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D*-Cone Metric Space and Fixed Point Theorem

H. S. Al-Saadi and R. M. Badagaish
Mathematics Sciences Department, Faculty of Applied Sciences,
Umm Al-Qura University, Makkah 21955, P. O Box 11155, Saudi Arabia

Abstract

Con metric spaces are considered as mathematical tools and play a paramount role in several areas. The purpose of this paper is to introduce a notion of generalized con metric space and some convergence properties of sequences are proved. we also discuss the fixed point extended result of contractive mappings.
keyword: Cone metric spaces; contractive mappings; Fixed point theory.

1. Introduction

The study of fixed point theory has been at the center of vigorous activity although they arise in many other areas of mathematics. We start with the notion of Dhage [4] in 1992, he introduced the notion of generalized metric space or D-metric space. Later on, Rhoades [9], generalized Dhages contractive condition and he proved two general fixed point theorems for D-metric spaces. In 2006, Cho and Saadati [3] generalized the concept of a matric space using a δ-distance on a complete D-metric spaces and some properties of convergence and bounded. Furthermore, Hunge and Zhang [5] introduced the notion of a cone metric space in which the real numbers is replaced by an ordering Banch space and they proved some fixed point theorems for mappings satisfying different contractive condition. Abbas and Rhoades [1] proved some fixed point theorems in cone metric spaces, including results which generalized in [5]. Mustafa and Sims [6, 7] introduced a notion of G-metric space, replacing the tetrahedral by an inequality involving repetition of indices. In 2010, Beg, Abbas and Nazir introduced a notion of generalized cone metric space. Also, they proved some convergence properties of sequence and some fixed point theorems for contractive mappings. In this paper, we give a generalization of cone metric space and discuss some properties of convergence of sequence. our results generalized some fixed point theorems in metric spaces using some generalized contractive condition in D*-cone metric space as a probable modification of definition of D-metric spaces introduced by Dhage (1992).

Definition 1.1. *[5]. Let E be a real Banach space, $P \subseteq E$, P is called a cone if and only if*

(i) P *is closed, non-empty,* $P \neq \{0\}$,

(ii) for each $a, b \geq 0$, $x, y \in P$, *then* $ax + by \in P$,

(iii) $P \cap (-P) = \{0\}$.

For a given cone $P \subseteq E$ we denote $x \ll y$ if $y - x \in IntP$, where $IntP$ is the interior P, The cone P is called normal if there is number $M > 0$ such that for all $0 \leq x \leq y$ implies $\|x\| \leq M\,\|y\|$, M is called the normal element.

Definition 1.2. *[5]. Let Xbe a nonempty set. Suppose that the mapping $d : X \times X \to E$,satisfies :*

(i) $0 \leq d\,(x, y)$ *,for all* $x, y \in X$ *and* $d\,(x, y) = 0$ *if and only if* $x = y$*;*

(ii) $d\,(x, y) = d\,(y, x)$*, for all* $x, y \in X$*;*

(iii) $d\,(x, y) \leq d\,(x, z) + d\,(z, y)$ *for all* $x, y, z \in X$.

Then d is called a cone metric space on X, and (X, d) is called a cone metric space.

Definition 1.3. *[3]. Let X be non-empty set, a D-metric space is a function $D : X \times X \times X \to \mathbb{R}^+$ defined on X such that for any $x, y, z, a \in X$*

(i) $D(x, y, z) = 0$ *if and only if* $x = y = z$ *for each* $x, y, z \in X$,

(ii) $D(x, y, z) = D(x, z, y) = ...$ *(Symmetry in all three variable),*

(iii) $D(x, y, z) \leq D(x, y, a) + D(x, a, z) + D(a, y, z)$

Beg and et.el [2] are defined the concept of G-cone metric space by replacing the set of real numbers by an ordered Banach space.

Definition 1.4. *[2]. Let X be a non-empty set. Suppose a mapping $G : X \times X \times X \to E$ satisfies:*

(i) $G(x, y, z) = 0$ *if* $x = y = z$,

(ii) $0 < G(x, x, y)$*; whenever* $x \neq y$*, for all* $x, y \in X$

(iii) $G(x, x, y) \leq G(x, y, z)$*; whenever* $y \neq z$,

(iv) $G(x, y, z) = G(x, z, y) = G(y, x, z) = ...$ *(Symmetric in all three variables),*

(v) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ *for all* $x, y, z, a \in X$

Then G is called a ***Generalized cone metric*** *on X, and X is called a generalized cone metric space or more specifically a G-cone metric space.*

The concept of G-cone metric space is more general than that of G-metric space and cone metric space

Definition 1.5. *[4]. Suppose X be a non-empty set, a strong cone D-metric space is a function $D : X \times X \times X \to E$ defined on X such that for any $x, y, z, a \in X$:*

(i) $D(x, y, z) \geq 0$, *for each* $x, y, z \in X$, $D(x, y, z) = 0$ *if and only if* $x = y = z$,

(ii) $D(x, y, z) = D(p(x, y, z))$, p *is a permutation,*

(iii) $D(x, y, z) \leq D(x, y, a) + D(a, y, z)$.

