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General Quasilinear Problems Involving $(p_1(x), p_2(x))$-Laplace Type Equation With Robin Boundary

H. Belaouidel[1] , A. Ourraoui[2] and N. Tsouli[3]

[1]*Faculty of Sciences, University Mohamed I, Oujda, Morocco,*

[2,3]*Department of Mathematics and Computer Science, Faculty of Sciences, University Mohamed I, Oujda, Morocco,*

Abstract

In this article, we study the existence of infinitely many nontrivial solutions for the class of $(p_1(x), p_2(x))$-laplacian elliptic problem involving concave-convex nonlinearities with Robin boundary, our method is based on the Ljusternik-Schnirelman principle in Banach spaces.

Keywords:$p(x)$-laplacian, Ljusternik-Schnirelman principle; multiple solutions, critical point theory
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1. INTRODUCTION

This paper is concerned with the existence and multiplicity of solutions for the following equation

$$\begin{cases} -\Delta_{p_1(x)}u - \Delta_{p_2(x)}u = h(x)|u|^{r(x)-2}u & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial n_{p_1 p_2}} + a(x)|u|^{p_1(x)-2}u + b(x)|u|^{p_2(x)-2}u = \lambda g(x)|u|^{\alpha(x)-2}u & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded subset of $\mathbb{R}^N$ $(N \geq 2)$, with smooth boundary, by $-\Delta_{p_i(x)}, i = 1, 2$ we denote the $p_i(x)$-Laplacian differential operator $-\Delta_{p_i(x)}u = -div(\nabla u|^{p_i(x)-2}\nabla u)$ for all $u \in W^{1,p_i(x)}(\Omega), i = 1, 2$ and $p_i \in C_+(\overline{\Omega})$ with

$$1 < p_i^- := \inf_{\overline{\Omega}} p_i(x) \leq p_i^+ := \sup_{\overline{\Omega}} p_i(x) < N$$

and $p_i(x) < p_i^*(x)$ where

$$p_i^*(x) = \begin{cases} \frac{Np_i(x)}{N-p_i(x)} & \text{if } p_i(x) < N, \\ +\infty & \text{if } p_i(x) \geq N \end{cases}$$

for any $x \in \overline{\Omega}$ and $i = 1, 2.$ and $1 < \alpha^- := \inf_{x \in \overline{\Omega}} \alpha(x) \leq \alpha^+ := \sup_{x \in \overline{\Omega}} \alpha(x)$,

$$p_i^\partial(x) = (p_i(x))^\partial = \begin{cases} \frac{(N-1)p_i(x)}{Np_i(x)} & \text{if } p_i(x) < N, \\ +\infty & \text{if } p_i(x) \geq N, \end{cases}$$

and $1 < r^- := \inf_{x \in \overline{\Omega}} r(x) \leq r^+ := \sup_{x \in \overline{\Omega}} r(x)$.

In the boundary condition $\frac{\partial u}{\partial n_{p_1 p_2}}$ denotes the conormal derivative of u defined by extension of the map: $C^1(\overline{\Omega}) \ni u \to \left[|\nabla u|^{p_1(x)-2}\nabla u + |\nabla u|^{p_2(x)-2}\nabla u\right] \frac{\partial u}{\partial n}$ with $n(.)$ being the outward unit normal on $\partial\Omega$. This generalized normal derivative is dictated by the nonlinear Greens identity (see Gasinski-Papageorgiou ([[24], p.210]). It is also used by Lieberman [26].

Recently, the study of differential equations and variational problems involving $p(x)$-growth conditions have been extensively investigated and received much attention because they can be presented as models for many physical phenomena which arouse in the study of elastic mechanics [38], electro-rheological fluid dynamics [33, 32] and image processing [10], electrical resistivity and polycrystal plasticity [6, 7] and continuum mechanics [5] etc, for an overview of this subject, and for more details we refer readers to [16] and [9, 15] and the references therein. The existence of nontrivial solutions to nonlinear elliptic boundary value problems has been extensively studied by many researchers [1, 2, 12, 19, 23, 28] and references therein.

It is known that the extension $p(x)$-Laplace operator possesses more complicated structure than the p-Laplacian. For example, it is inhomogeneous and usually it does not have the so-called first eigenvalue, since the infimum of its spectrum is zero.

In recent years, the kind of problems of the form (1.1) with convex and concave nonlinearities have been extensively studied by many authors, using various methods.

We refer the reader to the celebrate paper of Ambrosetti-Brezis-Cerami [3] and the work recently studied by Bonder [8], Mashiyev,[29] , Liu [27] , Chen [11] with their references therein.

Our direction is based on the paper [25] where authors have been considered the following nonlinear elliptic equation with Robin Boundary

$$-\Delta_p u = \lambda |u|^{p-2}u + a(x)|u|^{q-2}u \quad \text{in } \Omega, \tag{1.2}$$

$$|\nabla u|^{p-2}\frac{\partial u}{\partial \eta} + b(x)|u|^{p-2}u = \mu\rho(x)|u|^{r-2}u \quad \text{on } \partial\Omega, \tag{1.3}$$

where $\frac{\partial u}{\partial \eta}(x)$ denotes the outward unit normal at $x \in \partial\Omega$, λ, μ are parameters, $\mu > 0$, $a : \Omega \to \mathbb{R}$, $b, \rho : \partial\Omega \to \mathbb{R}$ are essentially bounded functions, with $b(x) \geq 0$ and $meas\Big\{x \in \partial\Omega : b(\cdot) > 0\Big\} > 0$. and provided the existence results concerning positive solutions to (1.2)-(1.3) when q is either subcritical or critical, r is subcritical and $\lambda \leq \lambda_1$, whith λ_1 is the first eigenvalue of the associated eigenvalue problem.

The aim of this work is to extend the results obtained in [8] to the more general problems (1.1). Precisely, we prove the existence and multiplicity of nontrivial solutions for the non-linear elliptic problem (1.1).

