Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 15, Number 2 (2019), pp. 113-124

http://www.ripublication.com/gjpam.htm

The Stochastic Process and Fokker-Planck Equation for Elastic Fluid on S_s^7

Nurfarisha[1], Adhitya Ronnie Effendie[2], and Muhammad Farchani Rosyid[1]

[1]*Kelompok Penelitian Kosmologi, Astrofisika, Fisika Matematik dan Fisika Partikel (KAMP), Departemen Fisika, Fakultas Matematika dan Ilmu Pengetahuan Alam, Universitas Gadjah Mada Sekip Utara Bulaksumur, Yogyakarta, 55281, Indonesia.*

[2]*Departemen Matematika, Fakultas Matematika dan Ilmu Pengetahuan Alam, Universitas Gadjah Mada.*

Abstract

In this article, we discuss several topics concerning the relationship between the stochastic process and the geometric structure of the manifolds. The purpose of the present report is to survey of the recent researches related to the theory of stochastic differential equations and a standard 7-sphere S_s^7. Since S_s^7 is assumed to be a measurable space for the stochastic processes. Hence, the Fokker-Planck equation and elastic fluid equation on S_s^7 is discussed and also the corresponding entropy rate and Elastic Fluid is derived.

Keywords: Stochastic process; Stratonovich stochastic differential equations; Fokker-Planck equation; entropy rate; Killing vector fields.

Mathematical Subject Classification:60G05; 60D05; 51H25; 35R60

1. INTRODUCTION

A stochastic process is the mathematical abstraction of an empirical process whose evolution is governed by the probabilistic laws [1]. Let $(\Omega, \mathcal{F}, P)$ a complete probability space equipped with a right-continuous filtration $(\mathcal{F}_t)_{t\geq 0}$ such that each $\mathcal{F}_t$ contains all the events of probability (measure) zero from the σ-algebra $\mathcal{F}$. We denote by $Ef = \int_\Omega f dP$ the expectation of a random variable $f : \Omega \to \mathbb{R}$. The conditional expectation of f given a sub-σ-field $\mathcal{G} \subset \mathcal{F}$ will be denoted by $E(\mathcal{F}|\mathcal{G})$. Technically, a stochastic process as a family of random variables $\{X_t, t \in T\}$ defined on Ω and parameterised by time set T. Here, X_t is practically observed at time t [5, 8, 9].

The theory of stochastic differential equations was originally developed by mathematicians as a tool to explicitly construct the trajectories of diffusion processes for given of drift and diffusion coefficients. The differential equations for random functions (stochastic processes) arise in the investigation of numerous physics and engineering problems [2, 4]. They are usually one of the following two fundamentally different types. Stochastic differential equations (SDEs) play important roles in stochastic modeling. For examples, in Economics, the solutions of a SDEs are applied to model the share prices of a stock price and in Biology, the solutions of the stochastic partial differential equations describe the number of populations at every point of time [3].

A n-dimensional unit sphere is the interesting example of a compact Riemannian homogeneous space. The surface of the sphere is "parametrized" by n-coordinates. The surface of a sphere is differentiable everywhere. For $n \geq 0$, the (unit) n-sphere is the subset $S^n \subset \mathbb{R}^{n+1}$ defined by $S^n := \{(x^1, \cdots, x^{n+1}) \in \mathbb{R}^{n+1} | \sum_{i=1}^{n+1}(x^i)^2 = 1\}$. Sometimes it is useful to think of an odd-dimensional sphere S^{2n+1} as a subset of $\mathbb{C}^{n+1}$, by means of the usual identification of $\mathbb{C}^{n+1}$ with $\mathbb{R}^{2n+2}$ [7]. In mathematical physics, spheres are topological spaces which are interesting to investigate, and in many branches of physics, they provide for instance as models for configuration spaces of some mechanical systems. In Physics, for example, the standard 7-dimensional sphere S^7_s is particularly interested in related to supersymmetry breaking [6] and to the work of Witten [11] in which he used it to cancel the global gravitational anomalies in 1985.

The main aim of this research is to relate geometrical and topological properties of vector fields of manifolds to the stochastic processes on it. In this case, a standard 7-sphere S^7_s is assumed to be a measurable space for stochastic processes. The Fokker-Planck equation on S^7_s is discussed and the corresponding entropy rate is derived.

The organization of this paper is as follows. In Subsec. 2.1, we review the theory of stochastic differential equations on the Riemannian manifold. In Subsec. 2.2, we review the standard 7-sphere S^7. In Sec. 3, we discuss the result, that is, stochastic differential equations on S^7_s related to Killing vector fields and also formulate the Fokker-Planck differential equation associated with the SDEs. In this section, the concepts of information-theoretic entropy are defined. A natural issue to be addressed is how the entropy $S(f)$ behaves as a function of time when probability density function $f(z, t)$ satisfies a Fokker-Planck equation.

