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On Convergence, Stability and Data Dependence of Four-Step Implicit Fixed Point Iterative Scheme for Contractive-Like Operators in Convex Metric Spaces

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Abstract

In the present paper, we introduced a Four-step implicit fixed point iterative scheme (FIFPIS) and established its convergence, stability and data dependence results for contractive-like operators in convex metric space. Here it has been exposed that the rate of convergence of the Four-step implicit fixed point iterative scheme (FIFPIS) is better than that of implicit Mann iterative scheme (IMIS), explicit Mann iterative scheme (EMIS), implicit Ishikawa iterative scheme (IIIS), explicit Ishikawa iterative schemes (EIIS), explicit Noor iterative schemes (ENIS), Four-step explicit iterative scheme (FEIS) and Chugh *et al.* implicit iterative scheme (CIIS). We also set a numerical example to support the analytic proof.

Keywords: four-step implicit fixed point iterative scheme; rate of convergence; stability; data dependence; contractive-like operators; convex metric space; hyperbolic space

2010 Mathematics Subject Classification: 47H09; 47H10; 47H15; 47H17

1. INTRODUCTION AND PRELIMINARY NOTES

Fixed point iterative scheme plays an important role in computation analysis by using computer programming techniques. The use of fixed point iterative scheme minimized both of the calculating time and cost. In modern day implicit fixed point iterative scheme is more acceptable than explicit fixed point iterative scheme, because

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of the rate of convergence of implicit fixed point iterative scheme is better than that of the explicit fixed point iterative scheme in convex metric spaces (see for instance, [1-17] and their references). Inspiring by the above mentioned works here we introduced a more general Four-step implicit fixed point iterative scheme (FIFPIS) whose rate of convergence is better than that of Chugh *et al.* implicit iterative scheme [1] and its analogues implicit and explicit fixed point iterative schemes. Stability and data dependence are another two important things for a fixed point iterative scheme, (see for instance, [1, 16, 18, 19] and their references). Stability of a fixed point iterative scheme shows that it is stable at fixed point of given operator (see for instance, [1, 3, 16, 20-28] and their references) and data dependence of a fixed point iterative scheme help us to find the unknown fixed point of a given operator without any calculation hazard (see for instance, [1, 16, 18, 19, 29, 30, 31] and their references). From this context here we also study the stability and the data dependence of our Four-step fixed point iterative scheme along with its rate of convergence.

Throughout this paper $\mathbb{N}$ denotes the set of natural number.

Let C be a nonempty closed convex subset of a convex metric space M and $T: C \to C$ be a given operator. Then for $x_0 \in C$ the *Four-step implicit fixed point iterative scheme* (FIFPIS) is defined as follows

$$\left.\begin{aligned} x_n &= I(x_{n-1}, Ty_n, \alpha_n) \\ y_n &= I(z_n, Tz_n, \beta_n) \\ z_n &= I(u_n, Tu_n, \gamma_n) \\ u_n &= I(x_n, Tx_n, \delta_n);\ \forall\, n \in \mathbb{N} \end{aligned}\right\} \tag{1.1}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ are real sequences in $[0, 1]$.

Equivalently, in linear space the iterative scheme (1.1) can be written as

$$\left.\begin{aligned} x_n &= \alpha_n x_{n-1} + (1-\alpha_n)Ty_n, \\ y_n &= \beta_n z_n + (1-\beta_n)Tz_n, \\ z_n &= \gamma_n u_n + (1-\gamma_n)Tu_n, \\ u_n &= \delta_n x_n + (1-\delta_n)Tx_n;\ \forall\, n \in \mathbb{N}, \end{aligned}\right\} \tag{1.2}$$

If we put $\delta_n = 1$ in (1.2), then we get the following *Chugh et al. implicit iterative scheme (*CIIS*)* (Noor type implicit iterative scheme) [1]:

$$\left.\begin{aligned} x_n &= \alpha_n x_{n-1} + (1-\alpha_n)Ty_n, \\ y_n &= \beta_n z_n + (1-\beta_n)Tz_n, \\ z_n &= \gamma_n x_n + (1-\gamma_n)Tx_n\ ;\ \forall\, n \in \mathbb{N} \end{aligned}\right\} \tag{1.3}$$

If we put $\delta_n = \gamma_n = 1$ in (1.2), then we get the following *Implicit Ishikawa iterative scheme* (IIIS) [2]:

$$\left.\begin{aligned} x_n &= \alpha_n x_{n-1} + (1-\alpha_n)Ty_n, \\ y_n &= \beta_n x_n + (1-\beta_n)Tx_n;\ \forall\, n \in \mathbb{N} \end{aligned}\right\} \tag{1.4}$$

If we put $\delta_n = \gamma_n = \beta_n = 1$ in (1.2), then we get the following *Implicit Mann iterative scheme (*IMIS*)* [8, 9, 15]:

$$x_n = I(x_{n-1}, Tx_n, \alpha_n) = \alpha_n x_{n-1} + (1 - \alpha_n)Tx_n;\ \forall\, n \in \mathbb{N} \tag{1.5}$$

Therefore, the Four-step implicit fixed point iterative scheme (FIFPIS) defined by (1.1) and (1.2) is a general implicit iterative scheme among the implicit iterative schemes defined by (1.1)-(1.5).

Also, the *Four-step explicit iterative scheme* (FEIS) [19] is defined as follows:

$$\left.\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n Ty_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n Tz_n \\ z_n &= (1 - \gamma_n)x_n + \gamma_n Tu_n, \\ u_n &= (1 - \delta_n)x_n + \delta_n Tx_n;\ \forall\, n \in \mathbb{N} \end{aligned}\right\} \tag{1.6}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ are convergent real sequences in $[0, 1]$, such that

$\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \beta_n = 0$, $\lim_{n\to\infty} \gamma_n = 0$, $\lim_{n\to\infty} \delta_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

If we put $\delta_n = 1$ in (1.6), then we get the following *Explicit Noor iterative scheme* (ENIS) or *simply Noor iterative scheme* [2, 32]:

$$\left.\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n Ty_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n Tz_n, \\ z_n &= (1 - \gamma_n)x_n + \gamma_n Tx_n;\ \forall\, n \in \mathbb{N} \end{aligned}\right\} \tag{1.7}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are convergent real sequences in $[0, 1]$, such that $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \beta_n = 0$, $\lim_{n\to\infty} \gamma_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

If we put $\delta_n = \gamma_n = 1$ in (1.6), then we get the following *Explicit Ishikawa iterative scheme* (EIIS) *or simply Ishikawa iterative scheme* [1, 2, 31, 33]:

$$\left.\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n Ty_n,, \\ y_n &= (1 - \beta_n)x_n + \beta_n Tx_n;\ \forall\, n \in \mathbb{N} \end{aligned}\right\} \tag{1.8}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are convergent real sequences in $[0, 1]$, such that $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

If we put $\delta_n = \gamma_n = \beta_n = 1$ in (1.6), then we get the following *Explicit Mann iterative scheme* (EMIS) *or simply Mann iterative scheme* [1, 2, 34]:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n;\ \forall\, n \in \mathbb{N} \tag{1.9}$$

where $\{\alpha_n\}$ is a convergent real sequence in $[0, 1]$, such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Therefore, the Four-step explicit iterative scheme (FEIS) defined by (1.6) is a general explicit iterative scheme among the explicit iterative schemes defined by (1.6)-(1.9).

In 1972 Zamfirescu [35] introduced the following general contractive-like operator:

DEFINITION 1.1 [35] The operator $T: C \rightarrow C$ is called a *Zamfirescu operator* if it satisfies the condition **Z** (Zamfirescu condition) i.e., if and only if there exist the real numbers α, β, γ satisfying $0 < \alpha < 1, 0 < \beta < \frac{1}{2}, 0 < \gamma < \frac{1}{2}$ such that for each pair $x, y \in C$, at least one of the following three conditions is true:

$$\left.\begin{array}{ll}(z_1) & d(Tx, Ty) \leq \alpha d(x, y) \\ (z_2) & d(Tx, Ty) \leq \beta[d(x, Tx) + d(y, Ty)] \\ (z_3) & d(Tx, Ty) \leq \gamma[d(x, Ty) + d(y, Tx)]\end{array}\right\} \tag{1.10}$$

In 1977 Rhoades [36] proved that $(z_1), (z_2)$ and (z_3) are independent conditions.

If $x, y \in C$ and T is a Zamfirescu operator, then by using the conditions $(z_1), (z_2)$ and (z_3), we obtain

$$d(Tx, Ty) \leq 2\sigma d(x, Tx) + \sigma d(x, y) \tag{1.11}$$

where, $\sigma = max\left\{\alpha, \frac{\beta}{(1-\beta)}, \frac{\gamma}{(1-\gamma)}\right\}$ and $0 \leq \sigma < 1$.

Formula (1.11) was obtained by Berinde [37].

Osilikeb and Udomene [25] introduced a more general definition of a *quasi-contractive operator*, they considered the operator for which there exists $L \geq 0$ and $q \in (0, 1)$ such that

$$d(Tx, Ty) \leq L(x, Tx) + qd(x, y), \forall\, x, y \in C \tag{1.12}$$

In 2003, Imoru and Olatinwo [21] considered the following more general type of contractive operator but they are failed to name it. Later in 2008, Soltuz, & Grosan [30] used it as contractive-like operators. Recently, Chugh *et al.* [1] has been studied the convergence of three-step implicit iterative scheme for that operator.

DEFINITION 1.2 [1, 16, 21, 30] The operator T is called *contractive-like operator* if there exist a constant $\lambda \in (0, 1)$ and a strictly increasing and continuous function $\phi: [0, \infty) \rightarrow [0, \infty)$ with $\phi(0) = 0$ such that for each $x, y \in C$

$$d(Tx, Ty) \leq \phi\big(d(x, Tx)\big) + \lambda d(x, y) \tag{1.13}$$

DEFINITION 1.3 [1, 16, 38] A map $I: M^2 \times [0, 1] \rightarrow M$ is said to be in convex structure on M if for all $x, y, u \in M$ and $\mu \in [0, 1]$ the following inequality holds:

$$d\big(u, I(x, y, \mu)\big) \leq \mu d(u, x) + (1 - \mu)d(u, y) \tag{1.14}$$

A metric space (M, d) together with a convex structure I is known as *convex metric* *space* and it is denoted by (M, d, μ). A nonempty subset C of a convex metric space is convex if $I(x, y, \mu) \in C$ for all $x, y \in C$ and $\mu \in [0, 1]$. All normed spaces and their subsets are convex metric space but the converse is not always true (see for instance, [38]).