Definition 1.6. *[1]. A point $x \in X$ is a common fixed point of two maps $f, g : X \to Y$ if $f(x) = g(x) = x$.*

Lemma 1.7. *[1]. Let f and g be weakly compatible self maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of $f$and g.*

2. D*-cone metric spaces

Definition 2.1. *Suppose X be a nonempty set, a function $D : X \times X \times X \to E$ defined on X such that for any $x, y, z, a \in X$:*

(i) $D(x, y, z) \geq 0$, *for each* $x, y, z \in X$, $D(x, y, z) = 0$ *if and only if* $x = y = z$,

(ii) $D(x, y, z) = D(x, z, y) = D(y, x, z) = ...$, *(symmetric in all three variables)*

(iii) $D(x, y, z) \leq D(x, a, a) + D(a, y, z)$.

Then D is called D-cone metric space.*

Remark 2.2. *Any D*-cone metric space is a strong cone D-metric space. But the covers of the following need not be true and the following example show that.*

Example 2.3. *Suppose (X, d) be a cone metric space, Defined $D : X \times X \times X \to E$ by*

$$D(x, y, z) = d(x, y) + d(y, z) + d(x, z)$$

is a D-cone metric space.*

Example 2.4. *Let $E = \mathbb{R}^3$, $P = \{(x, y, z) \in E, x, y, z \geq 0\}$, where $X = \mathbb{R}$. Define $D : X \times X \times X \to E$, by*

$$D(x, y, z) = (|x - y|, |y - z|, |x - z|)$$

define a D-cone metric space.*

Example 2.5. *Let (X, d) be a cone metric space, Defined $D : X \times X \times X \to E$ by*

$$D(x, y, z) = \max\{d(x, y), d(y, z), d(x, z)\}$$

is D-cone metric space.*

Example 2.6. *Let $X = \mathbb{R}^n$, Defined*

$$D(x, y, z) = (\|x - y\|^p, \|y - z\|^p, \|z - x\|^p)$$

for every $p \in \mathbb{R}^+$ is a D-cone metric space.*

Remark 2.7. *In D*-cone metric space $D(x, y, y) = D(x, x, y)$.*

Proof. For

(i) $D(x, x, y) \leq D(x, x, x) + D(x, y, y) = D(x, y, y)$ also

(ii) $D(y, y, x) \leq D(y, y, y) + D(y, x, x) = D(y, x, x)$.

Hence by (i) and (ii) we get $D(x, x, y) = D(x, y, y)$ □

Definition 2.8. *An open ball in a D*-cone metric space X with center x and radius r is denote by*

$$B_D = \{y \in X : D(x, y, y) < r\}$$

.

Definition 2.9. *Let (X, D) be a D*-cone metric space and $A \subseteq X$. If for every $x \in A$, there exist $r > 0$ such that $B_D(x, r) \subseteq A$, then subset A is called open subset of X.*

Definition 2.10. *Let (X, D) be a D*-cone metric space and $A \subseteq X$. If for every $x \in A$, A subset is said to be D-bounded if there exist $r > 0$ such that*

$$D(x, y, y) < r \qquad for \quad all \quad x, y \in A$$

Definition 2.11. *Suppose (X, D) be a D*-cone metric space. We say that $\{x_n\}$ is D-convergent to x if $\lim_{n,m\to\infty} D(x, x_n, x_m) = 0$, that is for each $\epsilon > 0$ there exists a positive integer N such that $D(x, x_n, x_m) < \epsilon$ for all $m, n \geq N$. We call that x is the limit of the sequence and we write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.*

Definition 2.12. *Suppose (X, D) be a D*-cone metric space. A sequence $\{x_n\}$ is said to be D-Cauchy sequence for each $\epsilon > 0$ there exists a positive integer N such that $D(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \geq N$*

Definition 2.13. *A D*-cone metric space (X, D) is complete if every D-Cauchy sequence in X converges to x limit in X*

Theorem 2.14. *In any D*-cone metric space, limits of D-convergent sequences are unique.*

Proof. Let (X, D) be a D*-cone metric space and suppose that the sequence $\{x_n\}$ in X is D-converge to x and z. Let $\epsilon > 0$. Since $\{x_n\} \to x$, there exists an index $N_1 \in \mathbb{N}$ such that

$$D(x_n, y, x) < \frac{\epsilon}{2}$$

for $n \geq N_1$. Since $\{x_n\} \to z$, there exists $N_2 \in \mathbb{N}$ such that

$$D(x_n, y, z) < \frac{\epsilon}{2}$$

for $n \geq N_2$.
Let $N = max\{N_1, N_2\}$. Then both inequalities above hold and so the triangle inequality gives

$$D(x, y, z) \leq D(x, y, x_N) + D(x_N, y, z) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Thus for any $\epsilon > 0$, $D(x, y, z) < \epsilon$. This implies that $D(x, y, z) = 0$ which in turn implies that $x = y = z \Rightarrow x = z$. We conclude that the limits of D-convergent sequence is unique. □