2. PRELIMINARIES

2.1. Stochastic Differential Equations

The definition of stochastic differential equations considered here is similarly defined by Ikeda and Watanabe [9]. The stochastic differential equations and Fokker-Planck equation can be formulated for stochastic processes in any coordinate patch of a manifold in a way that is very similar to the case of $\mathbb{R}^d$. Let $\mathcal{M}$ be a differentiable manifold. Consider on $\mathcal{M}$ the stochastic differential equations (SDEs):

$$dx_t = X(x_t) \circ dB_t + A(x_t)dt. \tag{1}$$

Here B_t is a d-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ (also called as stochastic basis), X is C^3 from $\mathbb{R}^m \times \mathcal{M}$ to the tangent bundle $T\mathcal{M}$ with $X(x) : \mathbb{R}^m \to T_x\mathcal{M}$ a linear map for each x in $\mathcal{M}$, and A is a C^2 vector field on $\mathcal{M}$.

We are mainly interested in non-degenerate stochastic differential equations. Recall that (1) is said to be a Brownian system (with drift A) if it has generator $\frac{1}{2}\Delta(+A)$ for Δ the Laplacian. It is called a gradient Brownian system (with drift A) if X is given by an isometric immersion $j : \mathcal{M} \to \mathbb{R}^m$, i.e. for each $e \in \mathbb{R}^m$ and $x \in \mathcal{M}$, $X(x)(e)$ is given by $\Delta\langle j(x), e\rangle$. The solution flow to the stochastic differential equations is then a Brownian flow or gradient Brownian flow respectively. If $A = \nabla h$ for some function h on the manifold, then we have h-Brownian systems [9].

2.2. Standard 7-Sphere S_s^7

It is easy to construct a C^∞ atlas on S^7, showing that S^7 has some differential structures. The topological space S^7 equipped with a standard differential structure is called the standard 7-sphere S_s^7. Every vector fields on S_s^7 will be orthogonal to every point, it is clear that every vector fields on S_s^7 orthogonal to the radial vector field N_p defined by

$$N_p = \sum_{i=1}^{8} z^i \frac{\partial}{\partial z^i}\bigg|_p, \tag{2}$$

at every $p \in S_s^7$ with respect to the Euclidean product $\langle \cdot, \cdot \rangle$ of $\mathbb{R}^8$. Let $\mathfrak{X}(S_s^7)$ be the set of all differentiable vector fields on S_s^7. In particular, we have the following vector fields on S_s^7:

$$\begin{aligned}
U_1 &= z^2\partial_1 - z^1\partial_2 + z^4\partial_3 - z^3\partial_4 + z^6\partial_5 - z^5\partial_6 + z^8\partial_7 - z^7\partial_8 \\
&= -\left(z^1\frac{\partial}{\partial z^2} - z^2\frac{\partial}{\partial z^1}\right) - \left(z^3\frac{\partial}{\partial z^4} - z^4\frac{\partial}{\partial z^3}\right) - \left(z^5\frac{\partial}{\partial z^6} - z^6\frac{\partial}{\partial z^5}\right) - \left(z^7\frac{\partial}{\partial z^8} - z^8\frac{\partial}{\partial z^7}\right) \\
&= -U_{12} - U_{34} - U_{56} - U_{78} \\
U_2 &= z^3\partial_1 - z^4\partial_2 - z^1\partial_3 + z^2\partial_4 - z^7\partial_5 + z^8\partial_6 + z^5\partial_7 - z^6\partial_8
\end{aligned}$$

$$\begin{aligned}
&= -\left(z^1\frac{\partial}{\partial z^3}-z^3\frac{\partial}{\partial z^1}\right)+\left(z^2\frac{\partial}{\partial z^4}-z^4\frac{\partial}{\partial z^2}\right)+\left(z^5\frac{\partial}{\partial z^7}-z^7\frac{\partial}{\partial z^5}\right)-\left(z^6\frac{\partial}{\partial z^8}-z^8\frac{\partial}{\partial z^6}\right)\\
&= -U_{13}+U_{24}+U_{57}-U_{68}
\end{aligned}$$