We assume λ is postive parameter and a, b, h, g are continuous positive functions defined on $\mathbb{R}^N$ such that

$(\mathbf{A}_1)$ $a(\cdot) \geq 0, b(\cdot) \geq 0$ on $\partial\Omega$, $meas\Big\{x \in \partial\Omega : a(\cdot) > 0, b(\cdot) > 0\Big\} > 0$,

$(\mathbf{A}_2)$ $h(\cdot) \geq 0$ on Ω, $g(\cdot) \geq 0$ on $\partial\Omega$, $meas\Big\{x \in \Omega : h(\cdot) > 0\Big\} > 0$, and $meas\Big\{x \in \partial\Omega : g(\cdot) > 0\Big\} > 0$.

Denote

$$p_m(x) := \min(p_1(x), p_2(x)),\ p_M(x) := \max(p_1(x), p_2(x)),$$

for all $x \in \overline{\Omega}$, and

$$p_m^- := \min(p_1^-, p_2^-),\ p_M^+ := \max(p_1^+, p_2^+).$$

Depending on the relative ordering of the exponents $p_i(x)$, $r(x)$, $p_i^{\partial}(x)$ and $\alpha(x)$.

The main result of this paper can state as follow:

Theorem 1.1. *Assume that* $(\boldsymbol{A}_1) - (\boldsymbol{A}_2)$ *hold. In addition, if we assume the following conditions:*

$$1 < p_M^+ < \min\{r^-, \alpha^-\}.$$

Then the problem (1.1) *has infinitely many solutions for every* $\lambda > 0$.

The remainder of this paper is organized as follows, in Section 2 we introduce some technical results and required hypotheses in order to solve our problem, in Section 3 we prove the main results of this work.

2. PRELIMINARIES

In the sequel, let $p(x) \in C_+(\overline{\Omega})$, where

$$C_+(\overline{\Omega}) = \left\{h : h \in C(\overline{\Omega}), h(x) > 1 \text{ for any } x \in \overline{\Omega}\right\}.$$

The variable exponent Lebesgue space is defined by

$$L^{p(x)}(\Omega) = \{u : \Omega \to \mathbb{R} \text{ measurable and } \int_\Omega |u(x)|^{p(x)}\,dx < +\infty\}$$

furnished with the Luxemburg norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\{\sigma > 0 : \int_\Omega |\frac{u(x)}{\sigma}|^{p(x)}\,dx \leq 1\},$$

Remark 2.1. Variable exponent Lebesgue spaces resemble to classical Lebesgue spaces in many respects, they are separable Banach spaces and the Hölder inequality holds. The inclusions between Lebesgue spaces are also naturally generalized, that is, if $0 < mes(\Omega) < \infty$ and p, q are variable exponents such that $p(x) < q(x)$ a.e. in Ω, then there exists a continuous embedding $L^{q(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$.

The variable exponent Sobolev space is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}$$

equipped with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}.$$

Proposition 2.2. *[20, 21] The spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are separable, uniformly convex, reflexive Banach spaces. The conjugate space of $L^{p(x)}(\Omega)$ is $L^{q(x)}(\Omega)$, where $q(x)$ is the conjugate function of $p(x)$; i.e.,*

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1,$$

for all $x \in \Omega$. For $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$ we have

1. $$\Big|\int_\Omega u(x)v(x)dx\Big| \leq \Big(\frac{1}{p^-} + \frac{1}{q^-}\Big)|u|_{p(x)}|v|_{q(x)}.$$
2. *If $p_1, p_2 \in C_+(\overline{\Omega})$, $p_1(x) \leq p_2(x)$ for any $x \in \overline{\Omega}$ $L^{p_2(x)} \hookrightarrow L^{p_1(x)}$ and the embedding is continuous.*

Moreover, if $h_1, h_2, h_3 : \overline{\Omega} \to (1, \infty)$ are Lipschitz continuous functions such that $\frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_3} = 1$, then for any $u \in L^{h_1(x)}(\Omega)$, $v \in L^{h_2(x)}(\Omega)$, $w \in L^{h_3(x)}(\Omega)$, the following inequality holds see [19, Proposition 2.5]

$$\int_\Omega |uvw|dx \leq \left(\frac{1}{h_1^-} + \frac{1}{h_2^-} + \frac{1}{h_3^-}\right)|u|_{h_1(x)}|v|_{h_2(x)}|w|_{h_3(x)}. \tag{2.1}$$

Proposition 2.3. *[17] Let $p(x)$ and $q(x)$ be measurable functions such that $p(x) \in L^\infty(\Omega)$ and $1 \leq p(x)q(x) \leq \infty$, for a.e. $x \in \Omega$. Let $u \in L^{q(x)}(\Omega),\ u \neq 0$. Then*

$$|u|_{p(x)q(x)} \leq 1 \Rightarrow |u|^{p^+}_{p(x)q(x)} \leq ||u|^{p(x)}|_{q(x)} \leq |u|^{p^-}_{p(x)q(x)},$$
$$|u|_{p(x)q(x)} \geq 1 \Rightarrow |u|^{p^-}_{p(x)q(x)} \leq ||u|^{p(x)}|_{q(x)} \leq |u|^{p^+}_{p(x)q(x)}.$$

In particular if $p(x) = p$ is a constant, then

$$|\,|u|^p|_{q(x)} = |u|^p_{pq(x)}.$$

Proposition 2.4. *[20, 21] Assume that the boundary of Ω possesses the cone property and $p, r \in C_+(\overline{\Omega})$ such that $r(x) \leq p^*(x)$ $(r(x) < p^*(x))$ for all $x \in \overline{\Omega}$, then there is a continuous (compact) embedding*

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega),$$

Proposition 2.5. *[14] For $p \in C_+(\overline{\Omega})$ and such $r \in C_+(\partial\Omega)$ that $r(x) \leq p^\partial(x)$ $(r(x) < p^\partial(x))$ for all $x \in \overline{\Omega}$, there is a continuous (compact) embedding*

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\partial\Omega).$$

Proposition 2.6 ([13], Theorem 2.1)**.** *For any $u \in W^{1,p(x)}(\Omega)$, let*

$$\| u \|_\partial := |u|_{L^{p(x)}(\partial\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}.$$