Proposition 2.15. *A sequence $\{x_n\}$ in a D*-cone metric space (X, D) converges to x if and only if*

$$\lim_{n\to\infty} D(x_n, y, x) = 0$$

Proof. The $\{D(x_n, y, x)\}$ from a sequence of non negative numbers. This sequence D-converges to 0 if and only if there for every $\epsilon > 0$ exists an $N \in \mathbb{N}$ such that $D(x_n, y, x) < \epsilon$ when $n \geq N$. This what the definition says. □

Theorem 2.16. *Any D-convergent sequence in a D*-cone metric space is a D-Cauchy sequence.*

Proof. Let $\{x_n\}$ D-convergent to x. Let $\epsilon > 0$ be given. Then there is an element $N \in \mathbb{N}$ such that $D(x_n, x, z) < \frac{\epsilon}{2}$ for all $n \geq N$. Suppose $n, m \in \mathbb{N}$ be such that $m \geq N$, $n \geq N$, Then

$$D(x_m, x_n, z) \leq D(x_m, x, z) + D(x_n, x, z) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Hence $\{x_n\}$ is a D-Cauchy sequence. □

Remark 2.17. *Converse of the above theorem is not true in general and the following example show that.*

Example 2.18. *Let $X = (0, 1]$. Then $\{\frac{1}{n}\}$ is a D-Cauchy sequence which is not D-convergent in X.*

Definition 2.19. *A subsequence of sequence $\{x_n\}$ in a D*-cone metric space X is a sequence $\{x_{n_k}\}$ in X consisting of terms of the sequence $\{x_n\}$ such that $n_k > n_h$ if $k > h$.*

Theorem 2.20. *Suppose that $\{x_n\}$ is a D-convergent sequence in a D*-cone metric space X. Then any subsequence of $\{x_n\}$ D-converges to the same limits as $\{x_n\}$.*

Proof. Let $\{x_n\}$ be D-converges to $x \in X$. Suppose that $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ and let $\epsilon > 0$. Since $\{x_n\} \to x$, there exists $N_1 \in \mathbb{N}$ such that

$$D(x_n, x, z) < \epsilon \qquad for \quad n \geq N_1$$

Choose $N \in \mathbb{N}$ large enough so that $n_k \geq N_1$ for $k > N_i$ in other words, choose an index N for the subsequence large enough so that the N-term in the subsequence is beyond the N_i-th term in the original sequence, then for $k \geq N$, $n_k \geq N_1$ so $D(x_{n_k}, x, z) < \epsilon$ We conclude that $\{x_{n_k}\}$ D-converges to x. □

Definition 2.21. *Let (X, D_X) and (Y, D_Y) be a D*-cone metric spaces and let $f : (X, D_X) \to (Y, D_Y)$ be a function then f is said to be D- continuous at a point $a \in X$ if given $\epsilon > 0$, there exists $\delta > 0$ such that $x, y \in X$, $D_X(a, x, y) < \delta$ implies $D_Y(f(a), f(x), f(y)) < \epsilon$. A function f is D-continuous on X if and only if it is D-continuous at all $a \in X$*

Theorem 2.22. *Let $f : (X, D_X) \to (Y, D_Y)$ be a function between D*-cone metric spaces, the the following are equivalent.*

(i) f is D-continuous at a point $a \in X$.

(ii) For all sequences $\{x_n\}$ D-converging to a, the sequence $\{f(x_n)\}$ is D-continuous to $f(a)$.

Proof. **(i) implies (ii)**: We show that for any $\epsilon > 0$ there is an element $N \geq \mathbb{N}$ such that $D_Y(f(x_n), f(a), f(z)) < \epsilon$ when $n \geq N$. Since f is D-continuous at a , there is a $\delta > 0$ such that $D_Y(f(x_n), f(a), F(z)) < \epsilon$ whenever $D(x, a, z) < \delta$. Since x_n D-converges to a, there is an $N \in \mathbb{N}$ such that $D_X(x_n, a, z) < \delta$ when $n \geq N$. But then $D_Y(f(x_n), f(a), f(z)) < \epsilon$ for all $n \geq N$.
(ii) implies (i): Suppose that f is not D-continuous at a. We must show that there is a sequence $\{x_n\}$ D-converging to a such that $\{f(x_n)\}$ dose not D-converges to $f(a)$. That f is not D-continuous at a means that there is an $\epsilon > 0$ such that no matter how small we choose $\delta > 0$, there is an element x such that $D_X(x, a, z) < \delta$, but $D_Y(f(x), f(a), f(z)) \geq \epsilon$. In particular, we can for each $n \in \mathbb{N}$ find x_n such that $D_X(x_n, a, z) < \frac{1}{n}$, but $D_Y(f(x_n), f(a), f(z)) \geq \epsilon$.
Then $\{x_n\}$ D-converges to a, but $\{f(x_n)\}$ dose not D-converge to $f(a)$.
The composition of two D-continuous functions is D-continuous. □

3. Fixed Point Theorems

In this section we prove some fixed point theorems in D*-cone metric space.