$$\begin{aligned}
U_3 &= z^4\partial_1+z^3\partial_2-z^2\partial_3-z^1\partial_4+z^8\partial_5+z^7\partial_6-z^6\partial_7-z^5\partial_8\\
&= -\left(z^1\frac{\partial}{\partial z^4}-z^4\frac{\partial}{\partial z^1}\right)-\left(z^2\frac{\partial}{\partial z^3}-z^3\frac{\partial}{\partial z^2}\right)-\left(z^5\frac{\partial}{\partial z^8}-z^8\frac{\partial}{\partial z^5}\right)-\left(z^6\frac{\partial}{\partial z^7}-z^7\frac{\partial}{\partial z^6}\right)\\
&= -U_{14}-U_{23}-U_{58}-U_{67}\\
U_4 &= z^5\partial_1-z^6\partial_2+z^7\partial_3-z^8\partial_4-z^1\partial_5+z^2\partial_6-z^3\partial_7+z^4\partial_8\\
&= -\left(z^1\frac{\partial}{\partial z^5}-z^5\frac{\partial}{\partial z^1}\right)+\left(z^2\frac{\partial}{\partial z^6}-z^6\frac{\partial}{\partial z^2}\right)-\left(z^3\frac{\partial}{\partial z^7}-z^7\frac{\partial}{\partial z^3}\right)+\left(z^4\frac{\partial}{\partial z^8}-z^8\frac{\partial}{\partial z^4}\right)\\
&= -U_{15}+U_{26}-U_{37}+U_{48}\\
U_5 &= z^6\partial_1+z^5\partial_2-z^8\partial_3-z^7\partial_4-z^2\partial_5-z^1\partial_6+z^4\partial_7+z^3\partial_8\\
&= -\left(z^1\frac{\partial}{\partial z^6}-z^6\frac{\partial}{\partial z^1}\right)-\left(z^2\frac{\partial}{\partial z^5}-z^5\frac{\partial}{\partial z^2}\right)+\left(z^3\frac{\partial}{\partial z^8}-z^8\frac{\partial}{\partial z^3}\right)+\left(z^4\frac{\partial}{\partial z^7}-z^7\frac{\partial}{\partial z^4}\right)\\
&= -U_{16}-U_{25}+U_{38}+U_{47}\\
U_6 &= z^7\partial_1-z^8\partial_2-z^5\partial_3+z^6\partial_4+z^3\partial_5-z^4\partial_6-z^1\partial_7+z^2\partial_8\\
&= -\left(z^1\frac{\partial}{\partial z^7}-z^7\frac{\partial}{\partial z^1}\right)+\left(z^2\frac{\partial}{\partial z^8}-z^8\frac{\partial}{\partial z^2}\right)+\left(z^3\frac{\partial}{\partial z^5}-z^5\frac{\partial}{\partial z^3}\right)-\left(z^4\frac{\partial}{\partial z^6}-z^6\frac{\partial}{\partial z^4}\right)\\
&= -U_{17}+U_{28}+U_{35}-U_{46}\\
U_7 &= z^8\partial_1+z^7\partial_2+z^6\partial_3+z^5\partial_4-z^4\partial_5-z^3\partial_6-z^2\partial_7-z^1\partial_8\\
&= -\left(z^1\frac{\partial}{\partial z^8}-z^8\frac{\partial}{\partial z^1}\right)-\left(z^2\frac{\partial}{\partial z^7}-z^7\frac{\partial}{\partial z^2}\right)-\left(z^3\frac{\partial}{\partial z^6}-z^6\frac{\partial}{\partial z^3}\right)-\left(z^4\frac{\partial}{\partial z^5}-z^5\frac{\partial}{\partial z^4}\right)\\
&= -U_{18}-U_{27}-U_{36}-U_{45}. \qquad (3)
\end{aligned}$$

All the vector fields mentioned above form a frame $\{U_1,U_2,\cdots,U_7\}$, i.e., at every point $p\in S^7_s$ $\{U_1(p),U_2(p),\cdots,U_7(p)\}$ is a basis of tangential space $T_pS^7_s$. All vector fields contained in the frame are Killing vector fields on S^7_s, for the vector fields of the form

$$z^i\frac{\partial}{\partial z^j}-z^j\frac{\partial}{\partial z^i},\quad i\neq j \qquad (4)$$

are Killing and the fact that the Lie derivative of the metric g satisfies $\mathcal{L}_{V+W}g=\mathcal{L}_Vg+\mathcal{L}_Wg$ for arbitrary vector fields V and W on S^7_s. For those vector fields in the frame, we obtain the following systems of Stratonovich stochastic differential equations

$$d\bar{z}_{\mu t}=U_\mu(\bar{z}_t)\circ dW_t, \qquad (\mu=1,2,\cdots,7) \qquad (5)$$