DEFINITION 1.4 [1, 16, 39] A convex metric space (M, d, μ) is said to be a hyperbolic space if it metric d and convexity mapping $I: M^2 \times [0, 1] \to M$ satisfy the following axioms:

$$(I_1)\ \ d\big(z, I(x, y, \mu)\big) \le (1-\mu)d(z, x) + \mu d(z, y)$$

$$(I_2)\ \ d\big(I(x, y, \mu_1), I(x, y, \mu_2)\big) = |\mu_1 - \mu_2| d(x, y)$$

$$(I_3)\ \ I(x, y, \mu) = I(x, y, 1-\mu)$$

$$(I_4)\ \ d\big(I(x, z, \mu), I(y, v, \mu)\big) \le (1-\mu)d(x, y) + \mu d(z, v)$$

for all $x, y, z, v \in M$ and $\mu, \mu_1, \mu_2 \in [0, 1]$.

T-stability of various implicit and explicit iterative schemes has broadly been explained by several researches (see for instance, [1, 16, 12, 20-23, 28, 30]) owing to its usefulness in computational mathematics, particularly in forming computer programming. The idea of T-stability in convex metric space was initially studied by Olatinwo [22]. Recently, the T-stability of three-step implicit iterative scheme in convex metric space has been studied by Chugh *et al.* [1]. The definition of T-stability in convex metric space is as follows.

DEFINITION 1.5 [1, 16, 22] Let (M, d, μ) be a convex metric space and T be a self-map on M. Let $\{x_n\}_{n=0}^{\infty} \subset M$ be a sequence generated by an iterative scheme involving T, which is defined by

$$x_{n+1} = f_{T,\alpha_n}^{x_n}, n = 0, 1, 2, \cdots \tag{1.15}$$

where $x_0 \in M$ is an initial approximation and $f_{T,\alpha_n}^{x_n}$ is some function having convex structure, such that $\alpha_n \in [0, 1]$ and the sequence $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point p of T. Again let $\{y_n\}_{n=0}^{\infty} \subset M$ be an arbitrary sequence and set $\xi_n = d\big(y_{n+1}, f_{T,\alpha_n}^{y_n}\big)$. Here the iterative scheme defined by (1.15) is said to be T-*stable* with respect to T if and only if $\lim_{n\to\infty} \xi_n = 0$, implies $\lim_{n\to\infty} y_n = p$.

LEMMA 1.6 [1, 16, 37, 12] If δ is a real number such that $0 \le \delta < 1$ and $\{\varepsilon_n\}_{n=0}^{\infty}$ is a sequence of positive numbers such that $\lim_{n\to\infty} \varepsilon_n = 0$, then for any sequence of positive numbers $\{v_n\}_{n=0}^{\infty}$ satisfying

$$v_{n+1} \le \delta v_n + \varepsilon_n,\ n = 0,1,2,\cdots \tag{1.16}$$

we have $\lim_{n\to\infty} v_n = 0$.

DEFINITION 1.7 [1, 16, 29] Let T and S be two operators on M. We say S is approximate operator of T if for all $x \in M$ and for a fixed $\varepsilon > 0$, we have $d(Tx, Sx) \leq \varepsilon$.

LEMMA 1.8 [1, 12, 18, 29, 37] Let $\{s_n\}_{n=0}^{\infty}$ be a nonnegative sequence for which one supposes there exists $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$ one has satisfied the following inequality:

$$s_{n+1} \leq (1 - r_n)s_n + r_n t_n \tag{1.17}$$

where $r_n \in (0, 1) \forall\, n \in \mathbb{N}, \sum_{n=1}^{\infty} r_n = \infty$ and $t_n \geq 0 \;\forall\, n \in \mathbb{N}$. Then,

$$0 \leq \lim_{n\to\infty} \sup s_n \leq \lim_{n\to\infty} \sup t_n. \tag{1.18}$$

The main objective of this paper is to establish a general implicit fixed point iterative scheme whose rate of convergence is better than that of analogous fixed point iterative scheme. From this point of view here, we considered the Four-step implicit fixed point iterative scheme (FIFPIS) and established its convergence and T-stability results for contractive-like operators in convex metric space. Moreover, the data dependence of same iterative scheme is constructed for contractive-like operators in hyperbolic spaces.

2. CONVERGENCE OF FOUR-STEP IMPLICIT FIXED POINT ITERATIVE SCHEME

In this section we stated and proved a convergence theorem of Four-step implicit fixed point iterative scheme (FIFPIS) defined by (1.1) in convex metric space.

THEOREM 2.1 *Let C be a nonempty closed convex subset of a convex metric space M and $T: C \to C$ be a contractive-like operator satisfying (1.13) with $F(T) \neq \emptyset$. Then, for $x_0 \in C$, the FIFPIS $\{x_n\}_{n=0}^{\infty}$ defined by (1.1) and (1.2) with $\sum(1 - \alpha_n) = \infty$, converges to the fixed point p of T.*

PROOF. Since $p \in F(T)$, then from (1.1) and (1.) we have

$$\begin{aligned} d(x_n, p) &= d(I(x_{n-1}, Ty_n, \alpha_n), p) = d\big((\alpha_n x_{n-1} + (1 - \alpha_n)Ty_n), p\big) \\ &\leq \alpha_n d(x_{n-1}, p) + (1 - \alpha_n) d(Ty_n, p) \\ &\leq \alpha_n d(x_{n-1}, p) + (1 - \alpha_n)\big[\phi\big(d(p, Tp)\big) + \lambda d(p, y_n)\big] \\ &= \alpha_n d(x_{n-1}, p) + (1 - \alpha_n)\lambda d(y_n, p) \end{aligned} \tag{2.1}$$

But, we have the following inequalities:

$$d(y_n,p) = d(I(z_n,Tz_n,\beta_n),p) = d\big((\beta_n z_n + (1-\beta_n)Tz_n),p\big)$$
$$\leq \beta_n d(z_n,p) + (1-\beta_n)d(Tz_n,p)$$
$$\leq \beta_n d(z_n,p) + (1-\beta_n)\lambda d(z_n,p) = [\beta_n + (1-\beta_n)\lambda]d(z_n,p) \qquad (2.2)$$

$$d(z_n,p) = d(I(u_n,Tu_n,\gamma_n),p) = d\big((\gamma_n u_n + (1-\gamma_n)Tu_n),p\big)$$
$$\leq \gamma_n d(u_n,p) + (1-\gamma_n)d(Tu_n,p)$$
$$\leq \gamma_n d(u_n,p) + (1-\gamma_n)\lambda d(u_n,p) = [\gamma_n + (1-\gamma_n)\lambda]d(u_n,p) \qquad (2.3)$$

and

$$d(u_n,p) = d(I(x_n,Tx_n,\delta_n),p) = d\big((\delta_n x_n + (1-\delta_n)Tx_n),p\big)$$
$$\leq \delta_n d(x_n,p) + (1-\delta_n)d(Tx_n,p)$$
$$\leq \delta_n d(x_n,p) + (1-\delta_n)\lambda d(x_n,p) = [\delta_n + (1-\delta_n)\lambda]d(x_n,p) \qquad (2.4)$$

Now, combining the inequalities (2.1), (2.2), (2.3) and (2.4), we obtain

$$d(x_n,p) \leq \alpha_n d(x_{n-1},p) + (1-\alpha_n)\lambda[\beta_n + (1-\beta_n)\lambda][\gamma_n + (1-\gamma_n)\lambda]$$
$$[\delta_n + (1-\delta_n)\lambda]d(x_n,p)$$
$$\Longrightarrow d(x_n,p) \leq \frac{\alpha_n}{1-(1-\alpha_n)\lambda[\beta_n+(1-\beta_n)\lambda][\gamma_n+(1-\gamma_n)\lambda]\,[\delta_n+(1-\delta_n)\lambda]}d(x_{n-1},p) \qquad (2.5)$$

Let

$$\frac{r_n}{s_n} = \frac{\alpha_n}{1-(1-\alpha_n)\lambda[\beta_n+(1-\beta_n)\lambda][\gamma_n+(1-\gamma_n)\lambda]\,[\delta_n+(1-\delta_n)\lambda]}$$

$$\Longrightarrow 1-\frac{r_n}{s_n} = \frac{1-(1-\alpha_n)\lambda[\beta_n+(1-\beta_n)\lambda][\gamma_n+(1-\gamma_n)\lambda]\,[\delta_n+(1-\delta_n)\lambda]-\alpha_n}{1-(1-\alpha_n)\lambda[\beta_n+(1-\beta_n)\lambda][\gamma_n+(1-\gamma_n)\lambda]\,[\delta_n+(1-\delta_n)\lambda]}$$

$$\Longrightarrow \frac{r_n}{s_n} \leq (1-\alpha_n)\lambda[\beta_n+(1-\beta_n)\lambda][\gamma_n+(1-\gamma_n)\lambda]\,[\delta_n+(1-\delta_n)\lambda]+\alpha_n$$

$$\leq (1-\alpha_n)\lambda + \alpha_n = 1-(1-\alpha_n)(1-\lambda) \qquad (2.6)$$

Now, using (2.6) in (2.5), we get

$$d(x_n,p) \leq [1-(1-\alpha_n)(1-\lambda)]d(x_{n-1},p)$$
$$\cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots$$
$$\leq \prod_{i=1}^{n}[1-(1-\alpha_i)(1-\lambda)]d(x_0,p) \qquad (2.7)$$

Taking limit as $n \to \infty$ on both sides of (2.7), we obtain $\lim_{n\to\infty} d(x_n,p) = 0$.

This implies that the sequence $\{x_n\}$ defined by the FIFPIS (1.1) converges to the fixed point p of T.

REMARK 2.2 Since, the contractive condition (1.13) is the most general contractive condition among the contractive conditions (1.10)-(1.13), then the convergence results of FIFPIS defined by (1.1) for the contractive conditions (1.10)-(1.11) can be easily accessible as special cases.