Then $\| u \|_\partial$ is a norm on $W^{1,p(x)}(\Omega)$ which is equivalent to

$$\|u\|_{W^{1,p(x)}(\Omega)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}.$$

Let us now consider the weighted variable exponent Lebesgue space. Let a measurable function $c : \Omega \to \mathbb{R}$. Define

$$L^{p(x)}_{c(x)}(\Omega) = \{u : \Omega \to \mathbb{R} \text{ measurable and } \int_\Omega c(x)|u(x)|^{p(x)}\,dx < +\infty\}$$

furnished with the Luxemburg norm

$$|u|_{L^{p(x)}_{c(x)}(\Omega)} = |u|_{(p(x),c(x))} = \inf\{\sigma > 0 : \int_\Omega c(x)|\frac{u(x)}{\sigma}|^{p(x)}\,dx \leq 1\}.$$

Then, $L^{p(x)}_{c(x)}(\Omega)$ is a Banach space.

Proposition 2.7 ([18])**.** *Set* $\rho(u) = \int_\Omega b(x)|u(x)|^{p(x)}dx$. *For* $u \in L^{p(x)}_{b(x)}(\Omega)$, *we have*

(i) $|u|_{(p(x),b(x))} < 1(= 1; > 1) \Leftrightarrow \rho(u) < 1(= 1; > 1)$;

(ii) If $|u|_{(p(x),b(x))} < 1 \Rightarrow |u|^{p^+}_{(p(x),b(x))} \leq \rho(u) \leq |u|^{p^-}_{(p(x),b(x))}$;

(iii) If $|u|_{(p(x),b(x))} > 1 \Rightarrow |u|^{p^-}_{(p(x),b(x))} \leq \rho(u) \leq |u|^{p^+}_{(p(x),b(x))}$.

Proposition 2.8 ([18])**.** *Assume that the boundary of* Ω *possesses the cone property and* $p \in C(\overline{\Omega})$. *Suppose that* $h \in L^{\gamma(x)}(\Omega)$, $h(x) > 0$ *for* $x \in \Omega$, $\gamma \in C(\overline{\Omega})$ *and* $\gamma^- > 1$, $\gamma_0^- \leq \gamma_0(x) \leq \gamma_0^+ (1/\gamma(x) + 1/\gamma_0(x) = 1)$. *If* $r \in C(\overline{\Omega})$ *and*

$$1 < r(x) < \frac{\gamma(x) - 1}{\gamma(x)} p^*(x) \quad \text{for all } x \in \overline{\Omega} \tag{2.2}$$

or

$$1 < \gamma(x) < \frac{N\gamma(x)}{N\gamma(x) - r(x)(N - p(x))},$$

then the embedding from $W^{1,p(x)}(\Omega)$ *to* $L^{r(x)}_{h(x)}(\Omega)$ *is compact. Moreover, there is a constant* $c_0 > 0$ *such that the inequality*

$$\int_\Omega h(x)|u|^{r(x)}dx \leq c_0(\|u\|^{r^-} + \|u\|^{r^+}) \tag{2.3}$$

holds.

Remark 2.9. Let $u \in W^{1,p(x)}(\partial\Omega)$. Then, by Proposition 2.8, there are positive constants $c_1,\ c_2 > 0$ such that the following inequalities hold:

$$\int_\Omega h(x)|u|^{r(x)}dx \leq \begin{cases} c_1\|u\|^{r^+} & \text{if } \|u\| > 1, \\ c_2\|u\|^{r^-} & \text{if } \|u\| < 1. \end{cases}$$

$$L^{p(x)}_{c(x)}(\partial\Omega) = \{u : \partial\Omega \to \mathbb{R} \text{ measurable and } \int_{\partial\Omega} c(x)|u(x)|^{p(x)}\, d\sigma x < +\infty\}$$

furnished with the Luxemburg norm

$$|u|_{L^{p(x)}_{c(x)}(\partial\Omega)} = |u|_{(p(x),c(x))} = \inf\{\sigma > 0 : \int_{\partial\Omega} c(x)|\frac{u(x)}{\sigma}|^{p(x)}\, d\sigma x \leq 1\},$$

Proposition 2.10 ([18])**.** *Set* $\rho(u) = \int_{\partial\Omega} b(x)|u(x)|^{p(x)}d\sigma x$. *For* $u \in L^{p(x)}_{b(x)}(\partial\Omega)$, *we have*

(i) $|u|_{(p(x),b(x))} < 1(= 1; > 1) \Leftrightarrow \rho(u) < 1(= 1; > 1)$;

(ii) *If* $|u|_{(p(x),b(x))} < 1 \Rightarrow |u|^{p^+}_{(p(x),b(x))} \leq \rho(u) \leq |u|^{p^-}_{(p(x),b(x))}$;

(iii) *If* $|u|_{(p(x),b(x))} > 1 \Rightarrow |u|^{p^-}_{(p(x),b(x))} \leq \rho(u) \leq |u|^{p^+}_{(p(x),b(x))}$.

Proposition 2.11 ([30])**.** *Assume that the boundary of Ω possesses the cone property and $p \in C(\overline{\Omega})$. Suppose that $g \in L^{\gamma(x)}(\partial\Omega)$, $g(x) > 0$ for $x \in \Omega$, $\gamma \in C(\overline{\Omega})$ and $\gamma^- > 1$, $\gamma_0^- \leq \gamma_0(x) \leq \gamma_0^+ (1/\gamma(x) + 1/\gamma_0(x) = 1)$. If $\alpha \in C(\overline{\Omega})$ and*

$$1 < \alpha(x) < \frac{\gamma(x) - 1}{\gamma(x)} p^*_\partial(x) \quad \textit{for all } x \in \overline{\Omega} \tag{2.4}$$

or

$$1 < \gamma(x) < \frac{N\gamma(x)}{N\gamma(x) - \alpha(x)(N - p(x))},$$

then the embedding from $W^{1,p(x)}(\Omega)$ to $L^{\alpha(x)}_{g(x)}(\partial\Omega)$ is compact. Moreover, there is a constant $c_3 > 0$ such that the inequality

$$\int_{\partial\Omega} g(x)|u|^{\alpha(x)} d\sigma x \leq c_3(\|u\|^{\alpha^-} + \|u\|^{\alpha^+}) \tag{2.5}$$

holds.

