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Grammars and multifactorial numbers

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Abstract

In this paper, the formal derivative operator defined with respect to context-free grammars is used to prove some properties involving binomial coefficients and multifactorial numbers. In addition, we extend the formal derivative operator to matrix grammars and show that multifactorial numbers can also be generated.

1. INTRODUCTION

Let Σ be an alphabet, whose letters are regarded as independent commutative indeterminates. Following [4], a formal function over Σ is defined recursively as follows:

1. Every letter in Σ is a formal function.
2. If u, v are formal functions, then $u + v$ and uv are formal functions.
3. If $f(x)$ is an analytic function, and u is a formal function, then $f(u)$ is a formal function.
4. Every formal function is constructed as above in a finite number of steps.

A context free grammar G over Σ is defined as a set of substitution rules replacing a letter in Σ by a formal function over Σ.

Definition 1.1. *Given a context-free grammar G over Σ, the formal derivative operator D, with respect to G, is defined in the following way:*

1. *For u, v formal functions, $D(u + v) = D(u) + D(v)$ and $D(uv) = D(u)v + uD(v)$.*
2. *If $f(x)$ is an analytic function and u is a formal function, $D(f(u)) = \dfrac{\partial f(u)}{\partial u} D(u)$.*

3. *For $a \in \Sigma$, if $a \to w$ is a production in G, w being a formal function, then $D(a) = w$; in other cases a is called a constant and $D(a) = 0$.*

We next define the iteration of the formal derivative operator.

Definition 1.2. *For a formal function u, we define $D^{n+1}(u) = D(D^n(u))$ for $n \geq 0$, and $D^0(u) = u$.*

For instance, given the context-free grammar $G = \{a \to a + b; b \to b\}$, we have $D^0(a) = a$, $D(a) = a + b$, $D(b) = b$, $D(ab) = D(a)b + aD(b) = (a + b)b + ab = b^2 + 2ab$, and $D^2(a) = D(D(a))$ so $D^2(a) = D(a + b) = D(a) + D(b) = a + 2b$. The generalized product rule as well as Leibniz's formula from calculus also hold for formal functions cf. [4].

Proposition 1.1 (Generalized product rule)**.** *If $u_1, u_2, \ldots, u_n$ are formal functions, then*

$$D(u_1u_2 \ldots u_n) = D(u_1)u_2 \ldots u_n + D(u_2)u_1u_3 \ldots u_n + \cdots + D(u_n)u_1u_2 \ldots u_{n-1}.$$

Proposition 1.2 (Leibniz's formula)**.** *If u, v are formal functions, then*

$$D^n(uv) = \sum_{k=0}^{n} \binom{n}{k} D^k(u)D^{n-k}(v).$$

In [4], formal functions and the formal derivative operator were used to study formal power series in combinatorics. In addition, the formal derivative operator, defined with respect to context-free grammars, has been used to study increasing trees [5], triangular arrays [9], permutations [11], Stirling permutations of the second kind [15], and for generating some combinatorial numbers such as Whitney numbers [3], Ramanujan's numbers [7], Eulerian numbers [5], Stirling numbers [14], among others [12]. In the same way, some families of polynomials such as Bessel polynomials [12], Eulerian polynomials [13], and other types of polynomials [6], have been studied by grammatical methods.

In this paper, we prove some properties involving binomial coefficients and multifactorial numbers by means of the formal derivative operator defined with respect to context-free grammars; consequently, most proofs are carried out by induction rather than by combinatorial arguments. In addition, we introduce the formal derivative operator with respect to matrix grammars and we show that multifactorial numbers can also be generated.

2. BINOMIAL COEFFICIENTS AND MULTIFACTORIAL NUMBERS VIA CONTEXT-FREE GRAMMARS

In this section we consider the family of context-free grammars $G = \{a \to a^{r+1}\}$ and use the formal derivative operator defined on them to prove some properties about binomial coefficients and multifactorial numbers.

Lemma 2.1. *If* $G = \{a \to a\}$, *then* $D^n(a^m) = m^n a^m$ *for all* $m, n \geq 0$.

Proof. We argue by induction on n. By definition 1.2, $D^0(a^m) = a^m$. Assuming that $D^n(a^m) = m^n a^m$, $D^{n+1}(a^m) = D(D^n(a^m)) = D(m^n a^m) = m^n(m a^{m-1} D(a)) = m^{n+1} a^m$. □

The following example states two well known properties regarding binomial coefficients for which we give new proofs based on the context-free grammar of Lemma 2.1. Standard combinatorial proofs are ready available, see for instance [2].