REMARK 2.3 Since, CIIS defined by (1.3), IIIS defined by (1.4) and IMIS defined by (1.5) are obtainable as special cases of defined by FIFPIS defined by (1.1), so the Theorem 2.1 will remain true for defined by CIIS defined by (1.3), IIIS defined by (1.4) and IIIS defined by (1.5).

3. RATE OF CONVERGENCE OF FOUR-STEP IMPLICIT FIXED POINT ITERATIVE SCHEME

To compare the rate of convergence of two fixed point iterative schemes $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ which are converge to a certain fixed point p of a given operator T, Rhoades [40] considered that $\{x_n\}_{n=0}^{\infty}$ is better than $\{y_n\}_{n=0}^{\infty}$ if

$$\|x_n - p\| \leq \|y_n - p\|, \ \forall\, n \in \mathbb{N}. \tag{3.1}$$

But after Rhoades, in 2004 Berinde [37] established the following technique to compare the rate of convergence of two fixed point iterative schemes:

DEFINITION 3.1 [1, 2, 16, 37, 41] Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$be two sequences of real numbers that converge to a and b, respectively, and assume that there exists a limit

$$l = \lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|} \tag{3.2}$$

(i) If $l = 0$, then it can be said that $\{a_n\}_{n=0}^{\infty}$ converges faster to a than $\{b_n\}_{n=0}^{\infty}$ to b.

(ii) If $0 < l < \infty$, then it can be said that $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ have the same rate of convergence.

In the case (i), the notation $a_n - a = \boldsymbol{o}(b_n - b)$ will be used and if $l = \infty$, then the sequence $\{b_n\}_{n=0}^{\infty}$ converges faster than $\{a_n\}_{n=0}^{\infty}$, that is $b_n - b = \boldsymbol{o}(a_n - a)$.

Suppose that, for two fixed point iterative schemes $\{x_n\}_{n=0}^{\infty}$ and$\{y_n\}_{n=0}^{\infty}$, both converging to the same fixed point p of a given operator T, the error estimates

$$\|x_n - p\| \leq a_n, \ n = 0, 1, 2, \ldots, \tag{3.3}$$

and

$$\|y_n - p\| \leq \mathrm{b}_n, \ n = 0, 1, 2, \ldots, \tag{3.4}$$

are available, where $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are sequences of positive numbers (converging to zero). Then, in view of Definition 3.1 Berinde [37] adopted the following concept.

DEFINITION 3.2 [1, 2, 16, 37, 40] Let $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ be two fixed point iterative schemes that converge to the same fixed point p and satisfy (3.3) and (3.4), respectively. If $\{a_n\}_{n=0}^{\infty}$ converges faster than $\{b_n\}_{n=0}^{\infty}$, then it can be said that $\{x_n\}_{n=0}^{\infty}$ converges faster than $\{y_n\}_{n=0}^{\infty}$ to p.

In recent years, Definition 3.2 has been used as a standard tool to compare the rate of convergence of two fixed point iterative schemes.

Now we state and prove a theorem which shows that the rate of convergence of the proposed FIFPIS defined by (1.1) is better than that of IMIS defined by (1.5), EMIS defined by (1.9), IIIS defined by (1.4), EIIS defined by (1.8), ENIS defined by (1.7), CIIS defined by (1.3) and FEIS defined by (1.6).

THEOREM 3.3 *Let C be a nonempty closed convex subset of a convex metric space M and $T: C \to C$ be a quasi-contractive operator satisfying (1.) with $F(T) \neq \emptyset$. Then, for $x_0 \in C$, the FIFPIS $\{x_n\}$ defined by (1.1) and (1.2) with $\sum(1-\alpha_n) = \infty$, converges faster than the following iterative schemes:*

(a) IMIS defined by (1.5), (b) EMIS defined by (1.9), (c) IIIS defined by (1.4)

(d) EIIS defined by (1.8), (e) CIIS defined by (1.3), (f) ENIS defined by (1.7)

(g) FXIS defined by (1.6).

PROOF. For $p \in F(T)$, from the FIFPIS $\{x_n\}$ defined by (1.1) and (1.2) we have

$$\begin{aligned} d(x_n, p) &= d(I(x_{n-1}, Ty_n, \alpha_n), p) = d\big((\alpha_n x_{n-1} + (1-\alpha_n)Ty_n), p\big) \\ &\le \alpha_n d(x_{n-1}, p) + (1-\alpha_n) d(Ty_n, p) \\ &= \alpha_n d(x_{n-1}, p) + (1-\alpha_n) d(Tp, Ty_n) \\ &\le \alpha_n d(x_{n-1}, p) + (1-\alpha_n)\big[\phi\big(d(p, Tp)\big) + \lambda d(p, y_n)\big] \\ &= \alpha_n d(x_{n-1}, p) + (1-\alpha_n)\lambda d(y_n, p) \qquad (3.1) \end{aligned}$$

But, we have the following inequalities:

$$\begin{aligned} d(y_n, p) &= d(I(z_n, Tz_n, \beta_n), p) = d\big((\beta_n z_n + (1-\beta_n)Tz_n), p\big) \\ &\le \beta_n d(z_n, p) + (1-\beta_n) d(Tz_n, p) \le \beta_n d(z_n, p) + (1-\beta_n)\lambda d(z_n, p) \\ &= [\beta_n + (1-\beta_n)\lambda] d(z_n, p) \qquad (3.2) \end{aligned}$$

$$\begin{aligned} d(z_n, p) &= d(I(u_n, Tu_n, \gamma_n), p) = d\big((\gamma_n u_n + (1-\gamma_n)Tu_n), p\big) \\ &\le \gamma_n d(u_n, p) + (1-\gamma_n) d(Tu_n, p) \\ &\le \gamma_n d(u_n, p) + (1-\gamma_n)\lambda d(u_n, p) = [\gamma_n + (1-\gamma_n)\lambda] d(u_n, p) \qquad (3.3) \end{aligned}$$

and

$$\begin{aligned} d(u_n, p) &= d(I(x_n, Tx_n, \delta_n), p) = d\big((\delta_n x_n + (1-\delta_n)Tx_n), p\big) \\ &\le \delta_n d(x_n, p) + (1-\delta_n)\lambda d(x_n, p) = [\delta_n + (1-\delta_n)\lambda] d(x_n, p) \qquad (3.4) \end{aligned}$$

Now, combining the inequalities (3.1), (3.2), (3.3) and (3.4), we obtain

$$d(x_n,p) \leq \alpha_n d(x_{n-1},p) + (1-\alpha_n)\lambda \begin{bmatrix} \beta_n + \\ (1-\beta_n)\lambda \end{bmatrix} \begin{bmatrix} \gamma_n + \\ (1-\gamma_n)\lambda \end{bmatrix} \begin{bmatrix} \delta_n + \\ (1-\delta_n)\lambda \end{bmatrix} d(x_n,p)$$

$$\Rightarrow d(x_n,p) \leq \frac{\alpha_n}{1-(1-\alpha_n)\lambda \begin{bmatrix} \beta_n + \\ (1-\beta_n)\lambda \end{bmatrix} \begin{bmatrix} \gamma_n + \\ (1-\gamma_n)\lambda \end{bmatrix} \begin{bmatrix} \delta_n + \\ (1-\delta_n)\lambda \end{bmatrix}} d(x_{n-1},p)$$

$$\leq \begin{bmatrix} \alpha_n + (1-\alpha_n)\lambda[\beta_n + (1-\beta_n)\lambda][\gamma_n + (1-\gamma_n)\lambda] \\ .[\delta_n + (1-\delta_n)\lambda] \end{bmatrix} d(x_{n-1},p) \quad (3.5)$$

This implies that,

$$d(x_n,p) \leq \prod_{i=1}^{n} \begin{bmatrix} \alpha_i + (1-\alpha_i)\lambda[\beta_i + (1-\beta_i)\lambda] \\ .[\gamma_i + (1-\gamma_i)\lambda]\ [\delta_i + (1-\delta_i)\lambda] \end{bmatrix} d(x_0,p) \quad (3.6)$$

Now, for the implicit Mann iterative scheme (1.5), we can write

$$\begin{aligned} d(x_n,p) &= d(I(x_{n-1},Tx_n,\alpha_n),p) = d(\alpha_n x_{n-1} + (1-\alpha_n)Tx_n,\ p) \\ &\leq \alpha_n d(x_{n-1},p) + (1-\alpha_n)d(Tx_n,\ p) \\ &\leq \alpha_n d(x_{n-1},p) + (1-\alpha_n)\lambda d(x_n,\ p) \end{aligned}$$

which yields that, $d(x_n,p) \leq \frac{\alpha_n}{1-(1-\alpha_n)\lambda} d(x_{n-1},p)$ (3.7)

Now, if we put $\frac{R_n}{S_n} = \frac{\alpha_n}{1-(1-\alpha_n)\lambda}$, then we obtain

$$1 - \frac{R_n}{S_n} = 1 - \frac{\alpha_n}{1-(1-\alpha_n)\lambda} = \frac{1-[(1-\alpha_n)\lambda+\alpha_n]}{1-(1-\alpha_n)\lambda} \geq 1 - [(1-\alpha_n)\lambda + \alpha_n]$$

This implies that, $\frac{R_n}{S_n} \leq [(1-\alpha_n)\lambda + \alpha_n] = 1-(1-\alpha_n)(1-\lambda)$ (3.8)

Combining (3.7) and (3.8), we get

$$d(x_n,p) \leq [1-(1-\alpha_n)(1-\lambda)]d(x_{n-1},p)$$

This implies that,

$$d(x_n,p) \leq \prod_{i=1}^{n}[1-(1-\alpha_i)(1-\lambda)]d(x_0,p) \quad (3.9)$$

Now if we put,

$$p_n = \prod_{i=1}^{n}\big[\alpha_i + (1-\alpha_i)\lambda[\beta_i + (1-\beta_i)\lambda][\gamma_i + (1-\gamma_i)\lambda]\ [\delta_i + (1-\delta_i)\lambda]\big]$$

and

$$q_n = \prod_{i=1}^{n}[1-(1-\alpha_i)(1-\lambda)]$$

then we obtain, $\lim_{n\to\infty} \frac{p_n}{q_n} = 0$.

Using the definitions (3.1) and (3.2), we yield that the convergence of FIFPIS (1.1) is faster than IMIS (1.5). This proves (a).