Remark 2.12. Let $u \in W^{1,p(x)}(\Omega)$. Then, by Proposition 2.11, there are positive constants $c_4,\ c_5 > 0$ such that the following inequalities hold*:*

$$\int_{\partial\Omega} g(x)|u|^{\alpha(x)} d\sigma x \leq \begin{cases} c_4\|u\|^{\alpha^+} & \text{if } \|u\| > 1, \\ c_5\|u\|^{\alpha^-} & \text{if } \|u\| < 1. \end{cases}$$

Define:

$$p^\partial_{\alpha(x)}(x) := \frac{\alpha(x) - 1}{\alpha(x)} p^\partial(x),$$

for all $x \in \partial\Omega$.

Proposition 2.13 ([14]Theorem 2.1.)**.** *Assume that the boundary of Ω possesses the cone property and $p \in C(\overline{\Omega})$ with $p^- > 1$. Suppose that $g \in L^{\alpha(x)}(\partial\Omega)$, with $\alpha \in C(\partial\Omega)$ with $\alpha(x) > \frac{p^\partial(x)}{p^\partial(x)-1}$ for all $x \in \partial\Omega$. If $q \in C(\partial\Omega)$ and $1 < q(x) < p^\partial_{\alpha(x)}(x)$ for all $x \in \partial\Omega$.*

Then, there exists a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}_{g(x)}(\Omega)$. In particular, there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q_0(x)}(\Omega)$ where $1 \leq q_0(x) < p^\partial(x)$, for all $x \in \partial\Omega$.

Now, for any $u \in X_i := W^{1,p_i(x)}(\Omega)$ define

$$\|u\|_i := \inf\Big\{\sigma > 0 : \int_\Omega \big(|\frac{\nabla u(x)}{\sigma}|^{p_i(x)}dx + \int_{\partial\Omega} c(x)|\frac{u(x)}{\sigma}|^{p_i(x)}\big)d\sigma_x \leq 1\Big\}.$$

Where $c \in L^\infty(\Omega)$ and $d\sigma_x$ is the measure on the boundary $\partial\Omega$.Then by Proposition 2.6, $\|.\|_i$ is also a norm on $W^{1,p_i(x)}(\Omega)$ which is equivalent to $\|.\|_{W^{1,p_i(x)}(\Omega)}$ and $\|.\|_\partial$. Now, we introduce the modular $\rho : X_i \to \mathbb{R}$ defined by

$$\rho(u) = \int_\Omega |\nabla u|^{p_i(x)}dx + \int_{\partial\Omega} c(x)|u(x)|^{p_i(x)}d\sigma_x$$

for all $u \in X_i$. Here, we give some relations between the norm $||.||_i$ and the modular ρ.

Proposition 2.14 ([20])**.** *For $u \in X$ we have*

(i) $\|u\|_i < 1(= 1; > 1) \Leftrightarrow \rho(u) < 1(= 1; > 1)$;

(ii) If $\|u\|_i < 1 \Rightarrow \|u\|_i^{p_i^+} \leq \rho(u) \leq \|u\|_i^{p_i^-}$;

(iii) If $\|u\|_i > 1 \Rightarrow \|u\|_i^{p_i^-} \leq \rho(u) \leq \|u\|_i^{p_i^+}$.

Definition 2.15. Assume that spaces E, F are Banach spaces, we define the norm on the space $E \cap F$ as $\|u\| = \|u\|_E + \|u\|_F$.

We denote

$$X := X_1 \cap X_2,$$

Remark 2.16. Note that :

1. The weak solutions of problem (1.1) are considered in the generalized Sobolev space X equipped with the norm

$$\|u\| = \|u\|_{p_1} + \|u\|_{p_2}.$$

We observe that X endowed with the above norm is a separable and reflexive Banach space.

2. By Proposition 2.4 and Proposition 2.5 there is are continuous (compact) embeddings

$$X \hookrightarrow L^{p_M(x)}(\Omega) \ and \ \ X \hookrightarrow L^{p_M(x)}(\partial\Omega).$$

Definition 2.17. *We say that $u \in X$ is weak solution of* (1.1) *if*

$$\begin{aligned}
&\int_{\Omega} |\nabla u|^{p_1(x)-2}\nabla u \nabla v dx + \int_{\Omega} |\nabla u|^{p_2(x)-2}\nabla u \nabla v dx \\
&+ \int_{\partial\Omega} a(x)|u|^{p_1(x)-2}uv dx + \int_{\partial\Omega} b(x)|u|^{p_2(x)-2}uv dx \\
&= \int_{\Omega} h(x)|u|^{r(x)-2}uv dx + \lambda \int_{\partial\Omega} g(x)|u|^{\alpha(x)-2}uv dx. \qquad (2.6)
\end{aligned}$$

for all $v \in X$.