Example 2.1. *For* $n \in \mathbb{N}$:

1. $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^n.$

2. $\displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$

Let G be the grammar $\{a \to a\}$.

1. By Leibniz's formula,

$$D^n(a^2) = \sum_{k=0}^{n} \binom{n}{k} D^{n-k}(a) D^k(a). \tag{1}$$

By taking $m = 1$ and $m = 2$ in Lemma 2.1 we have $D^n(a) = a$ and $D^n(a^2) = 2^n a^2$; substituting in (1):

$$2^n a^2 = \sum_{k=0}^{n} \binom{n}{k} (a)(a).$$

By equating the coefficient of a^2 it follows $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

2. By Leibniz's formula,

$$D^n(aa^{-1}) = \sum_{k=0}^{n} \binom{n}{k} D^{n-k}(a) D^k(a^{-1}). \tag{2}$$

By taking $m = 1$ and $m = -1$ in Lemma 2.1 we have $D^n(a) = a$ and $D^n(a^{-1}) = (-1)^n a^{-1}$; substituting in (2):

$$D^n(a^0) = \sum_{k=0}^{n} \binom{n}{k} (a)((-1)^k a^{-1}).$$

Since $a^0 = 1$, $D^n(a^0) = 0$. Thus $\displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$.

The multifactorial numbers $n!_r$ are given by the recurrence relation,

$$n!_r = n(n-r)!_r \text{ with } (1-r)!_r = \cdots = (-1)!_r = 0!_r = 1.$$

For instance, $(17)!_5 = (17)(12)(7)(2) = 2856$. When $r = 1$ we get factorial numbers i.e., $n!_1 = n!$; when $r = 2$ we get double factorial numbers i.e., $n!_2 = n!!$. The following lemma establishes a connection between the context-free grammar $G = \{a \to a^{r+1}\}$ and multifactorial numbers.

Lemma 2.2. *If* $G = \{a \to a^{r+1}\}$*, then* $D^n(a^m) = \dfrac{(m+(n-1)r)!_r}{(m-r)!_r} a^{m+nr}$ *for* $n \geq 0$ *and* $r \geq 1$.

Proof. We argue by induction on n. Since $D^0(a^m) = a^m$, the lemma is true for $n = 0$. Assuming that $D^n(a^m) = \dfrac{(m+(n-1)r)!_r}{(m-r)!_r} a^{m+n}$, $D^{n+1}(a^m)$ is calculated as follows:

$$\begin{aligned} D^{n+1}(a^m) &= D(D^n(a^m)) \\ &= D\left(\frac{(m+(n-1)r)!_r}{(m-r)!_r} a^{m+nr}\right) \\ &= \frac{(m+(n-1)r)!_r}{(m-r)!_r}(m+nr)a^{m+nr-1}D(a) \\ &= \frac{(m+nr)!_r}{(m-r)!_r} a^{m+nr-1}(a^{r+1}) \\ &= \frac{(m+nr)!_r}{(m-r)!_r} a^{m+(n+1)r}. \end{aligned}$$

Hence $D^n(a^m) = \dfrac{(m+(n-1)r)!_r}{(m-r)!_r} a^{m+n}$. □

Proposition 2.1. *For* m, n, r *integers,* $m, n \geq 0$ *and* $r \geq 1$*, we have*

$$\frac{(2m+(n-1)r)!_r}{(2m-r)!_r} = \sum_{k=0}^{n} \binom{n}{k} \frac{(m+(k-1)r)!_r}{(m-r)!_r} \frac{(m+(n-k-1)r)!_r}{(m-r)!_r}.$$

Proof. Let G be the grammar $\{a \to a^{r+1}\}$. By Leibniz's formula,

$$D^n(a^{2m}) = \sum_{k=0}^{n} \binom{n}{k} D^k(a^m) D^{n-k}(a^m). \tag{3}$$

By Lemma 2.2, $D^n(a^{2m}) = \dfrac{(2m+(n-1)r)!_r}{(2m-r)!_r} a^{2m+nr}$; substituting in (3) we get

$$\frac{(2m+(n-1)r)!_r}{(2m-r)!_r} a^{2m+nr} = \sum_{k=0}^{n} \binom{n}{k} D^k(a^m) D^{n-k}(a^m). \tag{4}$$