Now for EMIS (1.9), we have

$$\begin{aligned} d(x_n,p) &\le \alpha_n d(x_{n-1},p) + (1-\alpha_n) d(Tx_{n-1},p) \\ &\le \alpha_n d(x_{n-1},p) + (1-\alpha_n)\lambda d(x_{n-1},p) \\ &\le \prod_{i=1}^{n}[1-(1-\alpha_i)(1-\lambda)] d(x_0,p) \qquad (3.10) \end{aligned}$$

Using (3.6) and (3.10), we conclude that FIFPIS (1.1) converges faster than the EMIS (1.9). This proves (b).

In the similar way for IIIS (1.4), EIIS (1.8), CIIS (1.3), ENIS (1.7) and FEIS (1.6) we obtain the following inequalities respectively:

$$d(x_n,p) \le \prod_{i=1}^{n}[(1-\alpha_i)\lambda[1-(1-\lambda)(1-\beta_i)] + \alpha_i] d(x_0,p) \qquad (3.11)$$

$$d(x_n,p) \le \prod_{i=1}^{n}[1-(1-\lambda)\alpha_i(1+\lambda\beta_i] d(x_0,p) \qquad (3.12)$$

$$d(x_n,p) \le \prod_{i=1}^{n}\begin{bmatrix}(1-\alpha_i)\lambda[1-(1-\lambda)(1-\beta_i)] \\ [1-(1-\lambda)(1-\gamma_i)] + \alpha_i\end{bmatrix} d(x_0,p) \qquad (3.13)$$

$$d(x_n,p) \le \prod_{i=1}^{n}[1-(1-\lambda)\alpha_i(1+\lambda\beta_i+\lambda^2\beta_i\gamma_i] d(x_0,p) \qquad (3.14)$$

$$d(x_n,p) \le \prod_{i=1}^{n}[1-(1-\lambda)\alpha_i(1+\lambda\beta_i+\lambda^2\beta_i\gamma_i+\lambda^3\beta_i\gamma_i\delta_i] d(x_0,p) \qquad (3.15)$$

Using (3.6) and (3.11), we conclude that the FIFPIS (1.1) converges faster than the IIIS (1.4) and using (3.6) and (3.12), we conclude that FIFPIS (1.1) converges faster than the EIIS (1.8). These prove (c) and (d) respectively.

Also, from (3.6), and (3.13), it is clear that the FIFPIS (1.1) converges faster than the CIIS (1.3) and from (3.6) and (3.14), it is clear that the FIFPIS (1.1) converges faster than the ENIS (1.7). These prove (e) and (f) respectively.

Finally, from (3.6) and (3.15), it is proved that the FIFPIS (1.1) converges faster than the FEIS (1.6). This completes the proof.

Now we give an example which demonstrates the Theorem 3.3 numerically.

EXAMPLE 3.4 Let $C = [0,1]$, $Tx = \frac{x}{3}, x \neq 0$ and $\alpha_n = \beta_n = \gamma_n = \delta_n = 1 - \frac{3}{\sqrt{n}}, n \ge 16$ and for $n = 1,2,3,\cdots,15, \alpha_n = \beta_n = \gamma_n = \delta_n = 0$, then for the FIFPIS (1.1), we have

$$u_n = \delta_n x_n + (1-\delta_n)Tx_n = \left(1-\frac{3}{\sqrt{n}}\right)x_n + \left(\frac{3}{\sqrt{n}}\right)\frac{x_n}{3} = \left(1-\frac{2}{\sqrt{n}}\right)x_n$$

$$z_n = \left(1-\frac{3}{\sqrt{n}}\right)\left(1-\frac{2}{\sqrt{n}}\right)x_n + \left(\frac{3}{\sqrt{n}}\right)\frac{\left(1-\frac{2}{\sqrt{n}}\right)x_n}{3} = \left(1-\frac{2}{\sqrt{n}}\right)^2 x_n$$

$$y_n = \left(1-\frac{3}{\sqrt{n}}\right)\left(1-\frac{2}{\sqrt{n}}\right)^2 x_n + \left(\frac{3}{\sqrt{n}}\right)\frac{\left(1-\frac{2}{\sqrt{n}}\right)^2 x_n}{3} = \left(1-\frac{2}{\sqrt{n}}\right)^3 x_n$$

thus

$$x_n = \left(1-\frac{3}{\sqrt{n}}\right)x_{n-1} + \left(\frac{3}{\sqrt{n}}\right)T\left(\left(1-\frac{2}{\sqrt{n}}\right)^3 x_n\right) = \left(1-\frac{3}{\sqrt{n}}\right)x_{n-1} + \left(\frac{3}{\sqrt{n}}\right)\frac{\left(1-\frac{2}{\sqrt{n}}\right)^3 x_n}{3}$$

$$= \left(1-\frac{3}{\sqrt{n}}\right)x_{n-1} + \frac{\left(1-\frac{2}{\sqrt{n}}\right)^3}{\sqrt{n}}x_n$$

which implies $x_n\left[1-\frac{\left(1-\frac{2}{\sqrt{n}}\right)^3}{\sqrt{n}}\right] = \left(1-\frac{3}{\sqrt{n}}\right)x_{n-1}$

Therefore,

$$x_n = \frac{n^2-3n^{\frac{3}{2}}}{n^2-n^{\frac{3}{2}}+6n-12n^{\frac{1}{2}}+8}x_{n-1} = \prod_{i=16}^{n}\left(\frac{i^2-3i^{\frac{3}{2}}}{i^2-i^{\frac{3}{2}}+6i-12i^{\frac{1}{2}}+8}\right)x_{16}. \tag{3.16}$$

Also, for CIIS (1.3), we have

$$z_n = \gamma_n x_n + (1-\gamma_n)Tx_n = \left(1-\frac{3}{\sqrt{n}}\right)x_n + \left(\frac{3}{\sqrt{n}}\right)\frac{x_n}{3} = \left(1-\frac{2}{\sqrt{n}}\right)x_n$$

$$y_n = \left(1-\frac{3}{\sqrt{n}}\right)\left(1-\frac{2}{\sqrt{n}}\right)x_n + \left(\frac{3}{\sqrt{n}}\right)\frac{\left(1-\frac{2}{\sqrt{n}}\right)x_n}{3} = \left(1-\frac{2}{\sqrt{n}}\right)^2 x_n$$

thus

$$x_n = \left(1-\frac{3}{\sqrt{n}}\right)x_{n-1} + \left(\frac{3}{\sqrt{n}}\right)T\left(\left(1-\frac{2}{\sqrt{n}}\right)^2 x_n\right) = \left(1-\frac{3}{\sqrt{n}}\right)x_{n-1} + \left(\frac{3}{\sqrt{n}}\right)\frac{\left(1-\frac{2}{\sqrt{n}}\right)^2 x_n}{3}$$

$$= \left(1-\frac{3}{\sqrt{n}}\right)x_{n-1} + \frac{\left(1-\frac{2}{\sqrt{n}}\right)^2}{\sqrt{n}}x_n$$

which implies

$$x_n\left[1-\frac{\left(1-\frac{2}{\sqrt{n}}\right)^2}{\sqrt{n}}\right] = \left(1-\frac{3}{\sqrt{n}}\right)x_{n-1}$$

Therefore,

$$x_n = \frac{n^{\frac{3}{2}}-3n}{n^{\frac{3}{2}}-n+4n^{\frac{1}{2}}-4}x_{n-1} = \prod_{i=16}^{n}\left(\frac{i^{\frac{3}{2}}-3i}{i^{\frac{3}{2}}-i+4i^{\frac{1}{2}}-4}\right)x_{16}. \tag{3.17}$$

Similarly, for IIIS (1.4) and IMIS (1.5), we have

$$x_n = \prod_{i=16}^{n}\left(\frac{i-3i^{\frac{1}{2}}}{i-i^{\frac{1}{2}}+2}\right)x_{16} \tag{3.18}$$

and

$$x_n = \prod_{i=16}^{n}\left(\frac{i^{\frac{1}{2}}-3}{i^{\frac{1}{2}}-1}\right)x_{16} \tag{3.19}$$

respectively.

Now, for FEIS (1.6), we have

$$u_n = (1-\delta_n)x_{n-1} + \delta_n T x_{n-1} = \frac{3}{\sqrt{n}}x_{n-1} + \left(1-\frac{3}{\sqrt{n}}\right)\frac{x_{n-1}}{3} = \left(\frac{6+\sqrt{n}}{3\sqrt{n}}\right)x_{n-1}$$

$$z_n = \frac{3}{\sqrt{n}}x_{n-1} + \left(1-\frac{3}{\sqrt{n}}\right)\frac{\left(\frac{6+\sqrt{n}}{3\sqrt{n}}\right)x_{n-1}}{3} = \left(\frac{n+30\sqrt{n}-18}{9n}\right)x_{n-1}$$

$$y_n = \frac{3}{\sqrt{n}}x_{n-1} + \left(1-\frac{3}{\sqrt{n}}\right)\frac{\left(\frac{n+30\sqrt{n}-18}{9n}\right)x_{n-1}}{3} = \left(\frac{n\sqrt{n}+108n-108\sqrt{n}+54}{27n\sqrt{n}}\right)x_{n-1}$$

Thus

$$\begin{aligned} x_n &= \frac{3}{\sqrt{n}}x_{n-1} + \left(1-\frac{3}{\sqrt{n}}\right)\frac{\left(\frac{n\sqrt{n}+108n-108\sqrt{n}+54}{27n\sqrt{n}}\right)x_{n-1}}{3} \\ &= \left(\frac{n^2+348n\sqrt{n}-432n+378\sqrt{n}-162}{81n^2}\right)x_{n-1} \\ &= \prod_{i=16}^{n}\left(\frac{i^2+348i^{\frac{3}{2}}-432i+378i^{\frac{1}{2}}-162}{81i^2}\right)x_{16}. \end{aligned} \tag{3.20}$$

In the similar way, for ENIS (1.7), EIIS (1.8) and EMIS (1.9), we obtain the following:

$$x_n = \prod_{i=16}^{n}\left(\frac{i^{\frac{3}{2}}+108i-108i^{\frac{1}{2}}+54}{27i^{\frac{3}{2}}}\right)x_{16} \tag{3.21}$$

$$x_n = \prod_{i=16}^{n}\left(\frac{i+30i^{\frac{1}{2}}-18}{9i}\right)x_{16} \tag{3.22}$$

and

$$x_n = \prod_{i=16}^{n}\left(\frac{i^{\frac{1}{2}}+6}{3i^{\frac{1}{2}}}\right)x_{16} \tag{3.23}$$

respectively.

