^2(\beta)}{\lambda^{2\beta}} \sup_{t\ge 0} E\|u(t)-v(t)\|^2. \end{aligned}$$

By assumption (H_2)

$$\begin{aligned} Q_3 &= E\Big\| \int_0^t T(t-s)[f(s,u_s)-f(s,v_s)]ds\Big\|^2 \\ &\le M^2 L_1 \int_0^t e^{-\lambda(t-s)}ds \int_0^t e^{-\lambda(t-s)}E\|u_s-v_s\|^2 ds \\ &\le M^2 L_1 \lambda^{-1}\int_0^t e^{-\lambda(t-s)}E\|u_s-v_s\|^2 ds \\ &\le M^2 L_1 \lambda^{-2} \sup_{t\ge 0} E\|u(t)-v(t)\|^2. \end{aligned}$$

By assumption (H_7)

$$\begin{aligned} Q_4 &= E\Big\| \int_0^t \int_Z T(t-s)[h(s,u_s,z)-h(s,v_s,z)]\tilde{N}(ds,dz)\Big\|^2 \\ &\le \int_0^t T(t-s)ds \int_0^t \int_z T(t-s)E\|h(s,u_s,z)-h(s,v_s,z)\|^2 \tilde{N}(ds,dz) \\ &\le M^2 \int_0^t e^{-\lambda(t-s)} \int_0^t e^{-\lambda(t-s)}\|u_s-v_s\|^2 ds \\ &\le M^2\lambda^{-1} \int_0^t e^{-\lambda(t-s)}\|u_s-v_s\|^2 ds \le M^2\lambda^{-2}\sup_{t\ge 0} E\|u(t)-v(t)\|^2. \end{aligned}$$

By assumption (H_5)

$$\begin{aligned} Q_5 &= E\Big\| \sum_{0<t_k<t} s(t-t_k)[I_k(u(t_k))-I_k(v(t_k))]\Big\|^2 \\ &\le M^2\Big(\sum_{0<t_k<t} e^{-\lambda(t-t_k)}q_k E\|u(t_k)-v(t_k)\|\Big)^2 \\ &\le M^2\Big(\sum_{k=1}^{\infty} q_k\Big)^2 \sup_{t\ge 0} E\|u(t)-v(t)\|^2. \end{aligned}$$

Thus

$$E\|(\Psi u)(t)-(\Psi v)(t)\|^2 \leq 4\bigg(L_2\|(-A)^{-\beta}\|^2 + M_{1-\beta}^2 L_2\Gamma^2(\beta)\lambda^{-2\beta} + M^2L_1\lambda^{-2} + M^2\lambda^{-2} + L_4M^2e^{-\lambda t} + M^2\bigg(\sum_{k=1}^{\infty} q_k\bigg)^2\bigg)\sup_{t\geq 0} E\|u(t)-v(t)\|^2.$$

By the condition (4.1), we claim that Ψ is contractive. So, applying the Banach fixed point principle, the proof is complete. ■

Theorem 2. *(Infinite Delays), Under the conditions of Theorem 3.1, the mild solution to (1.1) exists uniquely and converges to zero in mean square, i.e.,*

$$\lim_{t\to\infty} E\|u(t)\|^2 = 0. \tag{4.18}$$

Proof: Denote by $\mathbb{S}'$ the space of all stochastic processes $x(t,\omega):(-\infty,\infty)\times\Omega\to X$ satisfying $u(t)=\pi(t), t\in(-\infty,0]$ and the conditions (i), (ii) in Definition 3.1 and

$$\lim_{t\to\infty} E\|u(t)\|^2 = 0. \tag{4.19}$$

We define the operator ψ on $\mathbb{S}'$ by $(\psi u)(t)=\pi(t), t\in(-\infty,0]$ and

$$\begin{aligned}(\psi u)(t) &= T(t)[u_0 - G(u) + g(0,\phi)] - g(t,u_t) - \int_0^t AT(t-s)g(s,u_s)ds \\ &\quad + \int_0^t T(t-s)f(s,u_s)ds + \int_0^t T(t-s)\sigma(s)dW^H(s) \\ &\quad + \int_0^t T(t-s)\int_Z h(s,u_s,y)\tilde{N}(ds,dy) \\ &\quad + \sum_{0<t_k<t} T(t-t_k)I_k(u(t_k)) := \sum_{i=1}^{6} P_i(t), t\geq 0. \end{aligned} \tag{4.20}$$