On the other hand, by taking k and $n-k$ instead of n, respectively, in Lemma 2.2, we have $D^k(a^m) = \frac{(m+(k-1)r)!_r}{(m-r)!_r}a^{m+kr}$ and $D^{n-k}(a^m) = \frac{(m+(n-k-1)r)!_r}{(m-r)!_r}a^{m+(n-k)r}$. Substituting in the right hand side of (4):

$$\sum_{k=0}^{n}\binom{n}{k}\left[\frac{(m+(k-1)r)!_r}{(m-r)!_r}a^{m+kr}\right]\left[\frac{(m+(n-k-1)r)!_r}{(m-r)!_r}a^{m+(n-k)r}\right]$$
$$=\sum_{k=0}^{n}\binom{n}{k}\frac{(m+(k-1)r)!_r}{(m-r)!_r}\frac{(m+(n-k-1)r)!_r}{(m-r)!_r}a^{2m+nr}.$$

By equating the coefficients of a^{2m+nr} we obtain

$$\frac{(2m+(n-1)r)!_r}{(2m-r)!_r}=\sum_{k=0}^{n}\binom{n}{k}\frac{(m+(k-1)r)!_r}{(m-r)!_r}\frac{(m+(n-k-1)r)!_r}{(m-r)!_r}. \qquad \square$$

Corollary 2.1. *For* $n \geq 0$, $((n+1)r)!_r = r\sum_{k=0}^{n}\binom{n}{k}(kr)!_r((n-k)r)!_r$.

Proof. By taking $r=m$ in Proposition 2.1. $\square$

Corollary 2.2. *For* $n \geq 0$, $(2n)!! = \sum_{k=0}^{n}\binom{n}{k}(2(n-k)-1)!!(2k-1)!!$.

Proof. By taking $r=2$ and $m=1$ in Proposition 2.1, $\square$

A combinatorial proof of Corollary 2.2 can be found in [8], Theorem 3.

3. A FAMILY OF CONTEXT-FREE GRAMMARS

The following proposition shows a connection between the family of context-free grammars $G=\{a_1 \to [a_1\dots a_r]^m a_1\ ;\ a_2 \to [a_1\dots a_r]^m a_2\ ;\dots;\ a_r \to [a_1\dots a_r]^m a_r\}$ and multifactorial numbers.

Proposition 3.1. *If* $G=\{a_1 \to [a_1\dots a_r]^m a_1\ ;\dots;\ a_r \to [a_1\dots a_r]^m a_r\}$, *then for* $r, n \geq 1$*:*

1. $D^n(a_i) = ((n-1)mr+1)!_{mr}[a_1\dots a_r]^{nm}a_i$.
2. $D^n([a_1\dots a_r]^m a_s a_t) = \frac{(nmr+2)!_{mr}}{2}[a_1\dots a_r]^{(n+1)m}a_s a_t$, *with* $s,t \leq r$.

Proof. Here we prove (1); (2) can be similarly proved.

Since $D(a_i) = (0mr+1)!_{mr}[a_1 \cdots a_r]^m a_i$, the proposition is true for $n = 1$. Assuming that $D^n(a_i) = ((n-1)mr+1)!_{mr}[a_1 a_2 \ldots a_r]^{nm} a_i$, $D^{n+1}(a_i)$ is calculated as follows

$$\begin{aligned}
&D\left(((n-1)mr+1)!_{mr}[a_1 a_2 \cdots a_r]^{nm} a_i\right)\\
&= ((n-1)mr+1)!_{mr} D([a_1 \cdots a_r]^{nm} a_i)\\
&= ((n-1)mr+1)!_{mr}(D([a_1 \cdots a_r]^{nm}) a_i + [a_1 \cdots a_r]^{nm} D(a_i))\\
&= ((n-1)mr+1)!_{mr}(nm[a_1 \cdots a_r]^{nm-1} D(a_1 \cdots a_r) a_i + [a_1 \cdots a_r]^{nm}[a_1 \cdots a_r]^m a_i)\\
&= ((n-1)mr+1)!_{mr}(nm[a_1 \cdots a_r]^{nm-1} D(a_1 \cdots a_r) + [a_1 \cdots a_r]^{(n+1)m}) a_i.
\end{aligned}$$

By Proposition 1.1, we have

$$\begin{aligned}
D(a_1 \cdots a_r) &= D(a_1) a_2 \cdots a_r + \cdots + D(a_r) a_1 \cdots a_{r-1}\\
&= [a_1 \cdots a_r]^m a_1 \cdots a_r + \cdots + [a_1 \cdots a_r]^m a_1 \cdots a_r\\
&= r[a_1 a_2 \cdots a_r]^{m+1}.
\end{aligned}$$