Now, using (3.16) and (3.17), we get

$$\frac{x_n(\text{FIFPIS})}{x_n(\text{CIIS})} = \prod_{i=16}^{n}\left(\frac{i^2-3i^{\frac{3}{2}}}{i^2-i^{\frac{3}{2}}+6i-12i^{\frac{1}{2}}+8}\right)\left(\frac{i^{\frac{3}{2}}-i+4i^{\frac{1}{2}}-4}{i^{\frac{3}{2}}-3i}\right)$$

$$= \prod_{i=16}^{n}\left(\frac{i^{\frac{1}{2}}\left(i^{\frac{3}{2}}-3i\right)}{i^2-i^{\frac{3}{2}}+6i-12i^{\frac{1}{2}}+8}\right)\left(\frac{i^{\frac{3}{2}}-i+4i^{\frac{1}{2}}-4}{i^{\frac{1}{2}}-3i}\right) = \prod_{i=16}^{n}\left(\frac{i^2-i^{\frac{3}{2}}+4i-4i^{\frac{1}{2}}}{i^2-i^{\frac{3}{2}}+6i-12i^{\frac{1}{2}}+8}\right)$$

$$= \prod_{i=16}^{n}\left(1-\frac{2i-8i^{\frac{1}{2}}+8}{i^2-i^{\frac{3}{2}}+6i-12i^{\frac{1}{2}}+8}\right)$$

But

$$0 \le \lim_{n\to\infty}\prod_{i=16}^{n}\left(1-\frac{2i-8i^{\frac{1}{2}}+8}{i^2-i^{\frac{3}{2}}+6i-12i^{\frac{1}{2}}+8}\right)$$

$$\le \lim_{n\to\infty}\prod_{i=16}^{n}\left(1-\frac{1}{i}\right) = \lim_{n\to\infty}\frac{15}{16}\times\frac{16}{17}\times\cdots\times\frac{n-1}{n} = \lim_{n\to\infty}\frac{15}{n} = 0.$$

That is $\lim_{n\to\infty}\left|\frac{x_n(\text{FIFPIS})-0}{x_n(\text{CIIS})-0}\right| = 0.$

Therefore, by definition 3.1 we can say that the FIFPIS defined by (1.1) converges faster than the CIIS defined by (1.3) to the fixed point $p = 0$.

Also, using (3.16) and (3.18), we get

$$\frac{x_n(\text{FIFPIS})}{x_n(\text{IIIS})} = \prod_{i=16}^{n}\left(\frac{i^2-3i^{\frac{3}{2}}}{i^2-i^{\frac{3}{2}}+6i-12i^{\frac{1}{2}}+8}\right)\left(\frac{i-i^{\frac{1}{2}}+2}{i-3i^{\frac{1}{2}}}\right)$$

$$= \prod_{i=16}^{n}\left(\frac{i\left(i-3i^{\frac{1}{2}}\right)}{i^2-i^{\frac{3}{2}}+6i-12i^{\frac{1}{2}}+8}\right)\left(\frac{i-i^{\frac{1}{2}}+2}{i-3i^{\frac{1}{2}}}\right) = \prod_{i=16}^{n}\left(\frac{i^2-i^{\frac{3}{2}}+2i}{i^2-i^{\frac{3}{2}}+6i-12i^{\frac{1}{2}}+8}\right)$$

$$= \prod_{i=16}^{n}\left(1-\frac{4i-12i^{\frac{1}{2}}+8}{i^2-i^{\frac{3}{2}}+6i-12i^{\frac{1}{2}}+8}\right)$$

But

$$0 \le \lim_{n\to\infty}\prod_{i=16}^{n}\left(1-\frac{4i-12i^{\frac{1}{2}}+8}{i^2-i^{\frac{3}{2}}+6i-12i^{\frac{1}{2}}+8}\right)$$

$$\le \lim_{n\to\infty}\prod_{i=16}^{n}\left(1-\frac{1}{i}\right) = \lim_{n\to\infty}\frac{15}{16}\times\frac{16}{17}\times\cdots\times\frac{n-1}{n} = \lim_{n\to\infty}\frac{15}{n} = 0.$$

That is $\lim_{n\to\infty}\left|\frac{x_n(\text{FIFPIS})-0}{x_n(\text{IIIS})-0}\right| = 0.$

Therefore, by definition 3.1 we can say that the FIFPIS defined by (1.1) converges faster than the IIIS defined by (1.4) to the fixed point $p = 0$.

Similarly, using (3.16), (3.19) and definition 3.1 we can show that the FIFPIS defined by (1.1) converges faster than the IMIS defined by (1.5) to the fixed point $p = 0$.

Now, from (3.16) and (3.20), we obtain

$$\frac{x_n(\text{FIFPIS})}{x_n(\text{FEIS})} = \prod_{i=16}^{n}\left(\frac{i^2 - 3i^{\frac{3}{2}}}{i^2 - i^{\frac{3}{2}} + 6i - 12i^{\frac{1}{2}} - 8}\right)\left(\frac{81i^2}{i^2 + 348i^{\frac{3}{2}} - 432i + 378i^{\frac{1}{2}} - 162}\right)$$

$$= \prod_{i=16}^{n}\left(\frac{81i^4 - 243i^{\frac{7}{2}}}{i^4 + 347i^{\frac{7}{2}} - 774i^3 + 2886i^{\frac{5}{2}} - 7316i^2 + 4506i^{\frac{3}{2}} - 2052i - 4968i^{\frac{1}{2}} + 1296}\right)$$

$$= \prod_{i=16}^{n}\left(1 - \frac{-80i^4 + 590i^{\frac{7}{2}} - 774i^3 + 2886i^{\frac{5}{2}} - 7316i^2 + 4506i^{\frac{3}{2}} - 2052i - 4968i^{\frac{1}{2}} + 1296}{i^4 + 347i^{\frac{7}{2}} - 774i^3 + 2886i^{\frac{5}{2}} - 7316i^2 + 4506i^{\frac{3}{2}} - 2052i - 4968i^{\frac{1}{2}} + 1296}\right)$$

But

$$0 \leq \lim_{n\to\infty}\prod_{i=16}^{n}\left(1 - \frac{-80i^4 + 590i^{\frac{7}{2}} - 774i^3 + 2886i^{\frac{5}{2}} - 7316i^2 + 4506i^{\frac{3}{2}} - 2052i - 4968i^{\frac{1}{2}} + 1296}{i^4 + 347i^{\frac{7}{2}} - 774i^3 + 2886i^{\frac{5}{2}} - 7316i^2 + 4506i^{\frac{3}{2}} - 2052i - 4968i^{\frac{1}{2}} + 1296}\right)$$

$$\leq \lim_{n\to\infty}\prod_{i=16}^{n}\left(1 - \frac{1}{i}\right) = \lim_{n\to\infty}\frac{15}{16} \times \frac{16}{17} \times \cdots \times \frac{n-1}{n} = \lim_{n\to\infty}\frac{15}{n} = 0.$$

That is $\lim_{n\to\infty}\left|\frac{x_n(\text{FIFPIS}) - 0}{x_n(\text{FEIS}) - 0}\right| = 0.$

Therefore, by Definition 3.1 we can say that the FIFPIS defined by (1.1) converges faster than the FEIS defined by (1.6) to the fixed point $p = 0$.

By maintaining the above procedure and using Definition 3.1 and (3.16) and (3.21), (3.16) and (3.22), and (3.16) and (3.23), we can show that the FIFPIS defined by (1.1) converges faster than the ENIS defined by (1.7), EIIS defined by (1.8) and EMIS defined by (1.9) to the fixed point $p = 0$ respectively.

Therefore, in the case of rate of convergence at fixed point of a contractive-like operator, we obtained the following inequalities:

FIFPIS $>$ CIIS, FIFPIS $>$ IIIS, FIFPIS $>$ IMIS, FIFPIS $>$ FEIS, FIFPIS $>$ ENIS,

FIFPIS $>$ EIIS and FIFPIS $>$ EMIS.

This shows that the authenticity of the Theorem 3.3 effectively.

Now we give a comparison table of the rate of convergence of different implicit and explicit iterative schemes which is obtained by using computer programming language MATLAB-7. Here we take the initial approximation $x_{16} = 0.5, Tx = \frac{x}{3}$, and $\alpha_n = \beta_n = \gamma_n = \delta_n = 1 - \frac{3}{\sqrt{n}}$, $n \geq 16$. This comparison table confirms that the rate of convergence of newly introduced FIFPIS defined by (1.1) is better than that of

CIIS defined by (1.3), IIIS defined by (1.4), IMIS defined by (1.5) as well as FEIS defined by (1.6), ENIS defined by (1.7), EIIS defined by (1.8) and EMIS defined by (1.9).