Theorem 3.1. *Let (X, D) be a complete D*-cone metric space, P be a cone normal with a normal constant $M \leq 1$, suppose the mapping $T : X \to X$ satisfy the contractive condition, thus $D(Tx, Ty, Tz) \leq kD(x, y, z)$ for all $x, y, z \in X$, $k \in [0, 1)$ is constant. Then T has a unique fixed point in X.*

Proof. Let x_0 be an arbitrary point in X, define the iterative sequences $\{x_n\}$ by $x_0, x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, ..., x_{n+1} = Tx_n = T^{n+1}x_0$. So, we have

$$\begin{aligned} D(x_{n+1}, x_{n+1}, x_n) &= D(Tx_n, Tx_n, Tx_{n-1}) = D(x_{n+1}, x_{n+1}, x_n) \\ &\leq kD(x_n, x_n, x_{n-1}) \leq k^2 D(x_{n-1}, x_{n-1}, x_{n-2}) \\ &\leq ... \leq k^n D(x_1, x_1, x_0). \end{aligned}$$

So, for $n > m$ we have

$$D(x_n, x_n, x_m) \leq D(x_n, x_n, x_{n-1}) + D(x_{n-1}, x_n, x_m).$$

By definition, we obtain

$$\begin{aligned} D(x_n, x_n, x_m) \leq\ & D(x_n, x_n, x_{n-1}) + D(x_{n-1}, x_{n-1}, x_{n-2}) \\ & + ... + D(x_{m+1}, x_{m+1}, x_m) \end{aligned}$$

Therefore

$$D(x_n, x_n, x_m) \leq (k^{n-1} + k^{n-2} + ... + k^m)D(x_1, x_1, x_0)$$

So

$$D(x_n, x_n, x_m) \leq (k^m/1 - k)D(x_1, x_1, x_0)$$

And since P is normal we have

$$\|D(x_n, x_n, x_m)\| \leq M(k^m/1 - k)\|D(x_1, x_1, x_0)\|$$

as $m \to \infty, k^m/1 - k \to 0$ And we have $\|D(x_n, x_n, x_m)\| \to 0, n, m \to \infty$ so $D(x_n, x_n, x_m) \to 0, n, m \to \infty$ therefore $\{x_n\}$ is a D-Cauchy sequence, since X is complete metric space, there exist a point x in X such that $x_n \to x$.
Now we shows x is a fixed point of the mapping T. It follows from

$$D(Tx, Tx, x) \leq D(Tx, Tx, Tx_n) + D(Tx_n, Tx, x)$$

By the definition we obtain

$$D(Tx, Tx, x) \leq D(Tx_n, Tx, Tx) + D(Tx_n, Tx_n, x)$$

This implies that

$$D(Tx, Tx, x) \leq D(x_n, x, x) + D(x_{n+1}, x_{n+1}, x)$$

And we get

$$\|D(Tx, Tx, x)\| \leq M\|D(x_n, x, x) + D(x_{n+1}, x_{n+1}, x))\|$$

since $\{x_n\}$ is a D-Cauchy sequence in the complete D*-cone metric space, there exist $c \gg 0$ such that $D(x_n, x, x) \ll c$ and so $D(x_n, x, x) \to 0$, as $n \to \infty$ and similarly $D(x_{n+1}, x_{n+1}, x) \ll c, D(x_{n+1}, x_{n+1}, x) \to 0$, as $n \to \infty$, then $\|D(Tx, Tx, x)\| \to 0$ and we have $Tx = x$. This show that x is a fixed point of T.
If y is another fixed point,

$$D(x, y, y) = D(Tx, Ty, Ty) \leq kD(x, y, y)$$

And we have the following inequality $\|D(x, y, y)\| \leq kM\|D(x, y, y)\|$, since $kM \leq 1$ we get $\|D(x, y, y)\| \leq \|D(x, y, y)\|$ and so $D(x, y, y) = 0$, and $x = y$, so x is unique. □

Theorem 3.2. *Let (X, D) be a complete D*-cone metric space on, P be a cone normal with a normal constant $M \leq 1$, and T contraction mapping on X, satisfy*

$$D(Tx, Ty, Tz) \leq k(D(x, x, x) + D(y, y, y) + D(z, z, z))$$

for all $x, y, z \in X, k \in [0, 1)$ is a constant. Then T has a unique fixed point in X.