Therefore, $D^{n+1}(a_i)$ is given by

$$\begin{aligned}
&((n-1)mr+1)!_{mr}(nm[a_1 \cdots a_r]^{nm-1} D(a_1 \cdots a_r) + [a_1 \cdots a_r]^{(n+1)m}) a_i\\
&= ((n-1)mr+1)!_{mr}(nm[a_1 \cdots a_r]^{nm-1}(r[a_1 \cdots a_r]^{m+1}) + [a_1 \cdots a_r]^{(n+1)m}) a_i\\
&= ((n-1)mr+1)!_{mr}(nmr[a_1 \cdots a_r]^{(n+1)m} + [a_1 \cdots a_r]^{(n+1)m}) a_i\\
&= ((n-1)mr+1)!_{mr}(nmr+1)[a_1 \cdots a_r]^{(n+1)m} a_i\\
&= (nmr+1)!_{mr}[a_1 \cdots a_r]^{(n+1)m} a_i.
\end{aligned}$$

Hence, $D^n(a_i) = ((n-1)mr+1)!_{mr}[a_1 \ldots a_r]^{nm} a_i$. □

By using Leibniz's formula and Proposition 3.1 we can get the following result

$$((n-1)mr+2)!_{mr} = \sum_{k=0}^{n} \binom{n}{k} ((n-k-1)mr+1)!_{mr}((k-1)mr+1)!_{mr}.$$

This can also be obtained as a particular case of Proposition 2.1.

4. FORMAL DERIVATIVE OPERATOR WITH RESPECT TO MATRIX GRAMMARS

Matrix grammars were introduced in [1] as a generalization of standard context-free grammars. Following [10], a matrix grammar is defined as follows.

Definition 4.1. *A matrix grammar G is a quadruple $G = (V_N, V_T, S, M)$, such that:*

1. *V_N is a finite set of objects, called variables.*
2. *V_T is a finite set of objects, called terminal symbols.*
3. *$S \in V_N$ is a symbol called the start variable.*

4. *M is a finite set of sequences of the form $[A_1 \to x_1, \dots, A_n \to x_n]$, with $A_i \in (V_N \cup V_T)^+$ and $x_i \in (V_N \cup V_T)^*$, $i = 1, \dots, n$.*

G is a matrix grammar of type k if and only if each $[A_1 \to x_1, \dots, A_n \to x_n]$ in M is a grammar of type k in the Chomsky hierarchy [10]. For instance, the grammars of type 2 $g_1 = \{a \to a + b\ ;\ b \to b\}$ and $g_2 = \{a \to a\ ;\ b \to a - b\}$ can be unified in the matrix grammar of type 2, $G = \{[a \to a + b\ ;\ b \to b]\ ,\ [a \to a\ ;\ b \to a - b]\}$. Matrix grammars of type 2 are named context-free matrix grammars in [16], and we use that terminology henceforth.

We now proceed to extend the definition of the formal derivative operator D (Definition 1.1) to context-free matrix grammars.

Definition 4.2. *Let $G = \{g_1, \dots, g_n\}$ be a context-free matrix grammar, i.e., each g_i is a context-free grammar: $g_i = [a_1 \to w_{i1}\ ; \dots ; a_m \to w_{im}]$, each w_{ik} being a formal function. If $a_j \to w_{ij}$ is a production in g_i, then $D_i(a_j) = w_{ij}$. In other cases, $D_i(a_j) = 0$. Moreover, $D_{ik}(a_j) = D_i(D_k(a_j))$, $D_{ik}^{n+1}(a_j) = D_{ik}(D_{ik}^n(a_j))$ and $D_{ik}^0(a_j) = a_j$ for any formal function a_j.*

The following example shows that $D_{ik}(w)$ does not necessarily agrees with $D_{ki}(w)$.