Table 3.5: Rate of convergence comparison of FIFPIS with different implicit and explicit iterative schemes

Step No.	EMIS	EIIS	ENIS	FEIS	IMIS	IIIS	CIIS	FIFPIS
16	0.5000000000	0.5000000000	0.5000000000	0.5000000000	0.5000000000	0.5000000000	0.5000000000	0.5000000000
17	0.4166666667	0.4097222222	0.4091435185	0.4090952932	0.1666666667	0.1428571429	0.1333333333	0.1290322581
18	0.3410019098	0.3285628612	0.3279494606	0.3273920445	0.0599352995	0.0444666840	0.0388152416	0.0363512500
19	0.2744171451	0.2581433635	0.2575318407	0.2565546486	0.0229683301	0.0148776085	0.0121702321	0.0110310521
20	0.2173836032	0.1989167743	0.1983374012	0.1970569190	0.0092922234	0.0052955983	0.0040673809	0.0035687191
21	0.1696781038	0.1504739961	0.1499484190	0.1484892832	0.0039397698	0.0019890649	0.0014370928	0.0012208626
22	0.1306129636	0.1118422027	0.1113831186	0.1098618536	0.0017403635	0.0007832388	0.0005332529	0.0004387578
23	0.0992312094	0.0817422309	0.0813544213	0.0798681446	0.0007971833	0.0003216075	0.0002066811	0.0001647565
24	0.0744593447	0.0587892279	0.0584713080	0.0570919765	0.0003771524	0.0001370928	0.0000832959	0.0000643527
25	0.0552176818	0.0416340371	0.0413804028	0.0401539203	0.0001836903	0.0000604418	0.0000347733	0.0000260466
26	0.0404929666	0.0290513059	0.0288539297	0.0278027727	0.0000918451	0.0000274735	0.0000149885	0.0000108890
27	0.0293803038	0.0199846398	0.0198345258	0.0189623877	0.0000470319	0.0000128400	0.0000066519	0.0000046889
28	0.0211019188	0.0135603243	0.0134485596	0.0127457000	0.0000246152	0.0000061555	0.0000030322	0.0000020747
29	0.0150097485	0.0090803095	0.0089987339	0.0084470749	0.0000131436	0.0000030207	0.0000014167	0.0000009413
30	0.0105777302	0.0060032655	0.0059448235	0.0055222221	0.0000071490	0.0000015147	0.0000006772	0.0000004371
31	0.0073883510	0.0039202828	0.0038791424	0.0035626003	0.0000039555	0.0000007749	0.0000003306	0.0000002074
32	0.0051167577	0.0025296774	0.0025011931	0.0022690075	0.0000022236	0.0000004039	0.0000001646	0.0000001004
33	0.0035146329	0.0016135972	0.0015941834	0.0014271850	0.0000012686	0.0000002141	0.0000000835	0.0000000495
34	0.0023951824	0.0010177988	0.0010047639	0.0008868516	0.0000007339	0.0000001154	0.0000000431	0.0000000249
35	0.0016199349	0.0006350551	0.0006264272	0.0005446161	0.0000004300	0.0000000631	0.0000000226	0.0000000127
36	0.0010876163	0.0003920858	0.0003864524	0.0003306235	0.0000002551	0.0000000350	0.0000000120	0.0000000066
37	0.0007250775	0.0002396080	0.0002359776	0.0001984761	0.0000001531	0.0000000197	0.0000000065	0.0000000035
38	0.0004800965	0.0001449757	0.0001426653	0.0001178516	0.0000000928	0.0000000112	0.0000000036	0.0000000018
39	0.0003157961	0.0000868720	0.0000854193	0.0000692358	0.0000000569	0.0000000065	0.0000000020	0.0000000010
40	0.0002064010	0.0000515663	0.0000506635	0.0000402535	0.0000000352	0.0000000038	0.0000000011	0.0000000005
41	0.0001340701	0.0000303291	0.0000297743	0.0000231664	0.0000000220	0.0000000022	0.0000000006	0.0000000003
42	0.0000865665	0.0000176791	0.0000173419	0.0000132007	0.0000000138	0.0000000013	0.0000000004	0.0000000002
43	0.0000555705	0.0000102157	0.0000100128	0.0000074492	0.0000000088	0.0000000008	0.0000000002	0.0000000001
44	0.0000354723	0.0000058528	0.0000057320	0.0000041638	0.0000000056	0.0000000005	0.0000000001	0.0000000001
45	0.0000225194	0.0000033254	0.0000032542	0.0000023058	0.0000000036	0.0000000003	0.0000000001	0.0000000000
46	0.0000142205	0.0000018741	0.0000018325	0.0000012653	0.0000000024	0.0000000002	0.0000000000	0.0000000000

4. STABILITY OF FOUR-STEP IMPLICIT FIXED POINT ITERATIVE SCHEME

In this section we established the stability result of Four-step implicit fixed point iterative scheme (FIFPIS) defined by (1.1) for contractive-like operators in convex metric spaces.

THEOREM 4.1 *Let C be a nonempty closed convex subset of a convex metric space M and $T: C \to C$ be a contractive-like operator satisfying (1.13) with $F(T) \neq \emptyset$. Then, for $x_0 \in C$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by the FIFPIS (1.1) with the converging point at $j \in F(T)$ and $\alpha_n \leq \alpha < 1$, $\sum(1-\alpha_n) = \infty$, is T-stable.*

PROOF. First suppose that, $\{j_n\}_{n=0}^{\infty} \subset C$ is an arbitrary sequence such that

$$\xi_n = d\big(j_n, I(j_{n-1}, Tk_n, \alpha_n)\big)$$

where $k_n = I(l_n, Tl_n, \beta_n)$, $l_n = I(m_n, Tm_n, \gamma_n), m_n = I(j_n, Tj_n, \delta_n)$ and let $\lim_{n\to\infty} \xi_n = 0$.

Now, since T is a contractive-like operator, then using (1.13), we have

$$\begin{aligned}
d(j_n, j) &\leq d\big(j_n, I(j_{n-1}, Tk_n, \alpha_n)\big) + d(I(j_{n-1}, Tk_n, \alpha_n), j) \\
&\leq \xi_n + \alpha_n d(j_{n-1}, j) + (1-\alpha_n) d(Tk_n, j) \\
&\leq \xi_n + \alpha_n d(j_{n-1}, j) + (1-\alpha_n)\big[\phi\big(d(Tj, j)\big) + \lambda d(k_n, j)\big] \\
&= \xi_n + \alpha_n d(j_{n-1}, j) + (1-\alpha_n)\lambda d(I(l_n, Tl_n, \beta_n), j) \\
&\leq \xi_n + \alpha_n d(j_{n-1}, j) + (1-\alpha_n)\lambda[\beta_n d(l_n, j) + (1-\beta_n) d(Tl_n, j)] \\
&\leq \xi_n + \alpha_n d(j_{n-1}, j) + (1-\alpha_n)\lambda\left[\beta_n d(l_n, j) + (1-\beta_n)\begin{bmatrix}\phi\big(d(Tj, j)\big) \\ +\lambda d(l_n, j)\end{bmatrix}\right] \\
&\leq \xi_n + \alpha_n d(j_{n-1}, j) + (1-\alpha_n)\lambda[\beta_n d(l_n, j) + (1-\beta_n)\lambda d(l_n, j)] \\
&\leq \xi_n + \alpha_n d(j_{n-1}, j) + \lambda(1-\alpha_n)\begin{bmatrix}\beta_n \\ +\lambda(1-\beta_n)\end{bmatrix}\begin{bmatrix}\gamma_n \\ +\lambda(1-\gamma_n)\end{bmatrix} d(m_n, j) \\
&\leq \xi_n + \alpha_n d(j_{n-1}, j) + \begin{bmatrix}\lambda(1-\alpha_n)\begin{bmatrix}\beta_n \\ +\lambda(1-\beta_n)\end{bmatrix} \\ \begin{bmatrix}\gamma_n \\ +\lambda(1-\gamma_n)\end{bmatrix}\begin{bmatrix}\delta_n \\ +\lambda(1-\delta_n)\end{bmatrix}\end{bmatrix} d(j_n, j)
\end{aligned}$$

$$\Rightarrow d(j_n, j) \leq \frac{\alpha_n}{1-\lambda(1-\alpha_n)[\beta_n+\lambda(1-\beta_n)][\gamma_n+\lambda(1-\gamma_n)][\delta_n+\lambda(1-\delta_n)]} d(j_{n-1}, j)$$

$$+ \frac{\alpha_n}{[1-\lambda(1-\alpha_n)[\beta_n+\lambda(1-\beta_n)][\gamma_n+\lambda(1-\gamma_n)][\delta_n+\lambda(1-\delta_n)]]} \times \frac{\xi_n}{\alpha_n} \tag{4.1}$$

But, according to (2.6), we get

$$\frac{\alpha_n}{1-\lambda(1-\alpha_n)[\beta_n+\lambda(1-\beta_n)][\gamma_n+\lambda(1-\gamma_n)][\delta_n+\lambda(1-\delta_n)]} \leq 1-(1-\alpha_n)(1-\lambda) \quad (4.2)$$

Thus (4.1) converts as follows

$$d(j_n, j) \leq [1-(1-\alpha_n)(1-\lambda)]d(j_{n-1}, j) + [1-(1-\alpha_n)(1-\lambda)] \times \frac{\xi_n}{\alpha_n} \quad (4.3)$$

Now, applying $\alpha_n \leq \alpha < 1$ and $\lambda \in (0,1)$, we have

$$1-(1-\alpha_n)(1-\lambda) < 1 \quad (4.4)$$

Since, $\lim_{n\to\infty} \xi_n = 0$, hence from (4.3), (4.4) and Lemma 1.6, we obtain

$$\lim_{n\to\infty} d(j_n, j) = 0$$

which implies that $\lim_{n\to\infty} j_n = j$.

Conversely, if we consider $\lim_{n\to\infty} j_n = j$, then by applying the contractive condition (1.13), it is easy to obtain that $\lim_{n\to\infty} \xi_n = 0$.

Hence, by the Definition 1.5, the FIFPIS (1.1) is T-stable. This completes the proof.

REMARK 4.2 Since, the contractive condition (1.13) is the most general contractive condition among the contractive conditions (1.10)-(1.13), then the stability results of FIFPIS (1.1) for the contractive conditions (1.10)-(1.11) can be easily accessible as special cases.

REMARK 4.3 Since, CIIS defined by (1.3), IIIS defined by (1.4) and IIIS defined by (1.5) are obtainable as special cases of FIFPIS defined by (1.1), so the Theorem 4.1 will remain true for CIIS defined by (1.3), IIIS defined by (1.4) and IIIS defined by (1.5).

5. DATA DEPENDENCE OF FOUR-STEP IMPLICIT FIXED POINT ITERATIVE SCHEME

In this section we established a data dependence result of Four-step implicit fixed point iterative scheme (FIFPIS) for contractive-like operator in hyperbolic space.