or

$$d\bar{z}_{1t}=\begin{bmatrix} z_t^2\circ dW_t\\ -z_t^1\circ dW_t\\ z_t^4\circ dW_t\\ -z_t^3\circ dW_t\\ z_t^6\circ dW_t\\ -z_t^5\circ dW_t\\ z_t^8\circ dW_t\\ -z_t^7\circ dW_t\end{bmatrix},\quad\cdots,\quad d\bar{z}_{7t}=\begin{bmatrix} z_t^8\circ dW_t\\ z_t^7\circ dW_t\\ z_t^6\circ dW_t\\ z_t^5\circ dW_t\\ -z_t^4\circ dW_t\\ -z_t^3\circ dW_t\\ -z_t^2\circ dW_t\\ -z_t^1\circ dW_t\end{bmatrix}, \qquad (6)$$

where W_t is a semimartingale. Semimartingales are the most general and natural setup for stochastic integration and differentiation, in the sense that stochastic differential equations formulated using semimartingales have semimartingales as solutions.

Since $U_1, U_2, \cdots$, and U_7 are Killing vector fields, the flows generated by equation (5) are isometric. The isometric stochastic flows $\bar{z}_{\mu t}$ will be called *frame isometric stochastic flows*. Due to equation (6) and properties of W_t, the process $\bar{z}_{\mu t}$ is a semimartingale for all $\mu = 1, \cdots, 7$.

3. RESULTS AND DISCUSSION

The equation (6) can be written as follows:

$$d\bar{z}_{1t}^{\,i} = \sum_{k=1}^{8} \delta_{ik} U_\mu^i(\bar{z}_t) \circ dW_t, \tag{7}$$

where $U_\mu^1, U_\mu^2, \cdots, U_\mu^8$ are the components of U_μ.

Associated to stochastic differential equations (7) there are the Fokker-Planck equations representing or controlling the probability density function for the position or the velocity of the particle whose motion is described by equation (7). The Fokker-Planck equations corresponding to (7) that describes the evolution of the probability density function $f(z,t)$ for this process are given by

$$\begin{aligned} \frac{\partial f(z,t)}{\partial t} &= -\frac{1}{2}\frac{1}{\sqrt{G(z)}} \sum_{i=1}^{8} \frac{\partial}{\partial z_i}\Bigg(\sum_{j,k=1}^{8} \delta_{kj} U_\mu^j(\bar{z}_t) \frac{\partial\big(\delta_{ik} U_\mu^i(\bar{z}_t)\big)}{\partial z_k}\; f(z,t)\; \sqrt{G(z)}\Bigg) \\ &\quad + \frac{1}{2}\frac{1}{\sqrt{G(z)}} \sum_{i,j=1}^{8} \frac{\partial^2}{\partial z_i \partial z_j}\Bigg(\sum_{k=1}^{8} \delta_{ik} U_\mu^i(\bar{z}_t) \delta_{kj} U_\mu^j(\bar{z}_t)\; f(z,t)\; \sqrt{G(z)}\Bigg), \end{aligned} \tag{8}$$

where $G(z)$ is the determinant of the metric tensor on the Riemannian manifold.

In physics, entropy is a measure of randomness or disorder in a statistical mechanical system. Entropy is related to distribution function or probability density function [3, 10]. The entropy of a probability density function (8) on S_s^7 is defined as

$$S(f(z,t)) = -\int_{z\in D} f(z,t) \log f(z,t) \sqrt{G(z)} d(z), \tag{9}$$

where $D \subset \mathbb{R}^8$ is the coordinate domain.

Now the vector fields $U_1, U_2, \cdots, U_7$ can be expressed in the spherical coordinate system by

