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Non Linear Dynamical Control Systems and Stability

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Abstract

In this paper, some types of stability such as Lyapunov, orbital and structural stabilities were considered. The relationship between stability, asymptotic stability and exponential stability was also stated. Conditions under which some systems fail to be stable were also explained.

Keywords: Equilibrium, Lyapunov stability, Orbital stability, Stability, Structural stability.

1. INTRODUCTION

A non linear system refers to a set of non linear equations used to describe a physical device or process that may not be described by a set of linear equations of any kind. Dynamical control system is used as a synonym for Mathematical or Physical system when the describing equations represent evolution of a solution with time control input parameters.

The Theory of non linear dynamical control systems has been greatly advanced since nineteenth century [9]. Today, non linear dynamical control systems are used to describe a great variety of scientific and engineering phenomenon ranging from social, life and physical sciences to engineering and technology. See [4], [7], [8].

Stability Theory plays a central role in system engineering, especially in the field of dynamical control systems with regard to both dynamics and control [5]. The basic concept of stability emerged from the study of an equilibrium state of mechanical systems such as the equilibrium of a rigid body under natural force of gravity [6].

2. PRELIMINARIES

A continuous-time non linear control system is generally described by a different equation of the form

$$\dot{x} = f(x, t, u), \qquad t \in [t_0, \infty) \tag{1.1}$$

where $x = x(t)$ is the state of the system usually in E^n, u is the control input vector belonging to E^m, $(with\, m \leq n)$ and f is a Lipschitz or continuously differentiable non linear function. We want to study the stability of (1.1). In studying the stability of (1.1), we have special interest in the following types of stabilities.

2.0 **Stabilities**

We intend to look into these three types of stabilities: - Lyapunov stability of a system with respect to its equilibra, the orbital stability of a system output trajectory, and the structural stability of a system itself. These are of fundamental importance in the studies of nonlinear dynamical control systems.

2.1 Lyapunov Stability

Lyapunov stability of a system with respect to its equilibrium of interest is roughly the behavior of the system output toward the equilibrium state. Without loss of generality, let us assume that the origin $x = 0$ is the system's equilibrium of interest. We note that Lyapunov stability theory is concerned with various stabilities of the system with respect to this equilibrium. We note that if another equilibrium is to be considered, the new equilibrium is first of all shifted to zero by a change of variables and then the transformed system is then studied in the same way.

Let us now consider a dynamical system without explicitly involving control inputs. This dynamical system is of the form

$$\dot{x} = f(x, t) \qquad x(t_0) = x_0 \in E^n \tag{2.1}$$

Definition 1. Lyapunov Stability [10]

System (2.1) is said to be stable in the sense of Lyapunov with respect to the equilibrium x = 0 if for $\epsilon > 0$ and any initial time $t_0 \geq 0$, there exists a constant $\delta = \delta(\varepsilon, t_0)$, such that

$$\|x^*(t_0)\| < \delta \Longrightarrow \|x^*(t)\| < \varepsilon \; for\; all\; t \geq t_0 \tag{2.2}$$

Here, we emphasize that the constant δ generally depends on both $\varepsilon\; and\; t_0$

We note also that the above stability, in the sense of Lyapunov, is said to be uniform with respect to the initial time if $\delta = \delta(\varepsilon)$ and does not depend explicitly on t_0 over the entire interval of time in $[0,\infty)$

Definition 2. Lyapunov Asymptotically Stability.

System (2.2) is said to be asymptotically stable about its equilibrium point $x^* = 0$ if it is stable in the sense of Lyapunov and furthermore, there exists a constant $\delta = \delta(t_0) > 0$ such that

$$\|x^*(t_0)\| < \delta \Longrightarrow \|x^*(t)\| \longrightarrow 0 \; as \; t \; \longrightarrow \infty. \qquad (2.3)$$

We have to note also that asymptotic stability is said to be uniform if δ is independent of t_0 over the finite interval $[0, \infty)$.

Definition 3. Exponential Stability [2]

System (2.2) is said to be exponential stable if for two positive constants c and σ, we have

$$\|x(t_0)\| < \delta \Longrightarrow \|x(t)\| \leq \; c \, e^{-\sigma t} for \; all \; t \; \geq t_0$$

Clearly, we have to observe that exponential stability implies asymptotic stability and asymptotic stability in turn implies stability in the sense of Lyapunov. Note that the reverse need not be true.

For Lyapunov stability, let us consider, for example, the following time varying system with discontinuous coefficient. Let us take for instance

$$\dot{x}(t) = \frac{1}{1-t} x(t), \;\; x(t_0) = x_0 \qquad (2.4)$$

Note that (2.4) has the solution of the form

$$x(t) = x_0 \frac{1-t_0}{1-t}, 0 \leq t_0 \leq t < \infty \qquad (2.5)$$

We also note (2.5) is stable in the sense of Lyapunov about the equilibrium $x^* = 0$ over the entire time domain $[0,\infty)$ if and only if $t_0 = 1$. This shows that the initial time t_0 plays an important role in the stability of a non-autonomous system.