Proof. Let x_0 be an arbitrary point in X, set the iterative sequences $\{x_n\}$ by

$$x_0, x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, ..., x_{n+1} = Tx_{n+1} = T^{n+1}x_0$$

So

$$\begin{aligned} D(x_{n+1}, x_{n+1}, x_n) &= D(Tx_n, Tx_n, Tx_{n-1}) \\ &\leq k(D(Tx_n, x_n, x_n) + D(Tx_n, x_n, x_n + D(Tx_{n-1}, x_{n-1}, x_{n-1})) \\ &= k(D(Tx_n, x_n, x_n) + D(Tx_n, x_n, x_n + D(x_n, x_{n-1}, x_{n-1})). \end{aligned}$$

This implies that

$$D(x_{n+1}, x_{n+1}, x_n) \leq k(D(x_{n+1}, x_n, x_n) + D(x_{n+1}, x_n, x_n) + D(x_n, x_{n-1}, x_{n-1}))$$

This implies that

$$D(x_{n+1}, x_{n+1}, x_n) \leq k(D(x_{n+1}, x_{n+1}, x_n) + D(x_n, x_{n-1}, x_{n-1}))$$

And we have

$$\|D(x_{n+1}, x_{n+1}, x_n)\| \leq kM\|(D(x_{n+1}, x_{n+1}, x_n) + D(x_n, x_{n-1}, x_{n-1})\|$$

By easy steps calculation we get

$$\|D(x_{n+1}, x_{n+1}, x_n)\| \leq kM/(1 - kM)\|D(x_n, x_{n-1}, x_{n-1})\|$$

$kM/(1-kM) := h \leq 1$, so $D(x_{n+1}, x_{n+1}, x_n) \leq hD(x_n, x_{n-1}, x_{n-1})$ and $D(x_{n+1}, x_{n+1}, x_n) \leq h^n D(x_1, x_1, x_0)$ Consequently we get

$$D(x_n, x_n, x_m) \leq D(x_n, x_n, x_{n-1}) + D(x_{n-1}, x_{n-1}, x_{n-2}) + ... + D(x_{m+1}, x_{m+1}, x_m)$$

Thus

$$D(x_n, x_n, x_m) \leq (h^{n-1} + h^{n-2} + ... + h^m)D(x_1, x_1, x_0) = h^m/(1 - h)D(x_1, x_1, x_0)$$

For $c \gg 0$, $c \in E$, choose natural number N such that $h^m/(1 - h)D(x_1, x_1, x_0) \ll c \quad \forall m > N$, so we get $D(x_n, x_n, x_m) \ll c \quad \forall n, m > N$ and this show that $\{x_n\}$ is a D-Cauchy sequence, and since X is complete D*-cone metric space, there exist a point x in X such that $x_n \to x$.
To show that x is a fixed point, we have

$$D(Tx, Tx, x) \leq D(Tx, Tx, Tx) + D(Tx, Tx, x)$$

So, since P is normal we have a normal element M satisfy

$$\|D(Tx, Tx, x)\| \leq M(\|D(Tx, Tx, Tx)\| + \|D(Tx, Tx, x)\|)$$

This implies that

$$\|D(Tx, Tx, x)\| \leq M/(1 - M)\|D(Tx, Tx, Tx)\|$$

From the first condition of the D*-cone metric space we get $D(Tx, Tx, Tx) = 0$, thus $\|D(Tx, Tx, x)\| \to 0$ and we have $Tx = x$.
We shall show x is unique, for another fixed point $y \in X$

$$D(Tx, Ty, Ty) = D(x, y, y) \leq k(D(Tx, x, x) + D(Ty, y, y) + D(Ty, y, y))$$

This implies that

$$\begin{aligned} D(Tx, Ty, Ty) = D(x, y, y) &\leq k(D(x, x, x) + D(y, y, y) \\ &+ D(y, y, y)) \to 0 \Rightarrow D(x, y, y) = 0 \Rightarrow x = y. \end{aligned}$$

□

Theorem 3.3. *Let (X, D) be a D*-cone metric space, and T contraction mapping on X, satisfy*

$$D(Tx, Ty, Tz) \leq k(D(Tx, y, z) + D(Ty, z, x) + D(Tz, x, y))$$

for all $x, y, z \in X, k \in [0, 1/2)$ is a constant. Then T has a unique fixed point in X.