The functional associated to (1.1) is given by

$$I(u) = I_1(u) + I_2(u) - I_3(u), \qquad (2.7)$$

where

$$I_1(u) = \int_{\Omega} \frac{1}{p_1(x)}|\nabla u|^{p_1(x)} dx + \int_{\partial\Omega} \frac{1}{p_1(x)} a(x)|u|^{p_1(x)} d\sigma(x), \qquad (2.8)$$

$$I_2(u) = \int_{\Omega} \frac{1}{p_2(x)}|\nabla u|^{p_2(x)} + \int_{\partial\Omega} \frac{1}{p_2(x)} b(x)|u|^{p_2(x)} d\sigma(x), \qquad (2.9)$$

$$I_3(u) = \int_{\Omega} \frac{1}{r(x)} h(x)|u|^{r(x)} dx + \lambda \int_{\partial\Omega} \frac{1}{\alpha(x)} g(x)|u|^{\alpha(x)} d\sigma(x). \qquad (2.10)$$

The functional I is of class $C^1(X, \mathbb{R})$ and

$$I'(u).v = I_1'(u)v + I_2'(u)v - I_3'(u)v,$$

for ever $v \in X$,

where

$$I_1'(u)v = \int_{\Omega} |\nabla u|^{p_1(x)-2}\nabla u \nabla v dx + \int_{\partial\Omega} a(x)|u|^{p_1(x)-2}uv d\sigma(x), \qquad (2.11)$$

$$I_2'(u)v = \int_{\Omega} |\nabla u|^{p_2(x)-2}\nabla u \nabla v dx + \int_{\partial\Omega} b(x)|u|^{p_2(x)-2}uv d\sigma(x), \qquad (2.12)$$

$$I_3'(u)v = \int_{\Omega} h(x)|u|^{r(x)-2}uv dx + \lambda \int_{\partial\Omega} g(x)|u|^{\alpha(x)-2}uv d\sigma(x). \qquad (2.13)$$

Then, we know that a weak solution of (1.1) corresponds to critical point of the functional I.

As in [37] we have the followings results

Lemma 2.18. *For $i = 1, 2, 3$*

1. *$I_i' : X \to X^*$ is a continuous bounded and strictly monotone operator;*
2. *$I_i' : X \to X^*$is of type (S_+), namely $u_n \rightharpoonup u$ and $\limsup_{n\to+\infty} L(u_n)(u_n - u) \leq 0$ implies $u_n \to u$;*
3. *$I_i' : X \to X^*$ is a homeomorphism.*

Remark 2.19. Through the Lemma 2.18, we can conclude that the operator I' is a strictly monotone, continuous bounded homeomorphism and is of type (S_+).

The main tools used in proving Theorem 1.1 are the well known the Ljusternik-Schnirelman principle in Banach spaces.

Let Y be an Banach space and $\Sigma := \{A \subseteq Y\backslash\{0\} : A \text{ is closed and } A = -A\}$. The genus of a set $A \in \Sigma$ is defined by

$$\gamma(A) := \min\{n \in \mathbb{N} : \exists\varphi \in C(A, \mathbb{R}^n\backslash\{0\}) \text{ with } \varphi(x) = -\varphi(-x)\}.$$

Theorem 2.20. *[4] Suppose that $I : Y \to \mathbb{R}$ is an even $C^1(Y, \mathbb{R})$ function such that:*

(i) I satisfies the Palais-Smale condition.

(ii) $I(u) > 0$ if $0 < \|u\| < \varrho$ and $I(u) \geq c > 0$ if $\|u\| = \varrho$, for some $\varrho > 0$.

(iii) There exists a subspace $Y_m \subseteq Y$ of dimension m and a compact subset $A_m \subseteq Y_m$ with $I < 0$ on A_m such that 0 lies in a bounded component of $Y_m - A_m$ in Y_m.

Let $\Gamma := \{h \in C(Y, Y) : h(0) = 0, h \text{ is an odd homeomorhism, } I(h(B_1)) \geq 0\}$, $K_m := \{K \subseteq Y : K \text{ is compact,} K = -K, \gamma(K \cap h(\partial B_1)) \geq m \text{ for every } h \in \Gamma\}$, where B_1 denotes the unit ball of Y. Then

$$c_m := \inf_{K\in K_m} \max_{u\in K} I(u)$$

is a critical value of I with $0 < c < c_m \leq c_{m+1} < +\infty$. Furthermore, if $c_m = c_{m+1} = \cdots = c_{m+n}$, then $\gamma(K_{c_m}) \geq n + 1$, where $K_{c_m} := \{u \in X : I'(u) = 0, I(u) = c_m\}$.

3. PROOF OF MAIN RESULT

Proof of Theorem 1.1

Proof. I satisfies the Palais-Smale condition on X i.e there exists a sequence $(u_n) \subset X$ which satisfies the :

$$I(u_n) \to d \text{ and } I'(u_n) \to 0 \text{ as } n \to +\infty. \tag{3.1}$$

possesses a convergent subsequence in X.

Let (u_n) be a sequence in X which satisfies (3.1). We prove that (u_n) possesses a convergent subsequence in X.

We assume by contradiction $\|u_n\| \to \infty$ as $n \to +\infty$. By (2.8)-(2.10), (3.1), Remark 2.9 and Remark 2.12, we have

$$\begin{aligned} I(u_n) &= \int_\Omega \frac{1}{p_1(x)}|\nabla u_n|^{p_1(x)}dx + \int_{\partial\Omega} \frac{1}{p_1(x)}a(x)|u_n|^{p_1(x)}d\sigma(x) \\ &+ \int_\Omega \frac{1}{p_2(x)}|\nabla u_n|^{p_2(x)} + \int_{\partial\Omega} \frac{1}{p_2(x)}b(x)|u_n|^{p_2(x)}d\sigma(x) \\ &- \int_\Omega \frac{1}{r(x)}h(x)|u_n|^{r(x)}dx - \lambda\int_{\partial\Omega} \frac{1}{\alpha(x)}g(x)|u_n|^{\alpha(x)}d\sigma(x) \\ &\geq \frac{1}{p_M^+}\|u_n\|^{p_M^-} - \int_\Omega \frac{1}{r(x)}h(x)|u_n|^{r(x)}dx - \lambda\int_{\partial\Omega} \frac{1}{\alpha(x)}g(x)|u_n|^{\alpha(x)}d\sigma(x), \end{aligned} \tag{3.2}$$

and by (2.11)-(2.13), (3.1), Remark 2.9 and Remark 2.12, we obtain

$$\begin{aligned} \frac{1}{K}I'(u_n)u_n &= \frac{1}{K}\int_\Omega |\nabla u_n|^{p_1(x)}dx + \frac{1}{K}\int_{\partial\Omega} a(x)|u_n|^{p_1(x)}d\sigma(x) \\ &+ \frac{1}{K}\int_\Omega |\nabla u_n|^{p_2(x)}dx + \frac{1}{K}\int_{\partial\Omega} b(x)|u_n|^{p_2(x)}d\sigma(x) \\ &- \frac{1}{K}\int_\Omega h(x)|u_n|^{r(x)}dx - \frac{\lambda}{K}\int_{\partial\Omega} g(x)|u_n|^{\alpha(x)}d\sigma(x) \\ &\geq \frac{1}{K}\|u_n\|^{p_M^-} - \frac{1}{K}\int_\Omega h(x)|u_n|^{r(x)}dx - \frac{\lambda}{K}\int_{\partial\Omega} g(x)|u_n|^{\alpha(x)}d\sigma(x), \end{aligned} \tag{3.3}$$

where $K = \max\{r^-, \alpha^-\}$.