THEOREM 5.1 *Let C be a nonempty closed convex subset of a hyperbolic space M, $T: C \to C$ be a contractive-like operator satisfying (1.13) with $F(T) \neq \emptyset$ and S be an* *approximate operator of* T *satisfying (1.13) with* $F(S) \neq \emptyset$. *Let* $\{x_n\}$ *and* $\{e_n\}$ *represent two four-step implicit iterative schemes related to* T *and* S *which are defined as follows*

$$\left.\begin{array}{r} x_n = I(x_{n-1}, Ty_n, \alpha_n) \\ y_n = I(z_n, Tz_n, \beta_n) \\ z_n = I(u_n, Tu_n, \gamma_n) \\ u_n = I(x_n, Tx_n, \delta_n);\ \forall\, n \in \mathbb{N}, \end{array}\right\} \quad (5.1)$$

and

$$\left.\begin{array}{r} e_n = I(e_{n-1}, Sg_n, \alpha_n) \\ g_n = I(h_n, Sh_n, \beta_n) \\ h_n = I(i_n, Si_n, \gamma_n) \\ i_n = I(e_n, Se_n, \delta_n);\ \forall\, n \in \mathbb{N}, \end{array}\right\} \quad (5.2)$$

respectively, where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty}$ *and* $\{\delta_n\}_{n=0}^{\infty}$ *are real sequences in* $[0,1]$ *such that* $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$. *Then for any* $\varepsilon > 0$, *the following estimation exist* $d(p_1, p_2) \leq \frac{\varepsilon}{(1-\lambda)^2}$ *where* $p_1 \in F(T)$, $p_2 \in F(S)$ *and* $\lambda \in (0,1)$.

Proof. Applying the Definition 1.4, in the iterative schemes defined by (5.1) and (5.2), we have

$$\begin{aligned} d(x_n, e_n) &= d\big(I(x_{n-1}, Ty_n, \alpha_n), I(e_{n-1}, Sg_n, \alpha_n)\big) \\ &\leq \alpha_n d(x_{n-1}, e_{n-1}) + (1-\alpha_n) d(Ty_n, Sg_n) \\ &\leq \alpha_n d(x_{n-1}, e_{n-1}) + (1-\alpha_n)\big[\varepsilon + \phi\big(d(y_n, Sy_n)\big) + \lambda d(y_n, g_n)\big] \\ &= \alpha_n d(x_{n-1}, e_{n-1}) + (1-\alpha_n)\varepsilon + (1-\alpha_n)\phi\big(d(y_n, Sy_n)\big) \\ &\quad + \lambda(1-\alpha_n) d(y_n, g_n) \end{aligned} \quad (5.3)$$

But, we have the following inequalities:

$$\begin{aligned} d(y_n, g_n) &= d\big(I(z_n, Tz_n, \beta_n), I(h_n, Sh_n, \beta_n)\big) \\ &\leq \beta_n d(z_n, h_n) + (1-\beta_n) d(Tz_n, Sh_n) \\ &\leq \beta_n d(z_n, h_n) + (1-\beta_n)\big[\varepsilon + \phi\big(d(z_n, Sz_n)\big) + \lambda d(z_n, h_n)\big] \\ &= [\beta_n(1-\lambda) + \lambda] d(z_n, h_n) + (1-\beta_n)\varepsilon + (1-\beta_n)\phi\big(d(z_n, Sz_n)\big) \end{aligned} \quad (5.4)$$

$$\begin{aligned} d(z_n, h_n) &= d\big(I(u_n, Tu_n, \gamma_n), I(i_n, Si_n, \gamma_n)\big) \\ &\leq \gamma_n d(u_n, i_n) + (1-\gamma_n) d(Tu_n, Si_n) \\ &\leq \gamma_n d(u_n, i_n) + (1-\gamma_n)\big[\varepsilon + \phi\big(d(u_n, Su_n)\big) + \lambda d(u_n, i_n)\big] \\ &= [\gamma_n(1-\lambda) + \lambda] d(u_n, i_n) + (1-\gamma_n)\varepsilon + (1-\beta_n)\phi\big(d(u_n, Su_n)\big) \end{aligned} \quad (5.5)$$

$$
\begin{aligned}
d(u_n, i_n) &= d\big(I(x_n, Tx_n, \delta_n), I(e_n, Se_n, \delta_n)\big) \\
&\le \delta_n d(x_n, e_n) + (1-\delta_n) d(Tx_n,\ Se_n\ ) \\
&\le \delta_n d(x_n, e_n) + (1-\delta_n)\big[\varepsilon + \phi\big(d(x_n,\ Sx_n)\big) + \lambda d(x_n, e_n)\big] \\
&= [\delta_n(1-\lambda) + \lambda] d(x_n, e_n) + (1-\delta_n)\varepsilon + (1-\delta_n)\phi\big(d(x_n,\ Sx_n)\big) \quad (5.6)
\end{aligned}
$$

Now, combining (5.3)-(5.6), we obtain

$$
\begin{aligned}
d(x_n, e_n\ ) &\le \alpha_n d(x_{n-1}, e_{n-1}\ ) + (1-\alpha_n)\varepsilon + (1-\alpha_n)\phi\big(d(y_n, Sy_n)\big) \\
&\quad + \lambda(1-\alpha_n)\begin{bmatrix} [\beta_n(1-\lambda)+\lambda] d(z_n, h_n) + (1-\beta_n)\varepsilon \\ +(1-\beta_n)\phi\big(d(z_n,\ Sz_n\ )\big) \end{bmatrix} \\
&= \alpha_n d(x_{n-1}, e_{n-1}\ ) + (1-\alpha_n)\varepsilon + (1-\alpha_n)\phi\big(d(y_n, Sy_n)\big) \\
&\quad + \lambda(1-\alpha_n)(1-\beta_n)\varepsilon + \lambda(1-\alpha_n)(1-\beta_n)\phi\big(d(z_n,\ Sz_n\ )\big) \\
&\quad + \lambda(1-\alpha_n)[\beta_n(1-\lambda)+\lambda] d(z_n, h_n) \\
&\le \alpha_n d(x_{n-1}, e_{n-1}\ ) + (1-\alpha_n)\varepsilon + (1-\alpha_n)\phi\big(d(y_n, Sy_n)\big) \\
&\quad + \lambda(1-\alpha_n)(1-\beta_n)\varepsilon + \lambda(1-\alpha_n)(1-\beta_n)\phi\big(d(z_n,\ Sz_n\ )\big) \\
&\quad + \lambda(1-\alpha_n)[\beta_n(1-\lambda)+\lambda]\begin{bmatrix} [\gamma_n(1-\lambda)+\lambda] d(u_n, i_n) \\ +(1-\gamma_n)\varepsilon + (1-\beta_n)\phi\big(d(u_n,\ Su_n)\big) \end{bmatrix} \\
&\le \alpha_n d(x_{n-1}, e_{n-1}\ ) + \begin{bmatrix} (1-\alpha_n) + \lambda(1-\alpha_n)(1-\beta_n) \\ +\big[\lambda(1-\alpha_n)[\beta_n(1-\lambda)+\lambda]\big](1-\gamma_n) \end{bmatrix}\varepsilon \\
&\quad + (1-\alpha_n)\phi\big(d(y_n, Sy_n)\big) + \lambda(1-\alpha_n)(1-\beta_n)\phi\big(d(z_n,\ Sz_n\ )\big) \\
&\quad + \lambda(1-\alpha_n)[\beta_n(1-\lambda)+\lambda](1-\beta_n)\phi\big(d(u_n,\ Su_n)\big) \\
&\quad + \lambda(1-\alpha_n)[\beta_n(1-\lambda)+\lambda][\gamma_n(1-\lambda)+\lambda] \\
&\quad \big[[\delta_n(1-\lambda)+\lambda] d(x_n, e_n) + (1-\delta_n)\varepsilon + (1-\delta_n)\phi\big(d(x_n,\ Sx_n)\big)\big] \\
&= \alpha_n d(x_{n-1}, e_{n-1}\ ) + \begin{bmatrix} (1-\alpha_n) + \lambda(1-\alpha_n)(1-\beta_n) \\ +\big[\lambda(1-\alpha_n)[\beta_n(1-\lambda)+\lambda]\big](1-\gamma_n) \\ +\lambda(1-\alpha_n)[\beta_n(1-\lambda)+\lambda][\gamma_n(1-\lambda)+\lambda](1-\delta_n) \end{bmatrix}\varepsilon \\
&\quad + (1-\alpha_n)\phi\big(d(y_n, Sy_n)\big) + \lambda(1-\alpha_n)(1-\beta_n)\phi\big(d(z_n,\ Sz_n\ )\big) \\
&\quad + \lambda(1-\alpha_n)[\beta_n(1-\lambda)+\lambda](1-\beta_n)\phi\big(d(u_n,\ Su_n)\big)
\end{aligned}
$$

$$+\lambda(1-\alpha_n)[\beta_n(1-\lambda)+\lambda][\gamma_n(1-\lambda)+\lambda](1-\delta_n)\phi\big(d(x_n,\ Sx_n)\big)$$

$$+\lambda(1-\alpha_n)[\beta_n(1-\lambda)+\lambda][\gamma_n(1-\lambda)+\lambda][\delta_n(1-\lambda)+\lambda]d(x_n,e_n)$$

which implies that,

$$d(x_n,e_n) \le \frac{\alpha_n}{1-\lambda(1-\alpha_n)[\beta_n(1-\lambda)+\lambda][\gamma_n(1-\lambda)+\lambda][\delta_n(1-\lambda)+\lambda]} d(x_{n-1},e_{n-1})$$

$$+\frac{\varepsilon(1-\alpha_n)\begin{bmatrix}1+\lambda(1-\beta_n)\\ +\lambda[\beta_n(1-\lambda)+\lambda](1-\gamma_n)\\ +\lambda[\beta_n(1-\lambda)+\lambda][\gamma_n(1-\lambda)+\lambda](1-\delta_n)\end{bmatrix}}{1-\lambda(1-\alpha_n)[\beta_n(1-\lambda)+\lambda][\gamma_n(1-\lambda)+\lambda][\delta_n(1-\lambda)+\lambda]}$$

$$+\frac{(1-\alpha_n)\begin{bmatrix}\phi\big(d(y_n,Sy_n)\big)+\lambda(1-\beta_n)\phi\big(d(z_n,Sz_n)\big)\\ +\lambda[\beta_n(1-\lambda)+\lambda](1-\beta_n)\phi\big(d(u_n,Su_n)\big)\\ +\lambda[\beta_n(1-\lambda)+\lambda][\gamma_n(1-\lambda)+\lambda](1-\delta_n)\phi\big(d(x_n,Sx_n)\big)\end{bmatrix}}{1-\lambda(1-\alpha_n)[\beta_n(1-\lambda)+\lambda][\gamma_n(1-\lambda)+\lambda][\delta_n(1-\lambda)+\lambda]} \quad (5.7)$$