Proof. We set for an arbitrary point in $x_0 \in X$,the above iterative sequence's $\{x_n\}$

$$x_0, x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, ..., x_{n+1} = Tx_{n+1} = T^{n+1}x_0$$

So

$$\begin{aligned} D(x_{n+1}, x_{n+1}, x_n) &= D(Tx_n, Tx_n, Tx_{n-1}) \\ &\leq k(D(Tx_n, x_n, x_{n-1}) + D(Tx_n, x_n, x_{n-1} + D(Tx_{n-1}, x_n, x_n) \end{aligned}$$

Therefore

$$D(x_{n+1}, x_{n+1}, x_n) \leq k(2D(Tx_n, x_n, x_{n-1}) + D(x_n, x_n, x_n))$$

We note that the last term in the right side is zero from the first condition of the D*-cone metric space, then

$$\begin{aligned} D(x_{n+1}, x_{n+1}, x_n) &\leq 2kD(Tx_n, x_n, x_{n-1}) = 2kD(Tx_{n+1}, x_n, x_{n-1}) \\ 2kD(Tx_{n-1}, x_n, x_{n+1}) &\leq 2k(D(x_n, x_{n-1}, x_n) + D(x_n, x_{n-1}, x_{n+1})) \\ &\leq 2k(D(x_n, x_n, x_{n-1}) + D(x_{n+1}, x_{n+1}, x_n) \end{aligned}$$

From that we get

$$\|D(x_{n+1}, x_{n+1}, x_n)\| \leq 2kM(\|(D(x_n, x_n, x_{n-1})\| + \|D(x_{n+1}, x_{n+1}, x_n)\|)$$

Hence

$$\|D(x_{n+1}, x_{n+1}, x_n)\| \leq 2kM/(1 - 2kM)\|(D(x_n, x_n, x_{n-1})\|$$

Denote $2kM/(1-2kM) =: h \leq 1$ this gives $D(x_{n+1}, x_{n+1}, x_n) \leq h\|D(x_n, x_n, x_{n-1})\|$ analogy as the above theorems we get $D(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$ and $\{x_n\}$ be a D-Cauchy sequence, and since X is complete D*-cone metric space, so x_n D-converge to a point x in X.

We show that x is a fixed point of the mapping T as follows

$$D(Tx, Tx, x) \leq D(Tx, Tx, Tx) + D(Tx, Tx, x)$$

I t follows analogy from the proof of the above theorems that $D(Tx, Tx, x) = 0$ and so $Tx = x$.

To prove the uniqueness of the fixed point x, we consider y is another fixed point, so

$$\begin{aligned} D(Tx, Ty, Tz) = D(x, y, z) &\leq k(D(Tx, y, y) + D(Ty, x, y) + D(Ty, x, y)) \\ &= k(D(x, y, y) + D(y, x, y) + D(y, x, y)) = 3kD(x, y, y). \end{aligned}$$

This means that $D(x, y, z) \leq 3kD(x, y, y)$ hence $D(x, y, z) = 0$ and so $x = y$. Therefore the fixed point x of T is unique.

□

Theorem 3.4. *Let (X, D) be a D*-cone metric space. Suppose the self maps $f, g : X \to X$ satisfy the contractive condition '*

$$D(fx, fy, fz) \leq kD(gx, gy, gz) \qquad for \quad all \quad x, y, z \in X \tag{1}$$

where $k \in [0, 1)$ is a constant. If the range of f is contained in range of g and rang of g is a complete subspace of X. More over if f and g are weakly compatible, f and g have a unique common fixed point.

Proof. Suppose that $x_0 \in X$ be arbitrary. Choose $x_1 \in X$ such that $f(x_0) = g(x_1)$. This possible since $f(X) \subseteq g(X)$.
Continuing this process, choose $x_{n+1} \in X$ such that

$$f(x_n) = g(x_{n+1}) \tag{2}$$

Now,

$$D(gx_{n+1}, gx_{n+1}, g_n) = D(fx_n, fx_n, fx_{n+1}) \leq kD(gx_n, gx_n, gx_{n-1}) \tag{3}$$

this is by using (1)
Now, by Repeated application of (1),we get

$$D(gx_{n+1}, gx_{n+1}, gx_n) < k^n D(gx_1, gx_1, gx_0) \tag{4}$$

For $n > m$,

$$\begin{aligned} D(gx_n, gx_n, gx_m) &\leq D(gx_n, gx_n, gx_{n-1}) + D(gx_{n-1}, gx_{n-1}, gx_{n-2}) \\ &\quad + ... + D(gx_{m+1}, gx_{m+1}, gx_m) \\ &\leq (k^{n-1} + k^{n-2} + ... + k^m)D(gx_1, gx_1, gx_0) \\ &\leq k^n(1 + k + k^2 + ... + ...)D(gx_1, gx_1, gx_0) \\ &= \frac{k^m}{1-k} D(gx_1, gx_1, gx_0) \end{aligned} \tag{5}$$

Let $0 \ll c$ be given. Choose $n_1 \in N$ such that

$$\frac{k^m}{1-k} D(gx_1, gx_1, gx_0) \ll c \qquad \forall m \geq n_1 \in N \tag{6}$$