$$\begin{aligned}
U_1 &= \cos\varphi_2 \frac{\partial}{\partial\varphi_1} - \cot\varphi_1 \sin\varphi_2 \frac{\partial}{\partial\varphi_2} + \cos\varphi_4 \frac{\partial}{\partial\varphi_3} - \cot\varphi_3 \sin\varphi_4 \frac{\partial}{\partial\varphi_4} + \cos\varphi_6 \frac{\partial}{\partial\varphi_5} \\
&\quad - \cot\varphi_5 \sin\varphi_6 \frac{\partial}{\partial\varphi_6} - \frac{\partial}{\partial\varphi_7} \\
U_2 &= \sin\varphi_2 \cos\varphi_3 \frac{\partial}{\partial\varphi_1} + \Big((\cot\varphi_3 \cos\varphi_3 - \csc\varphi_3)\cos\varphi_4 + \cot\varphi_1 \cos\varphi_2 \cos\varphi_3 \Big) \frac{\partial}{\partial\varphi_2} \\
&\quad + \Big((\cot\varphi_3 \cos\varphi_3 - \csc\varphi_3)\cot\varphi_1 \csc\varphi_2 - \cot\varphi_2 \cos\varphi_3 \cos\varphi_4 \Big) \frac{\partial}{\partial\varphi_3} \\
&\quad + \cot\varphi_2 \csc\varphi_3 \sin\varphi_4 \frac{\partial}{\partial\varphi_4} + \sin\varphi_6 \cos\varphi_7 \frac{\partial}{\partial\varphi_5} + \Big(\cot\varphi_5 \cos\varphi_6 \cos\varphi_7 + \sin\varphi_7 \Big) \frac{\partial}{\partial\varphi_6} \\
&\quad + \Big(\cot\varphi_6 \cos\varphi_7 - \cot\varphi_5 \csc\varphi_6 \sin\varphi_7 \Big) \frac{\partial}{\partial\varphi_7} \\
U_3 &= \sin\varphi_2 \sin\varphi_3 \cos\varphi_4 \frac{\partial}{\partial\varphi_1} + \Big(\cot\varphi_1 \cos\varphi_2 \sin\varphi_3 \cos\varphi_4 + \cos\varphi_3 \Big) \frac{\partial}{\partial\varphi_2} \\
&\quad + \Big(\cot\varphi_1 \csc\varphi_2 \cos\varphi_3 \cos\varphi_4 - \cot\varphi_2 \sin\varphi_3 \Big) \frac{\partial}{\partial\varphi_3} \\
&\quad - \cot\varphi_1 \csc\varphi_2 \csc\varphi_3 \sin\varphi_4 \frac{\partial}{\partial\varphi_4} + \sin\varphi_6 \sin\varphi_7 \frac{\partial}{\partial\varphi_5} + \Big(\cot\varphi_5 \cos\varphi_6 \sin\varphi_7 - \cos\varphi_7 \Big) \frac{\partial}{\partial\varphi_6} \\
&\quad + \Big(\cot\varphi_6 \sin\varphi_7 + \cot\varphi_5 \csc\varphi_6 \cos\varphi_7 \Big) \frac{\partial}{\partial\varphi_7}
\end{aligned}$$

$$\begin{aligned}
U_4 &= \sin\varphi_2 \sin\varphi_3 \sin\varphi_4 \cos\varphi_5 \frac{\partial}{\partial\varphi_1} \\
&\quad + \Big((\cot\varphi_1 \cos\varphi_2 + \cot\varphi_5 \cos\varphi_6 - \sec\varphi_5 \csc\varphi_5 \cos\varphi_6) \sin\varphi_3 \sin\varphi_4 \cos\varphi_5 \Big) \frac{\partial}{\partial\varphi_2} \\
&\quad + \Big(\big(\cos\varphi_6 \cot\varphi_6 \cos\varphi_7 + \csc\varphi_6 \cos\varphi_7 + \cot\varphi_2 \cos\varphi_3 \sin\varphi_6 \cot\varphi_6 \big) \sin\varphi_5 \\
&\qquad + \cot\varphi_1 \csc\varphi_2 \cos\varphi_3 \cos\varphi_5 \Big) \sin\varphi_4 \frac{\partial}{\partial\varphi_3} \\
&\quad + \Big(\big(\cot\varphi_1 \csc\varphi_2 - \cot\varphi_2 \sin\varphi_5 \big) \csc\varphi_3 \cos\varphi_5 - \big(\cot\varphi_3 \cos\varphi_7 + \sin\varphi_7 \big) \sin\varphi_5 \sin\varphi_6 \Big) \\
&\qquad \cos\varphi_4 \frac{\partial}{\partial\varphi_4} + \Big((\cot\varphi_5 \cos\varphi_5 - \csc\varphi_5) \cot\varphi_1 \csc\varphi_2 \csc\varphi_3 \csc\varphi_4 - \\
&\qquad \big((\cot\varphi_4 \sin\varphi_7 + \cot\varphi_3 \csc\varphi_4 \cos\varphi_7) \sin\varphi_6 + \cot\varphi_2 \csc\varphi_3 \csc\varphi_4 \cos\varphi_6 \big) \cos\varphi_5 \Big) \frac{\partial}{\partial\varphi_5} \\
&\quad + \Big(\big(\cot\varphi_5 \csc\varphi_6 - \cot\varphi_2 \cot\varphi_6 \cos\varphi_6 \big) \csc\varphi_3 \csc\varphi_4 \\
&\qquad - \big(\cot\varphi_4 \sin\varphi_7 + \cot\varphi_3 \csc\varphi_4 \cos\varphi_7 \big) \cos\varphi_6 \Big) \csc\varphi_5 \frac{\partial}{\partial\varphi_6} \\
&\quad + \Big(\cot\varphi_3 \csc\varphi_4 \sin\varphi_7 - \cot\varphi_4 \cos\varphi_7 \Big) \csc\varphi_5 \csc\varphi_6 \frac{\partial}{\partial\varphi_7}
\end{aligned} \tag{10}$$