By (3.2)-(3.3)

$$\begin{aligned} & d + 1 + \|u_n\| \\ \geq\ & I(u_n) - \frac{1}{K}I'(u_n)u_n \\ \geq\ & \frac{1}{p_M^-}\|u_n\|^{p_M^+} - \int_\Omega \frac{1}{r(x)}h(x)|u_n|^{r(x)}dx - \lambda\int_{\partial\Omega} \frac{1}{\alpha(x)}g(x)|u_n|^{\alpha(x)}d\sigma(x) \\ -\ & \frac{1}{K}\|u_n\|^{p_M^-} + \frac{1}{K}\int_\Omega h(x)|u_n|^{r(x)}dx + \frac{\lambda}{K}\int_{\partial\Omega} g(x)|u_n|^{\alpha(x)}d\sigma(x) \\ \geq\ & \left(\frac{1}{p_M^+} - \frac{1}{K}\right)\|u_n\|^{p_M^-} + c_1\left(\frac{1}{K} - \frac{1}{r^-}\right)\|u_n\|^{r^-} + \lambda c_4\left(\frac{1}{K} - \frac{1}{\alpha^-}\right)\|u_n\|^{\alpha^-}. \end{aligned}$$

Since $p_M^+ < \min\{r^-, \alpha^-\}$, the sequence (u_n) is bounded in X. Therefore, there exists a subsequence, again denoted by (u_n), and $u \in X$ such that

$$u_n \rightharpoonup u \text{ in } X,$$

$$u_n \to u \text{ in } L^{p_i(x)}(\Omega),\ u_n \to u \text{ in } L^{r(x)}(\Omega),$$

$$u_n \to u \text{ in } L^{\alpha(x)}(\partial\Omega) \text{ and } u_n \to u\ L^{p_i(x)}(\partial\Omega),$$

where $i = 1, 2, m, M$. Using relation (3.1), we have

$$I'(u_n)(u_n - u) \to 0 \text{ as } n \to +\infty.$$

On the other hand, we have

$$\begin{aligned}
& I'(u_n)(u_n - u) \\
= & \int_\Omega |\nabla u_n|^{p_1(x)-2}\nabla u_n(\nabla u_n - \nabla u)dx + \int_{\partial\Omega} a(x)|u_n|^{p_1(x)-2}u_n(u_n - u)d\sigma(x) \\
+ & \int_\Omega |\nabla u_n|^{p_2(x)-2}\nabla u_n(\nabla u_n - \nabla u)dx + \int_{\partial\Omega} b(x)|u_n|^{p_2(x)-2}u_n(u_n - u)d\sigma(x) \\
- & \int_\Omega h(x)|u_n|^{r(x)-2}u_n(u_n - u)dx - \int_{\partial\Omega} g(x)|u_n|^{\alpha(x)-2}u_n(u_n - u)d\sigma(x).
\end{aligned} \tag{3.4}$$

Then

$$\begin{aligned}
& I'(u_n)(u_n - u) + \int_\Omega h(x)|u_n|^{r(x)-2}u_n(u_n - u)dx \\
+ & \int_{\partial\Omega} g(x)|u_n|^{\alpha(x)-2}u_n(u_n - u)d\sigma(x) \\
= & \int_\Omega |\nabla u_n|^{p_1(x)-2}\nabla u_n(\nabla u_n - \nabla u)dx \\
+ & \int_{\partial\Omega} a(x)|u_n|^{p_1(x)-2}u_n(u_n - u)d\sigma(x) \\
+ & \int_\Omega |\nabla u_n|^{p_2(x)-2}\nabla u_n(\nabla u_n - \nabla u)dx \\
+ & \int_{\partial\Omega} b(x)|u_n|^{p_2(x)-2}u_n(u_n - u)d\sigma(x).
\end{aligned} \tag{3.5}$$

Since

$$u_n \to u \text{ in } L^{\alpha(x)}_{H(x)}(\partial\Omega) \text{ and } u_n \to u\ L^{r(x)}_{h(x)}(\Omega).$$

We obtain

$$\int_\Omega h(x)|u_n|^{r(x)-2}u_n(u_n-u)dx \to 0 \text{ as } n \to +\infty,$$

and

$$\int_{\partial\Omega} g(x)|u_n|^{\alpha(x)-2}u_n(u_n-u)d\sigma(x) \to 0 \text{ as } n \to +\infty.$$

Indeed,

$$\begin{aligned}\left|\int_\Omega h(x)|u_n|^{r(x)-2}u_n(u_n-u)dx\right| &\leq c_5|h|_{\gamma(x)}\left||u_n|^{r(x)-1}\right|_{\theta(x)}|u_n-u|_{r(x),h(x)}\\ &\leq c_6\left||u_n|^{r(x)-1}\right|_{\gamma_0(x)r(x)}|u_n-u|_{r(x),h(x)}\\ &\leq c_7\left||u_n|^{r^i-1}\right|_{(r^i-1)\gamma_0(x)r(x)}|u_n-u|_{r(x),h(x)}\\ &\leq c_8\|u_n\||u_n-u|_{r(x),h(x)},\end{aligned}$$

where $i=\pm$ and $\theta\in C_+(\overline{\Omega})$ such that $\frac{1}{\gamma(x)}+\frac{1}{\theta(x)}+\frac{1}{r(x)}=1$. Since

$$|u_n-u|_{r(x),h(x)} \to 0 \text{ as } n \to +\infty,$$

it follows that

$$\int_\Omega h(x)|u_n|^{r(x)-2}u_n(u_n-u)dx \to 0 \text{ as } n \to +\infty.$$

Similarly we have

$$\int_{\partial\Omega} g(x)|u_n|^{\alpha(x)-2}u_n(u_n-u)d\sigma(x) \to 0 \text{ as } n \to +\infty.$$