From (5) and (6),we get

$$D(gx_n, gx_n, gx_m) \ll c \qquad \forall m \geq n_1 \in N$$

Since $\{gx_n\}$ is an D-Cauchy sequence in $g(X)$, then $\{gx_n\}$ is an D-convergent sequence in $g(X)$, from $g(X)$ is complete in X.
Let $\{gx_n\}$ is D-convergent to $q \in g(X)$ is complete in X. Consequently, there is

$$p \in X such \quad that \quad gp = q \tag{7}$$

For the same given $c \in E$, choose $n_2 \in N$ such that

$$D(gx_{n-1}, gx_{n-1}, gp) \ll \frac{c}{k} \quad \forall n \geq n_2 \in N \tag{8}$$

Hence, using 8

$$D(gx_n, gx_n, fp) = D(fx_{n-1}, fx_{n-1}, fp) \leq kD(gx_{n-1}, gx_{n-1}, gp) \ll c, \tag{9}$$

Then $D(gx_n, gx_n, fp) \ll c \qquad \forall n \geq n_2 \in N.$
That mean $gx_n \to fp$.
Hence $\{gx_n\}$ D-converges to both q and fp. By uniqueness property of limit

$$fp = gp = q \tag{10}$$

Then q is the point of coincidence of f and g.
Let $r \in X$ be any other coincidence point of f and g, then

$$fr = gr \quad be \quad the \quad point \quad of \quad coincidence \quad f \quad and \quad g \tag{11}$$

Now,$D(gr, gr, gp) = D(fr, fr, fp) \leq kD(gr, gr, gp)$, then $D(gr, gr, gp) \leq kD(gr, gr, gp)$.
That mean $(k-1)D(gr, gr, gp) \in P$
Multiplying with positive real number $(1-k)$, we get $D(gr, gr, gp) \in P$. But, we have $D(gr, gr, gp) \in P$.
From the definition of cone and D*- cone metric, we get

$$gr = gp. \tag{12}$$

From (10),(11),(12), f and g have unique point of coincidence.
Finally, let f and g are weakly compatible self-maps having unique point of coincidence. Using the Lemma 1.7, f and g have a unique common fixed point. □

Now, let defined coupled fixed point in D*-cone metric space.

Definition 3.5. *Let (X, D) be a D*-cone metric space. An element $(x, y) \in X \times X$ is said to be a coupled fixed point of the mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.*

Theorem 3.6. *Suppose that* (X, D) *be a complete D*-cone metric space. Suppose that the mapping* $F : X \times X \to X$ *satisfies the following contractive condition for all* $x, y, w, z, u, v \in X$*:*

$$D(F(x,y), F(w,z), F(u,v)) \leq kD(x,w,u) + lD(y,z,v), \tag{13}$$

where k, l *are non-negative constants with* $k+l+m < 1$*. Then* F *has a unique coupled fixed point.*

Proof. Choose $x_0, y_0 \in X$ and set $x_1 = F(x_0, y_0)$,$y_1 = F(y_0, x_0), ..., x_{n+1}$= $F(x_n, y_n), y_{n+1} = F(y_n, x_n)$. Then by (13) we have

$$\begin{aligned} D(x_n, x_n, x_{n+1}) &= D(F(x_{n-1}, y_{n-1}), F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\ &\leq kD(x_{n-1}, x_{n-1}, x_n) + lD(y_{n-1}, y_n, y_{n+1}), \end{aligned}$$

and similarly,

$$\begin{aligned} D(y_n, y_n, y_{n+1}) &= D(F(y_{n-1}, x_{n-1}), F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \\ &\leq kD(y_{n-1}, y_{n-1}, y_n) + lD(x_{n-1}, x_{n-1}, x_n), \end{aligned}$$

Therefore,by letting

$$D_n = D(x_n, x_n, x_{n+1}) + D(y_n, y_n, y_{n+1}),$$

we have

$$\begin{aligned} D_n &= D(x_n, x_n, x_{n+1}) + D(y_n, y_n, y_{n+1}) \\ &\leq kD(x_{n-1}, x_{n-1}, x_n) + lD(y_{n-1}, y_{n-1}, y_n) + kD(y_{n-1}, y_{n-1}, y_n) + lD(x_{n-1}, x_{n-1}, x_n) \\ &\leq (k+l)(D(x_{n-1}, x_{n-1}, x_n) + D(y_{n-1}, y_{n-1}, y_n)) \\ &= (k+l)D_{n-1}. \end{aligned}$$

Consequently, if $\delta = k + 1$ then for each $n \in \mathbb{N}$ we have

$$0 \leq D_n \leq \delta D_{n-1} \leq \delta^2 D_{n-2} \leq ... \leq \delta^n D_0.$$