$$\begin{aligned}
U_5 \;=\;& \sin\varphi_2 \sin\varphi_3 \sin\varphi_4 \sin\varphi_5 \cos\varphi_6 \frac{\partial}{\partial\varphi_1} \\
&+\Big(\cos\varphi_5 + \big(\csc\varphi_5 - \cos\varphi_5 \cot\varphi_5\big) \cot\varphi_1 \cos\varphi_2 \cos\varphi_6 \Big) \sin\varphi_3 \sin\varphi_4 \frac{\partial}{\partial\varphi_2} \\
&+\Big(\cot\varphi_1 \csc\varphi_2 \cos\varphi_3 \sin\varphi_5 \cos\varphi_6 + \cot\varphi_2 \cos\varphi_3 \cos\varphi_5 - \sin\varphi_5 \sin\varphi_6 \sin\varphi_7 \Big) \sin\varphi_4 \frac{\partial}{\partial\varphi_3} \\
&+\Big(\big(\cot\varphi_1 \csc\varphi_2 \sin\varphi_5 \cos\varphi_6 + \cot\varphi_2 \cos\varphi_5\big) \csc\varphi_3 \cos\varphi_4 \\
&\quad + \big(\cot\varphi_3 \cos\varphi_4 \sin\varphi_7 + \cos\varphi_7\big) \sin\varphi_5 \sin\varphi_6 \Big) \frac{\partial}{\partial\varphi_4} \\
&+\Big(\big(\cot\varphi_1 \csc\varphi_2 \cos\varphi_5 \cos\varphi_6 - \cot\varphi_2 \csc\varphi_5 + \cos\varphi_2 \cos\varphi_5 \cot\varphi_5\big) \csc\varphi_3 \csc\varphi_4 \\
&\quad - \big(\cot\varphi_4 \cos\varphi_7 + \cot\varphi_3 \csc\varphi_4 \sin\varphi_7\big) \cos\varphi_5 \sin\varphi_6 \Big) \frac{\partial}{\partial\varphi_5} \\
&+\Big(\big(\cot\varphi_6 \cos\varphi_6 - \csc\varphi_6\big) \cot\varphi_1 \csc\varphi_2 \csc\varphi_3 \csc\varphi_4 - \big(\cot\varphi_4 \cos\varphi_7 \\
&\quad + \cot\varphi_3 \csc\varphi_4 \sin\varphi_7\big) \cos\varphi_6 \Big) \csc\varphi_5 \frac{\partial}{\partial\varphi_6} \\
&+\Big(\cot\varphi_4 \sin\varphi_7 - \cot\varphi_3 \csc\varphi_4 \cos\varphi_7 \Big) \csc\varphi_5 \csc\varphi_6 \frac{\partial}{\partial\varphi_7}
\end{aligned}$$

$$\begin{aligned}
U_6 \;=\;& \sin\varphi_2 \sin\varphi_3 \sin\varphi_4 \sin\varphi_5 \sin\varphi_6 \cos\varphi_7 \frac{\partial}{\partial\varphi_1} \\
&+\Big(\cot\varphi_1 \cos\varphi_2 \cos\varphi_7 + \sin\varphi_7 \Big) \sin\varphi_3 \sin\varphi_4 \sin\varphi_5 \sin\varphi_6 \frac{\partial}{\partial\varphi_2} \\
&+\Big(\big(\cot\varphi_2 \sin\varphi_7 + \cot\varphi_1 \csc\varphi_2 \cos\varphi_7\big) \cos\varphi_3 \sin\varphi_5 \sin\varphi_6 + \cos\varphi_5 \Big) \sin\varphi_4 \frac{\partial}{\partial\varphi_3} \\
&+\Big(\big(\csc\varphi_5 \csc\varphi_6 - \cot\varphi_5 \cos\varphi_5 \sin\varphi_6 - \csc\varphi_5 \cot\varphi_6 \cos\varphi_6\big) \cot\varphi_2 \csc\varphi_3 \sin\varphi_7 \\
&\quad + \cot\varphi_3 \cos\varphi_5 + \cot\varphi_1 \csc\varphi_2 \csc\varphi_3 \sin\varphi_5 \sin\varphi_6 \cos\varphi_7 \Big) \cos\varphi_4 \frac{\partial}{\partial\varphi_4} \\
&+\Big(\cot\varphi_3 \csc\varphi_4 \cot\varphi_5 \cos\varphi_5 + \big(\cot\varphi_4 \cos\varphi_6 \\
&\quad - (\cot\varphi_2 \sin\varphi_7 + \cot\varphi_1 \csc\varphi_2 \cos\varphi_7) \csc\varphi_3 \csc\varphi_4 \sin\varphi_6\big) \cos\varphi_5 \\
&\quad + \cot\varphi_3 \csc\varphi_4 \csc\varphi_5 \Big) \frac{\partial}{\partial\varphi_5} \\
&+\Big((\csc\varphi_6 - \cot\varphi_6 \cos\varphi_6) \cot\varphi_4 \\
&\quad + (\cot\varphi_1 \csc\varphi_2 + \cot\varphi_2 \sin\varphi_7) \csc\varphi_3 \csc\varphi_4 \cos\varphi_6 \Big) \csc\varphi_5 \frac{\partial}{\partial\varphi_6} \\
&+\Big(\cot\varphi_2 \cos\varphi_7 - \cot\varphi_1 \csc\varphi_2 \sin\varphi_7 \Big) \csc\varphi_3 \csc\varphi_4 \csc\varphi_5 \csc\varphi_6 \frac{\partial}{\partial\varphi_7}
\end{aligned}$$