2.2 Orbital Stability.

Let $Q_t(t_0)$ be a p-periodic solution, $P > 0$, of the autonomous system

$$\dot{x}(t) = f(x), \qquad x(t_0) = x_0 \in E^n \qquad (2.6)$$

and let Γ represent the closed orbit of $Q_t(x)$ in the state space.

i.e. $\Gamma = \{y | y = Q_t(x_0), 0 \leq t < P\}$ (2.7)

If, for any $\varepsilon > 0$, there exists a constant $\delta = \delta(\varepsilon) > 0, such\ that\ \ for\ any\ x_0$ satisfying

$$d(x_0\,,\Gamma) := \inf_{y \epsilon \Gamma} \|x_0 - y\| < \delta,$$

The solution of the system $Q_t(t_0)$ satisfies

$$d(Q_t(x_0)\,,\Gamma) < \varepsilon, for\ all\ t\ \geq 0$$

Then this P-periodic solution trajectory $Q_t(x_0)$ is said to be orbitally stable [3]. If we talk of equilibrium of certain point in the space, we may refer to property of the system located at such a point. We can see a kind of equilibrium we encounter in the earth motion around the sun. In this case, we do not have an equilibrium situation like the one that was described just above. Still, this system is also in some kind of equilibrium in the sense that this system repeats its motion periodically. We have a closed part which, after a year repeats itself. It is meaningful to ask the question about stability in both of these cases. However, these two phenomenon are so different that they require different concepts of stability in order to catch the important property of each system.

The orbital stability differs from Lyapunov stability described above in that it is concerned with the stability of a system output trajectory under small external perturbations. [6]

2.3 *Structural Stability*

Two systems are said to be topologically orbitally equivalent if there exists a homomorphism that transforms the family of trajectories of the first system to that of the second while preserving their motion directions [6]. For instance, systems $\dot{x} = x\ and\ \dot{x} = 2x$ are topologically equivalent, but are not so between $\dot{x} = x\ and\ \dot{x} = \sqrt{x}$.

So, if the dynamics of the system in the state space changes radically, for example by the appearance of a new equilibrium or a new periodic orbit, due to small external perturbation, then the system is said to be structurally unstable [5]

Let us consider the following set of functions.

$$\varphi = \left\{ g(x) | \|g(x)\| < \infty, \left\| \frac{\partial g(x)}{\partial x} \right\| < \infty\ for\ all\ x\ \epsilon\ E^n \right\} \tag{2.8}$$

If for any $g\ \epsilon\ \varphi$, there exists an $\varepsilon > 0$ such that orbits of the two systems

$$\dot{x} = f(x) \text{ and } \dot{x} = f(x) + \varepsilon g(x) \tag{2.9}$$

are topologically orbitally equivalent, then the autonomous system (2.6), the first unperturbed system, is said to be structurally stable.

Example

Let $\dot{x} = x$. This is structurally stable but $\dot{x} = x^2$ is not in the neighbourhood of the origin. This is because when the second system is slightly perturbed, to become, say $\dot{x} = x^2 + \varepsilon$, where $\varepsilon > 0$, then the resulting system has two equilibria $x_1^* = \sqrt{\varepsilon}$ *and* $x_2^* = -\sqrt{\varepsilon}$. This has more numbers of equilbria than the original system that has only one equilibrium $x^* = 0$

3. TOTAL STABILITY: STABILITY UNDER PERSISTENT PERTURBATIONS.

Consider a non-autonomous systems of the form

$$\dot{x} = f(x,t) + h(x,t),\ x(x_0) = x_0 \epsilon E^n \quad (3.1)$$

where f is continuously differential, with f(0,t) = 0 and h is a persistent perturbation, in the sense that for any $\varepsilon > 0$, there are two positive constants δ_1, δ_2 *such that if* $\|h(x^*,t)\| < \delta_1$ *for* $t \epsilon [t_0, \infty)$ *and if* $\|x^*(t_0)\| < \delta_2$, *then* $\|x^*(t)\| < \varepsilon$.

The equilibrium $x^* = 0$ of the unperturbed system, i.e. system

$$\dot{x} = f(x,t),\ x(x_0) = x_0 \epsilon E^n \quad (3.2)$$

is said to be totally stable if the persistently perturbed system (3.1) remains to be stable in the sense of Lyapunov [10] In this case, all uniformly and asymptotically stable systems with persistent perturbations are totally stable. i.e. stable orbit starting from a neighbourhood of another will stay nearly. See [4]. [5]. These lead us to the following theorems without proofs.

Theorem 3.1

If the unperturbed system (3.2) is uniformly and asymptotically stable about its equilibrium $x^* = 0$, then it is totally stable, namely, the persistently perturbed system (3.1) remains stable in the sense of Lyapunov.

If we consider an autonomous system have the following

$$\dot{x} = f(x) + h(x,t),\ x \epsilon E^n \quad (3.3)$$

We have the following;

Theorem 3.2 *(Perturbed orbital stability Theorem*)

If $Q_t(t_0)$ is an orbitally stable solution of the unperturbed autonomous systems (i.e. (3.3) with $h = 0$ there in, then it is totally stable. i.e. the perturbed system (3.3) remains orbitally stable under persistent perturbations.

4. CONCLUDING REMARKS

This paper has offered a brief introduction to the basic theory of methodology of Lyapunov stability, orbital stability and structural stability. Several important classes of nonlinear systems have been omitted in the discussion of various stability issues. Discussion of more advanced nonlinear systems, such as infinite-dimensional nonlinear systems, differential equations, non-linear stochastic systems are beyond the scope of this elementary expository paper.

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