If $D_0 = 0$ then (x_0, y_0) is a coupled fixed point of F. Now, let $D_0 > 0$. For each $n > m$ we have

$$D(x_n, x_n, x_m) \leq D(x_n, x_n, x_{n-1}) + D(x_{n-1}, x_{n-1}, x_{n-2}) + ... + D(x_{m+1}, x_{m+1} x_m),$$

$$D(y_n, y_n, y_m) \leq D(y_n, y_n, y_{n-1}) + D(y_{n-1}, y_{n-1}, y_{n-2}) + ... + D(y_{m+1}, y_{m+1}, y_m).$$

Therefore,

$$\begin{aligned} D(x_n, x_n, x_m) + D(y_n, y_n, y_m) &\leq D_{n-1} + D_{n-2} + ... + D_m \\ &\leq (\delta^{n-1} + \delta^{n-2} + ... + \delta^m) D_0 \\ &\leq \frac{\delta^m}{1-\delta} D_0 \end{aligned}$$

which implies that $\{x_n\}$ and $\{y_n\}$ are D-Cauchy sequences in X, and there exist $x^*, y^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$ and $\lim n \to \infty y_n = y^*$. Let $c \in E$ with $0 \ll c$. For every $m \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that $D(x_n, x_n, x^*) \ll c/2m$ and $d(y_n, y_n, y^*) \ll c/2m$ for all $n \geq N$. Thus

$$\begin{aligned} D(F(x^*, y^*), F(x^*, y^*), x^*) &\leq D(F(x^*, y^*), F(x^*, y^*), x_{N+1}) + D(x_{N+1}, x_{N+1}, x^*) \\ &= D(F(x^*, y^*), F(x^*, y^*), F(x_N, y_N)) + D(x_{N+1}, x_{N+1}, x^*) \\ &\leq kD(x_N, x_N, x^*) + lD(y_N, y_N, y^*) + D(x_{N+1}, x_{N+1}, x^*) \\ &\ll (k+l)\frac{c}{2m} + \frac{c}{2m} \leq \frac{c}{m}. \end{aligned}$$

Consequently, $D(F(x^*, y^*), F(x^*, y^*), x^*) \ll c/m$ for all $m \geq 1$. Thus , $D(F(x^*, y^*), F(x^*, y^*), x^*) = 0$ and hence $F(x^*, y^*) = x^*$. Similarly, we have $F(y^*, x^*) = y^*$ meaning that (x^*, y^*) is coupled fixed point of F.
Now, if (x', y') is another coupled fixed point of F, then

$$\begin{aligned} D(x', x', x^*) &= D(F(x', y'), F(x', y'), F(x^*, y^*)) \leq kD(x', x', x^*) + lD(y', y', y^*), \\ D(y', y', y^*) &= D(F(y', x'), F(x', y'), F(y^*, x^*)) \leq kD(y', y', y^*) + lD(x', x', x^*), \end{aligned}$$

and therefore,

$$D(x', x', x^*) + D(y', y', y^*) \leq (k+l)(D(x', x', x^*) + D(y', y', y^*)). \tag{14}$$

Since $k + l < 1$,(14) implies that $D(x', x', x^*) + D(y', y', y^*) = 0$. Hence, we have $(x', y') = (x^*, y^*)$ and the proof of the theorem is complete. □

Lemma 3.7. *Let (X, d) be a D*-cone metric space, Let $F : X \times X \to X$ satisfies the following contractive condition for all $x, y, w, z, u, v \in X$:*

$$D(F(x, y), F(w, z), F(u, v)) \leq \frac{k}{2}(D(x, w, u) + D(y, z, v)),$$

where $k \in [0, 1)$ is a constant. Then F has a unique fixed point.

Proof. The prove is the same of the theorem above. □

References

[1] M. Abbas and B. E. Rhoades, Fixed and periodic point results in cone metric spaces, Appl. Math. Letter, 22(2009), 511-515.

[2] I . Beg, M . Abbas and T . Nazir, Genralized cone metric spaces, J. Nonlinear Sci. Appl., 3(1) (2010), 21-31.

[3] Y. J. Cho and R. Saadati, A fixed point theorem in generalized D-metric spaces, Bull. of Iran. Math. Soci., 32(2)(2006), 13-19.

[4] B . C . Dhage, Generalized metric space and mapping with fixed point, Bull. Cal . Math. Soc, 84(1992), 329-336.

[5] L . G. Huang and X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl, 332(2007), 1468-1476.

[6] Z. Mustafa, B. Sims, A new approach to generalized metric spaces, J. Nonlinear Convex Anal. 7 (2006) 289-297.

[7] Z. Mustafa, B. Sims, Fixed point theorems for contractive mappings in complete G-metric spaces, Fixed Point Theory Appl. (2009), Art. ID 917175, 10 pp

[8] S . Rezapour and R . Hamlbarani, Some notes on the paper: Cone metric spaces and fixed point theorems of contractive mappings, Journal of Mathematical Analysis and Applications, 345(2)(2008), 719-724.

[9] B . E . Rhoades, A fixed point theorem for generalized metric spaces. Inter. J. Math. Sci., 19(1996), 457-460.