Setting $\frac{P_n}{Q_n} = \frac{\alpha_n}{1-\lambda(1-\alpha_n)[\beta_n(1-\lambda)+\lambda][\gamma_n(1-\lambda)+\lambda][\delta_n(1-\lambda)+\lambda]}$, we obtain

$$1-\frac{P_n}{Q_n} = 1-\frac{\alpha_n}{1-\lambda(1-\alpha_n)[\beta_n(1-\lambda)+\lambda][\gamma_n(1-\lambda)+\lambda][\delta_n(1-\lambda)+\lambda]}$$

$$=\frac{1-\alpha_n-\lambda(1-\alpha_n)[\beta_n(1-\lambda)+\lambda][\gamma_n(1-\lambda)+\lambda][\delta_n(1-\lambda)+\lambda]}{1-\lambda(1-\alpha_n)[\beta_n(1-\lambda)+\lambda][\gamma_n(1-\lambda)+\lambda][\delta_n(1-\lambda)+\lambda]} \ge 1-\alpha_n-\lambda(1-\alpha_n)$$

which further implies that

$$\frac{P_n}{Q_n} \le \alpha_n+\lambda(1-\alpha_n) = 1-(1-\alpha_n)(1-\lambda) \quad (5.8)$$

Hence, from (5.7) and (5.8), we get

$$d(x_n,e_n) \le [1-(1-\alpha_n)(1-\lambda)]\, d(x_{n-1},e_{n-1})$$

$$+\frac{\varepsilon(1-\alpha_n)\begin{bmatrix}1+\lambda(1-\beta_n)\\ +\lambda[\beta_n(1-\lambda)+\lambda](1-\gamma_n)\\ +\lambda[\beta_n(1-\lambda)+\lambda][\gamma_n(1-\lambda)+\lambda](1-\delta_n)\end{bmatrix}}{1-\lambda(1-\alpha_n)[\beta_n(1-\lambda)+\lambda][\gamma_n(1-\lambda)+\lambda][\delta_n(1-\lambda)+\lambda]}$$

$$+\frac{(1-\alpha_n)\begin{bmatrix}\phi\big(d(y_n,Sy_n)\big)+\lambda(1-\beta_n)\phi\big(d(z_n,Sz_n)\big)\\ +\lambda[\beta_n(1-\lambda)+\lambda](1-\beta_n)\phi\big(d(u_n,Su_n)\big)\\ +\lambda[\beta_n(1-\lambda)+\lambda][\gamma_n(1-\lambda)+\lambda](1-\delta_n)\phi\big(d(x_n,Sx_n)\big)\end{bmatrix}}{1-\lambda(1-\alpha_n)[\beta_n(1-\lambda)+\lambda][\gamma_n(1-\lambda)+\lambda][\delta_n(1-\lambda)+\lambda]}$$

$$\le [1-(1-\alpha_n)(1-\lambda)]\, d(x_{n-1},e_{n-1})$$

$$+\frac{4\varepsilon(1-\alpha_n)}{1-\lambda(1-\alpha_n)[\beta_n(1-\lambda)+\lambda][\gamma_n(1-\lambda)+\lambda][\delta_n(1-\lambda)+\lambda]}$$

$$+\frac{(1-\alpha_n)\begin{bmatrix}\phi(d(y_n,Sy_n))+\lambda(1-\beta_n)\phi(d(z_n,Sz_n))\\+\lambda[\beta_n(1-\lambda)+\lambda](1-\beta_n)\phi(d(u_n,Su_n))\\+\lambda[\beta_n(1-\lambda)+\lambda][\gamma_n(1-\lambda)+\lambda](1-\delta_n)\phi(d(x_n,Sx_n))\end{bmatrix}}{1-\lambda(1-\alpha_n)[\beta_n(1-\lambda)+\lambda][\gamma_n(1-\lambda)+\lambda][\delta_n(1-\lambda)+\lambda]}$$

$$\leq [1-(1-\alpha_n)(1-\lambda)]\, d(x_{n-1},e_{n-1})$$

$$+\frac{(1-\alpha_n)\begin{bmatrix}\phi(d(y_n,Sy_n))+\lambda(1-\beta_n)\phi(d(z_n,Sz_n))\\+\lambda[\beta_n(1-\lambda)+\lambda](1-\beta_n)\phi(d(u_n,Su_n))\\+\lambda[\beta_n(1-\lambda)+\lambda][\gamma_n(1-\lambda)+\lambda](1-\delta_n)\phi(d(x_n,Sx_n))+4\varepsilon\end{bmatrix}}{1-\lambda(1-\alpha_n)[\beta_n(1-\lambda)+\lambda][\gamma_n(1-\lambda)+\lambda][\delta_n(1-\lambda)+\lambda]}$$

$$\leq [1-(1-\alpha_n)(1-\lambda)]\, d(x_{n-1},e_{n-1})$$

$$+\frac{(1-\alpha_n)(1-\lambda)\begin{bmatrix}\phi(d(y_n,Sy_n))+\lambda(1-\beta_n)\phi(d(z_n,Sz_n))\\+\lambda[\beta_n(1-\lambda)+\lambda](1-\beta_n)\phi(d(u_n,Su_n))\\+\lambda[\beta_n(1-\lambda)+\lambda][\gamma_n(1-\lambda)+\lambda](1-\delta_n)\phi(d(x_n,Sx_n))+4\varepsilon\end{bmatrix}}{(1-\lambda)[1-\lambda(1-\alpha_n)[\beta_n(1-\lambda)+\lambda][\gamma_n(1-\lambda)+\lambda][\delta_n(1-\lambda)+\lambda]]} \quad (5.9)$$

But, $1-\lambda(1-\alpha_n)[\beta_n(1-\lambda)+\lambda][\gamma_n(1-\lambda)+\lambda][\delta_n(1-\lambda)+\lambda]$

$= 1-\lambda(1-\alpha_n)[1-(1-\beta_n)(1-\lambda)][1-(1-\gamma_n)(1-\lambda)][1-(1-\delta_n)(1-\lambda)]$

$\geq 1-\lambda$

Hence, $\frac{1}{1-\lambda(1-\alpha_n)[1-(1-\beta_n)(1-\lambda)][1-(1-\gamma_n)(1-\lambda)][1-(1-\delta_n)(1-\lambda)]} \leq \frac{1}{1-\lambda}$ (5.10)

Using (5.10) in (5.9), we have

$d(x_n,e_n) \leq [1-(1-\alpha_n)(1-\lambda)]\, d(x_{n-1},e_{n-1})$

$$+\frac{(1-\alpha_n)(1-\lambda)\begin{bmatrix}\phi(d(y_n,Sy_n))+\lambda(1-\beta_n)\phi(d(z_n,Sz_n))\\+\lambda[\beta_n(1-\lambda)+\lambda](1-\beta_n)\phi(d(u_n,Su_n))\\+\lambda[\beta_n(1-\lambda)+\lambda][\gamma_n(1-\lambda)+\lambda](1-\delta_n)\phi(d(x_n,Sx_n))+4\varepsilon\end{bmatrix}}{(1-\lambda)^2} \quad (5.11)$$

Now, if we put $s_n = d(x_n,e_n)$, $r_n = (1-\alpha_n)(1-\lambda)$

and

$$t_n = \frac{\begin{bmatrix}\phi(d(y_n,Sy_n))+\lambda(1-\beta_n)\phi(d(z_n,\ Sz_n))\\+\lambda[\beta_n(1-\lambda)+\lambda](1-\beta_n)\phi(d(u_n,\ Su_n))\\+\lambda[\beta_n(1-\lambda)+\lambda][\gamma_n(1-\lambda)+\lambda](1-\delta_n)\phi(d(x_n,\ Sx_n))+4\varepsilon\end{bmatrix}}{(1-\lambda)^2}$$

in (5.11), then by applying the Lemma 1.8, we have

$$\lim_{n\to\infty}\sup d(x_n, e_n) \leq \lim_{n\to\infty}\sup \frac{\begin{bmatrix} \phi(d(y_n,Sy_n))+\lambda(1-\beta_n)\phi(d(z_n,Sz_n)) \\ +\lambda[\beta_n(1-\lambda)+\lambda](1-\beta_n)\phi(d(u_n,Su_n)) \\ +\lambda[\beta_n(1-\lambda)+\lambda][\gamma_n(1-\lambda)+\lambda](1-\delta_n)\phi(d(x_n,Sx_n))+4\varepsilon \end{bmatrix}}{(1-\lambda)^2} \quad (5.12)$$

From, Theorem 2.1, we have $\lim_{n\to\infty} d(x_n,\ p_1) = 0$, $\lim_{n\to\infty} d(e_n,\ p_2) = 0$ and since ϕ is continuous, thus $\operatorname{Lim}_{n\to\infty}\phi(d(x_n,\ Sx_n)) = \lim_{n\to\infty}\phi(d(y_n,\ Sy_n)) = \lim_{n\to\infty}\phi(d(z_n,\ Sz_n))$ $= \lim_{n\to\infty}\phi(d(u_n,\ Su_n)) = 0$.

Therefore, (5.12) gives

$$d(p_1, p_2) \leq \frac{4\varepsilon}{(1-\lambda)^2} < \frac{\varepsilon}{(1-\lambda)^2}.$$

This completes the proof,

REMARK 5.2 If we put $\delta_n = 1$, $\delta_n = \gamma_n = 1$ and $\delta_n = \gamma_n = \beta_n = 1$ in Theorem 5.1 then respectively the data dependence results of CIIS defined by (1.3), IIIS defined by (1.4) and IIIS defined by (1.5) can smoothly be obtained.

6. CONCLUSION

We conclude that the FIFPIS is more general fixed point iterative scheme in case of rate of convergence at fixed point of contractive-like operators than CIIS, IIIS, IMIS, FEIS, ENIS, EIIS and EMIS and the Theorem 3.3 proved this fact analytically. Furthermore, we conclude that the stability result and data dependence result of FIFPIS obtained in Theorem 4.1 and Theorem 5.1 respectively, generalized the stability result and data dependence result of CIIS, IIIS, and IMIS. Finally, from the comparison Table-3.5 we comment that all implicit iterative schemes are faster than their corresponding explicit iterative schemes and among all iterative schemes FIFPIS, CIIS, IIIS, IMIS, FEIS, ENIS, EIIS and EMIS, our newly introduced iterative scheme FIFPIS is fastest.

Competing Interests

The authors declare that they have no any competing interests.

Authors' Contributions

All authors read and approved the final version of the manuscript.

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