Then according to (3.5) and Remark 2.19 we can obtain

$$\begin{aligned}&\int_\Omega |\nabla u_n|^{p_1(x)-2}\nabla u_n(\nabla u_n-\nabla u)dx+\int_{\partial\Omega} a(x)|u_n|^{p_1(x)-2}u_n(u_n-u)d\sigma(x)\\ +\ &\int_\Omega |\nabla u_n|^{p_2(x)-2}\nabla u_n(\nabla u_n-\nabla u)dx+\int_{\partial\Omega} b(x)|u_n|^{p_2(x)-2}u_n(u_n-u)d\sigma(x)\\ \to\ &0 \text{ as } n\to+\infty,\end{aligned}$$

and

$$u_n \to u \text{ in } X.$$

Where (i) is verified.

Let us now check (ii). Similarly to (3.2) we obtain

$$\begin{aligned}
I(u) &= \int_\Omega \frac{1}{p_1(x)}|\nabla u|^{p_1(x)}dx + \int_{\partial\Omega} \frac{1}{p_1(x)}a(x)|u|^{p_1(x)}d\sigma(x) \\
&+ \int_\Omega \frac{1}{p_2(x)}|\nabla u|^{p_2(x)} + \int_{\partial\Omega} \frac{1}{p_2(x)}b(x)|u|^{p_2(x)}d\sigma(x) \\
&- \int_\Omega \frac{1}{r(x)}h(x)|u|^{r(x)}dx - \lambda\int_{\partial\Omega} \frac{1}{\alpha(x)}g(x)|u|^{\alpha(x)}d\sigma(x) \\
&\geq \frac{1}{p_M^+}\|u\|^{p_M^+} - \frac{c_1}{r^+}\|u\|^{r^-} - \frac{\lambda c_4}{\alpha^+}\|u\|^{\alpha^-} \\
&= g(\|u\|),
\end{aligned}$$

where $g(t) = \frac{1}{p_M^+}t^{p_M^+} - \frac{c_1}{r^-}t^{r^-} - \frac{\lambda c_4}{\alpha^+}t^{\alpha^-}$. As $p_M^+ < \min\{r^-, \alpha^-\}$, (ii) follows for $\varrho > 0$ small enough.

Finally, to verify (iii), let us consider a sequence of subspace $Y_m \subseteq X$ of dimension m such that $u|\partial\Omega \not\equiv 0$ if $u \in Y_m \setminus \{0\}$. Hence

$$\min_{x\in B_m}\int_{\partial\Omega} g(x)|u|^{\alpha(x)}d\sigma(x) > 0,$$

where $B_m := \{u \in Y_m : \|u\| = 1\}$. Now, taking into account that $p_M^+ < \min\{r^-, \alpha^-\}$, we have

$$\begin{aligned}
I(tu) &= \int_\Omega \frac{1}{p_1(x)}|\nabla tu|^{p_1(x)}dx + \int_{\partial\Omega} \frac{1}{p_1(x)}a(x)|tu|^{p_1(x)}d\sigma(x) \\
&+ \int_\Omega \frac{1}{p_2(x)}|\nabla tu|^{p_2(x)} + \int_{\partial\Omega} \frac{1}{p_2(x)}b(x)|tu|^{p_2(x)}d\sigma(x) \\
&- \int_\Omega \frac{1}{r(x)}h(x)|tu|^{r(x)}dx - \lambda\int_{\partial\Omega} \frac{1}{\alpha(x)}g(x)|tu|^{\alpha(x)}d\sigma(x) \\
&\leq \frac{t^{p_M^+}}{p_M^-}\|u\|^{p_M^+} - \frac{t^{r^-}}{r^+}\int_\Omega h(x)|u|^{r(x)}dx - \lambda\frac{t^{\alpha^-}}{\alpha^+}\int_{\partial\Omega} g(x)|u|^{\alpha(x)}d\sigma(x) \\
&\leq \frac{t^{p_M^+}}{p_M^-}\|u\|^{p_M^+} - \lambda\frac{t^{\alpha^-}}{\alpha^+}\min_{x\in B_m}\int_{\partial\Omega} g(x)|u|^{\alpha(x)}d\sigma(x) \\
&< 0,
\end{aligned}$$

for $u \in B_m$ and $t \geq t_0$ sufficiently large.

Therefore (iii) follows by taking $A_m = t_0 B_m$. As the conditions of the Theorem 2.20 are verified, we deduce that we have many infinity of solutions, which completes the Theorem 1.1. □

REFERENCES

[1] G. A. Afrouzi, A. Hadjian, S. Heidarkhani, Steklove problems involving the $p(x)$-Laplacian, Electronic Journal of Differential Equations, Vol. 2014 (2014), No. 134, pp. 1–11.

[2] M. Allaoui, A. R, El Amrouss, A. Ourraoui, Existence and multiplicity of solutions for a Steklove problem involving the $p(x)$-Laplacian operator, Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 132, pp. 1–12.

[3] A. Ambrosetti, H. Brezis, G. Cerami; *Combined effects of concave and convexe nonlinearities in some elliptic problems*, Journal of Functional Analysis, **122** (1994), 519–543.

[4] A. Ambrosetti, P. H. Rabinowitz; *Dual variational methods in critical point theory and applications*, J. Funct. Analysis, 14(4) (1973), 349–381.

[5] S. Antontsev, S. Shmarev, Handbook of Differential Equations, Stationary Partial Differ. Equ. 3 (2006). Chap. 1.

[6] M. Bocea, M. Mihăilescu, Γ-convergence of power-law functionals with variable exponents, Nonlinear Anal. 73 (2010) 110–121.