Since $(\psi u)(t) = (\phi u)(t)$ on, $[0,\infty)$, this implies that ψ is contractive. Hence it remains to check $\psi(\mathbb{S}')\subset\mathbb{S}'$. In order to obtain this claim, we need to show that $\lim_{t\to\infty} E\|(\psi u)(t)\|^2 = 0$ for all $u\in\mathbb{S}'$.
By the definition of $\mathbb{S}'$, assumption (H_6) and the fact $t-r(t)\to\infty, t\to\infty$, we get

$$\lim_{t\to\infty} E\|P_1(t)\|^2 = \lim_{t\to\infty} E\|P_2(t)\|^2 = \lim_{t\to\infty} E\|P_5(t)\|^2 = 0. \tag{4.21}$$

We further have

$$\begin{aligned}E\|P_3(t)\|^2 &= E\bigg\|\int_0^t AT(t-s)g(s,u_s))ds\bigg\|^2\\ &\le M_{1-\beta}^2 L_2\int_0^t (t-s)^{\beta-1}e^{-\lambda(t-s)}ds\int_0^t (t-s)^{\beta-1}e^{-\lambda(t-s)}E\|u_s\|^2 ds\\ &\le M_{1-\beta}^2 L_2\Gamma(\beta)\lambda^{\beta}\int_0^t (t-s)^{\beta-1}e^{-\lambda(t-s)}E\|u_s\|^2 ds\end{aligned}$$

For any $x \in \mathbb{S}'$ and $\epsilon > 0$ it follows from (3.12) that there exists $s_1 > 0$ such that $E\|u(s-r(s)\|^2 < \epsilon$ for all $s \ge s_1$. Thus we obtain

$$E\|P_3(t)\|^2 \le M_{1-\beta}^2 L_2\Gamma(\beta)\lambda^{\beta}\int_0^t (t-s)^{\beta-1}e^{-\lambda(t-s)}E\|u_s)\|^2 ds + M_{1-\beta}^2 L_2\Gamma^2(\beta)\lambda^{-2\beta}\epsilon,$$

which proves that

$$E\|P_3(t)\|^2 \le M_{1-\beta}^2 L_2\Gamma^2(\beta)\lambda^{-2\beta}\epsilon, \forall\epsilon > 0,$$

and hence, $\lim_{t\to\infty} E\|P_3(t)\|^2 = 0$. In the same way we also have $\lim_{t\to\infty} E\|P_4(t)\|^2 = 0$. Furthermore, since

$$\begin{aligned}E\|P_6(t)\|^2 &= E\bigg\|\int_0^t\int_Z T(t-s)[h(s,u_s,z)]\tilde{N}(ds,dz)\bigg\|^2\\ &\le \int_0^t T(t-s)ds\int_0^t\int_z T(t-s)E\|h(s,u_s,z)\|^2\tilde{N}(ds,dz)\\ &\le M^2\int_0^t e^{-\lambda(t-s)}\int_0^t e^{-\lambda(t-s)}\|u_s\|^2 ds\\ &\le M^2\lambda^{-1}\int_0^t e^{-\lambda(t-s)}\|u_s\|^2 ds.\\ &\le M^2\lambda^{-1}\int_0^t e^{-\lambda(t-s)}\|u_s\|^2 ds + M^2\lambda^{-2}\epsilon.\end{aligned}$$

proves that,

$$E\|P_6(t)\|^2 \le M^2\lambda^{-2}\epsilon.$$

Then we have $\lim_{t\to\infty} E\|P_7(t)\|^2 = 0$. And,

$$\begin{aligned}E\|P_7(t)\|^2 &= E\bigg\|\sum_{0<t_k<t} T(t-t_k)I_k(u(t_k))\bigg\|^2\\ &\le M^2\sum_{0<t_k<t} q_k\sum_{0<t_k<t} q_k e^{-2\lambda(t-t_k)}E\|u(t_k)\|^2\\ &\le M^2\sum_{k=1}^{\infty} q_k\sum_{0<t_k<t} q_k e^{-2\lambda(t-t_k)}E\|u(t_k)\|^2 + M^2(\sum_{k=1}^{\infty} q_k)^2\epsilon,\end{aligned}$$

we can get that $\lim_{t\to\infty} E\|P_7(t)\|^2 = 0$. Once again, by applying the Banach fixed point principle we complete the proof of the theorem. ■

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