Example 4.1. *Given $G = \big\{[a \to ab^2\ ;\ b \to b^2]\ ,\ [a \to a^2b\ ;\ b \to b]\big\}$, we calculate $D_{21}(a)$ and $D_{12}(a)$.*

Here $g_1 = \{a \to ab^2\ ;\ b \to b^2\}$ and $g_2 = \{a \to a^2b\ ;\ b \to b\}$, hence

$$\begin{aligned} D_{12}(a) &= D_1(D_2(a)) \\ &= D_1(a^2b) \\ &= D_1(a^2)b + a^2D_1(b) \\ &= 2abD_1(a) + a^2D_1(b) \\ &= 2a^2b^3 + a^2b^2. \end{aligned} \qquad \begin{aligned} D_{21}(a) &= D_2(D_1(a)) \\ &= D_2(ab^2) \\ &= D_2(a)b^2 + aD_2(b^2) \\ &= D_2(a)b^2 + 2abD_2(b) \\ &= a^2b^3 + 2ab^2. \end{aligned}$$

As the following results show, we can also use matrix grammars for generating multifactorial numbers. For this purpose we need the following lemma.

Lemma 4.1. *For the grammar $G = \{[a \to a; b \to b], [a \to a^rb; b \to a^{r-1}b^2]\}$, we have*

1. *$D_1(a^mb^n) = (m + n)a^mb^n$.*
2. *$D_2(a^mb^n) = (m + n)a^{m+r-1}b^{n+1}$.*

Proof. Since $g_1 = \{a \to a; b \to b\}$, we have

$$\begin{aligned} D_1(a^mb^n) &= ma^{m-1}b^nD_1(a) + na^mb^{n-1}D_1(b) \\ &= ma^{m-1}b^n(a) + na^mb^{n-1}(b) \\ &= ma^mb^n + na^mb^n. \end{aligned}$$

On the other hand, since $g_2 = \{a \to a^r b; b \to a^{r-1}b^2\}$ we get:

$$\begin{aligned} D_2(a^m b^n) &= m a^{m-1} b^n D_2(a) + n a^m b^{n-1} D_2(b) \\ &= m a^{m-1} b^n (a^r b) + n a^m b^{n-1} (a^{r-1} b^2) \\ &= m a^{m+r-1} b^{n+1} + n a^{m+r-1} b^{n+1}. \end{aligned}$$

Hence $D_1(a^m b^n) = (m+n)a^m b^n$ and $D_2(a^m b^n) = (m+n)a^{m+r-1}b^{n+1}$. □

Proposition 4.1. *If* $G = \{[a \to a; b \to b], [a \to a^r b; b \to a^{r-1}b^2]\}$*, then for all* $n \geq 0$*:*

1. $D_{12}^n(a) = (nr+1)!_r((n-1)r+1)!_r a^{nr-(n-1)} b^n$.
2. $D_{21}^n(a) = ((n-1)r+1)!_r^2 a^{nr-(n-1)} b^n$.
3. $D_{12}^n(b) = (nr+1)!_r((n-1)r+1)!_r a^{nr-n} b^{n+1}$.
4. $D_{21}^n(b) = ((n-1)r+1)!_r^2 a^{nr-n} b^{n+1}$.

Proof. Here we prove (1); the other results can be similarly proved.

We argue by induction on n. Since $D_{12}^0(a) = a$, the proposition is true for $n = 0$. Assuming that $D_{12}^n(a) = (nr+1)!_r((n-1)r+1)!_r a^{nr-(n-1)} b^n$, $D_{12}^{n+1}(a) = D_{12}(D_{12}^n(a))$ can be expressed as

$$D_{12}^{n+1}(a) = (nr+1)!_r((n-1)r+1)!_r D_1(D_2(a^{nr-(n-1)}b^n)). \tag{5}$$

By Lemma 4.1, $D_2(a^{nr-(n-1)}b^n) = (nr+1)a^{(n+1)r-n}b^{n+1}$. Substituting in (5),

$$D_{12}^{n+1}(a) = (nr+1)!_r((n-1)r+1)!_r(nr+1)D_1(a^{(n+1)r-n}b^{n+1}). \tag{6}$$

By Lemma 4.1, $D_1(a^{(n+1)r-n}b^{n+1}) = ((n+1)r+1)a^{(n+1)r-n}b^{n+1}$. Substituting in (6), and using the identities $((n+1)r+1)!_r = ((n+1)r+1)(nr+1)!_r$ and $(nr+1)!_r = (nr+1)((n-1)r+1)!_r$ we conclude

$$D_{12}^{n+1}(a) = (nr+1)!_r((n-1)r+1)!_r(nr+1)((n+1)r+1)a^{(n+1)r-n}b^{n+1} \tag{7}$$

Thus $D_{12}^{n+1}(a) = ((n+1)r+1)!_r(nr+1)!_r a^{(n+1)r-n}b^{n+1}$. □

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