$$U_7 \;=\; \sin\varphi_2 \sin\varphi_3 \sin\varphi_4 \sin\varphi_5 \sin\varphi_6 \sin\varphi_7 \frac{\partial}{\partial\varphi_1}$$

$$
\begin{aligned}
&+\Big(\cot\varphi_1\cos\varphi_2\sin\varphi_7-\cos\varphi_7\Big)\sin\varphi_3\sin\varphi_4\sin\varphi_5\sin\varphi_6\frac{\partial}{\partial\varphi_2}\\
&+\Big(\cos\varphi_6-\big(\cot\varphi_2\cos\varphi_7-\cot\varphi_1\csc\varphi_2\sin\varphi_7\big)\cos\varphi_3\sin\varphi_6\Big)\sin\varphi_4\sin\varphi_5\frac{\partial}{\partial\varphi_3}\\
&+\Big(\Big[\big(\cos\varphi_6\cot\varphi_6-\csc\varphi_6\big)\cot\varphi_2\csc\varphi_3\cos\varphi_7+\cot\varphi_3\cos\varphi_6\\
&\quad+\big(\csc\varphi_6-\cos\varphi_6\cot\varphi_6\big)\cot\varphi_1\csc\varphi_2\csc\varphi_3\sin\varphi_7\Big]\cos\varphi_4\sin\varphi_5+\cos\varphi_5\Big)\frac{\partial}{\partial\varphi_4}\\
&+\Big(\Big[\cot\varphi_4\cos^2\varphi_5+\cot\varphi_3\csc\varphi_4\cos\varphi_6-\cot\varphi_2\csc\varphi_3\csc\varphi_4\sin\varphi_6\cos\varphi_7\\
&\quad+\big(\cot\varphi_1-\cot\varphi_1\cos\varphi_6\big)\csc\varphi_2\csc\varphi_3\csc\varphi_4\sin\varphi_7\Big]\cos\varphi_5-\cot\varphi_4\csc\varphi_5\Big)\frac{\partial}{\partial\varphi_5}\\
&+\Big((\cot\varphi_6\cos\varphi_6-\csc\varphi_6)\cot\varphi_3\\
&\quad+(\cot\varphi_1\csc\varphi_2\sin\varphi_7-\cot\varphi_2\cos\varphi_7)\csc\varphi_3\cos\varphi_6\Big)\csc\varphi_4\csc\varphi_5\frac{\partial}{\partial\varphi_6}\\
&+\Big(\cot\varphi_2\sin\varphi_7+\cot\varphi_1\csc\varphi_2\cos\varphi_7\Big)\csc\varphi_3\csc\varphi_4\csc\varphi_5\csc\varphi_6\frac{\partial}{\partial\varphi_7}. \qquad (11)
\end{aligned}
$$

The Fokker-Planck equation for Stratonovich version (8) in the spherical coordinate system is given by

$$
\begin{aligned}
\frac{\partial f}{\partial t} &= -\frac{1}{2}\Big(\sqrt{G(\varphi_1,\cdots,\varphi_7)}\Big)^{-1}\sum_{i=1}^{7}\frac{\partial}{\partial\varphi_i}\Big(U_\mu^i\frac{\partial U_\mu^i}{\partial\varphi_i}\,f\,\sqrt{G(\varphi_1,\cdots,\varphi_7)}\Big)\\
&\quad+\frac{1}{2}\Big(\sqrt{G(\varphi_1,\cdots,\varphi_7)}\Big)^{-1}\sum_{i=1}^{7}\frac{\partial^2}{\partial{\varphi_i}^2}\Big({U_\mu^i}^2 f\,\sqrt{G(\varphi_1,\cdots,\varphi_7)}\Big), \qquad (12)
\end{aligned}
$$