[7] M. Bocea, M. Mihăilescu, C. Popovici, On the asymptotic behavior of variable exponent power-law functionals and applications, Ric. Mat. 59 (2010) 207–238.

[8] J. F. Bonder , J. D. Rossi, Existence Results for the p-Laplacian with Nonlinear Boundary Conditions, Journal of Mathematical Analysis and Applications 263, 195–223 2001.

[9] J. Chabrowski, Y. Fu, Existence of solutions for $p(x)$-Laplacian problems on a bounded domain, J. Math. Anal. Appl. 306 (2005) 604–618.

[10] Y. Chen, S. Levine, R. Rao, Variable exponent, linear growth functionals in image restoration, SIAMJ. Appl. Math. 66 (2006) 1383–1406.

[11] C. Chen , S. Liu, H. Yao, Existence of solutions for quasilinear elliptic exterior problem with the concave-convex nonlinearities and the nonlinear boundary conditions,J. Math. Anal. Appl. 383 (2011) 111–119.

[12] G. Dai, Infinitely many solutions for a $p(x)$-Laplacian equation in $\mathbb{R}^N$, Nonlinear Anal. 71 (2009) 1133–1139.

[13] S.-G. Deng, Positive solutions for Robin problem involving the $p(x)$-Laplacian,J. Math. Anal. Appl. 360 (2009) 548–560.

[14] S.-G. Deng, Eigenvalues of the $p(x)$-Laplacian Steklov problem, J. Math. Anal. Appl. 339 (2008) 925–937.

[15] S.-G. Deng, A local mountain pass theorem and applications to a double perturbed $p(x)$-Laplacian equations, Appl. Math. Comput. 211 (2009) 234–241

[16] L. Diening, P. HÄSTÖ, A. Nekvinda, Open problems in variable exponent Lebesgue and Sobolev spaces, In FSDONA04 Proceedings (Drabek and Rakosnik (eds.); Milovy, Czech Republic, pages 38–58. Academy of Sciences of the Czech Republic, Prague, 2005, pp. 38–58.

[17] D. Edmunds, J. Rákosník; Sobolev embeddings with variable exponent, Studia Math. 143 (2000), no. 3, 267–293. MR1815935 (2001m:46072)

[18] X. Fan, Solutions for p(x)-Laplacian Dirichlet problems with singular coefficients ,J. Math. Anal. Appl. 312 (2005) 464–477.

[19] X. Fan, X. Han, Existence and multiplicity of solutions for $p(x)$-Laplacian equations in $\mathbb{R}^N$, Nonlinear Anal. 59 (2004) 173–188.

[20] X. Fan, D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$ J. Math. Anal. Appl. 263 (2001) 424–446.

[21] X. Fan, Q. Zhang, Existence of solutions for $p(x)$-Laplacian Dirichlet problem, Nonlinear Anal. 52 (2003) 1843–1852.

[22] M. Ferrara, S. Heidarkhani, Multiple solutions for perturbed p-Laplacian boundary-value problems with impulsive effects, Electronic journal of differential equations, n. 2014, 2014, pp.1–14, ISSN: 1072–6691.

[23] Y. Fu, X. Zhang, A multiplicity result for $p(x)$-Laplacian problem in $\mathbb{R}^N$, Nonlinear Anal. 70 (2009) 2261–2269.

[24] L. Gasinski, N. S. Papageorgiou; Nonlinear Analysis, Chapman and Hall/CRC Press, Boca Raton, (2006).

[25] D. A. Kandilakis, M. Magiropoulos, Robin Boundary-Value Problems For Quasilinear Elliptic Equations With Subcritical And Critical Nonlinearities, Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 100, pp. 1–10.

[26] G. M. Lieberman; The natural generalization of the natural conditions of Ladyzhenskaya and Uraltseva for elliptic equations, Comm. Partial Diff. Equations 16 (1991), 311-361.

[27] D. Liu, X Wang, J. Ya,On $p_1(x), p_2(x)$ Laplace Equation,Journal of Lanzhou University:Natural Sciences,2016,52(2).

[28] M. Mihăilescu, On a class of nonlinear problems involving a $p(x)$-Laplace type operator, Czechoslovak Math. J. 58 (2008) 155–172.

[29] R. A. Mashiyev, S. Ogras, Z. Yucedag, and M. Avci,The Nehari Manifold Approach For Dirichlet Problem Involving The $p(x)$-Laplacian Equation,J. Korean Math. Soc. 47 (2010), No. 4, pp. 845–860.

[30] S.H. Rasouli, *On a* PDE *involving the variable exponent operator with nonlinear boundary conditions*, Meditterr. J. Math. **12** (2015), 821–837.

[31] P.H. Rabinowitz, Minimax methods in critical point theory with applications to differential equations. In CBMS Regional Conference Series in Mathemat- ics, Vol. 65, American Mathematical Soceity, Providence (1986).

[32] M. Růžička, Electrorheological Fluids: Modeling and Mathematical Theory, Springer-Verlag, Berlin, 2002.

[33] M. Růžička, Flow of shear dependent electrorheological fluids, CR Math. Acad. Sci. Paris 329 (1999) 393–398.

[34] E. Zeidler, Nonlinear Functional Analysis and its Applications, vol. II/B, Berlin, Heidelberg, New York, 1985.

[35] D. Zhao, X. L. Fan; *O*n the Nemytskii operators from $L^{p_1(x)}(\Omega)$ to $L^{p_2(x)}(\Omega)$, J. Lanzhou Univ. 34 (1998), 1–5.

[36] J. F. Zhao; *S*tructure theory of Banach spaces, Wuhan Univ. Press, Wuhan, 1991 (in Chinese).

[37] Q. M. Zhou, B. Ge Multiple solutions for a Robin-type differential inclusion problem involving the $p(x)$-Laplacian. Math. Methods Appl. Sci. (2013).

[38] V. Zhikov, Averaging of functionals in the calculus of variations and elasticity, Math. USSR Izv. 29 (1987) 33–66.