where the volume element is given by

$$
\begin{aligned}
d_{S_s^7}V(\varphi_1,\cdots,\varphi_7) &= \sqrt{G(\varphi_1,\cdots,\varphi_7)}\,d\varphi_1 d\varphi_2 d\varphi_3 d\varphi_4 d\varphi_5 d\varphi_6 d\varphi_7\\
&= \sin^6(\varphi_1)\sin^5(\varphi_2)\sin^4(\varphi_3)\sin^3(\varphi_4)\sin^2(\varphi_5)\sin(\varphi_6)\\
&\quad d\varphi_1 d\varphi_2 d\varphi_3 d\varphi_4 d\varphi_5 d\varphi_6 d\varphi_7. \qquad (13)
\end{aligned}
$$

The entropy of the conditional transition probability density function $f(\varphi_1,\cdots,\varphi_7;t)$ describing the distribution of states of a random fields z on S_s^7 is given by the integral

$$S(f) = -\int_{S_s^7} f\,\log f\,d_{S_s^7}V. \qquad (14)$$

Quantities of the form $S(f(\varphi_1,\cdots,\varphi_7;t))$ is information-theoretic entropy playing a central role in information theory as measures of information, choice and uncertainty.

Differentiating (14) with respect to time gives

$$\frac{dS}{dt} = -\frac{1}{2}\int_{S_s^7}\sum_{i=1}^{7}{U_\mu^i}^2\left(-\frac{1}{f}\left(\frac{\partial f}{\partial\varphi_i}\right)^2+\frac{\partial^2 f}{\partial{\varphi_i}^2}\right)d_{S_s^7}V. \qquad (15)$$

For example, if we choose $\mu=1$ and we substitute $U_1^1=\cos\varphi_2$, $U_1^2=-\cot\varphi_1\ \sin\varphi_2$, $U_1^3=\cos\varphi_4$,

$U_1^4 = -\cot\varphi_3\ \sin\varphi_4$, $U_1^5 = \cos\varphi_6$, $U_1^6 = -\cot\varphi_5\ \sin\varphi_6$, $U_1^7 = -1$ to equation (12), the Fokker-Planck equation is given by

$$\begin{aligned}\frac{\partial f}{\partial t} = & -\frac{1}{2}\Big(\sqrt{G(\varphi_1,\cdots,\varphi_7)}\Big)^{-1}\Bigg[\frac{\partial}{\partial\varphi_1}\Big(\cos\varphi_2\frac{\partial\cos\varphi_2}{\partial\varphi_1}f\ \sqrt{G(\varphi_1,\cdots,\varphi_7)}\Big)\\
&+\frac{\partial}{\partial\varphi_2}\Big(\cot\varphi_1\ \sin\varphi_2\frac{\partial\big(\cot\varphi_1\ \sin\varphi_2\big)}{\partial\varphi_2}f\ \sqrt{G(\varphi_1,\cdots,\varphi_7)}\Big)\\
&+\frac{\partial}{\partial\varphi_3}\Big(\cos\varphi_4\frac{\partial\cos\varphi_4}{\partial\varphi_3}f\ \sqrt{G(\varphi_1,\cdots,\varphi_7)}\Big)\\
&+\frac{\partial}{\partial\varphi_4}\Big(\cot\varphi_3\ \sin\varphi_4\frac{\partial\big(\cot\varphi_3\ \sin\varphi_4\big)}{\partial\varphi_4}f\ \sqrt{G(\varphi_1,\cdots,\varphi_7)}\Big)\\
&+\frac{\partial}{\partial\varphi_5}\Big(\cos\varphi_6\frac{\partial\cos\varphi_6}{\partial\varphi_5}f\ \sqrt{G(\varphi_1,\cdots,\varphi_7)}\Big)\\
&+\frac{\partial}{\partial\varphi_6}\Big(\cot\varphi_5\ \sin\varphi_6\frac{\partial\big(\cot\varphi_5\ \sin\varphi_6\big)}{\partial\varphi_6}f\ \sqrt{G(\varphi_1,\cdots,\varphi_7)}\Big)\\
&+\frac{\partial}{\partial\varphi_5}\Big(\frac{\partial\, 1}{\partial\varphi_5}f\ \sqrt{G(\varphi_1,\cdots,\varphi_7)}\Big)\Bigg]\\
&+\frac{1}{2}\Big(\sqrt{G(\varphi_1,\cdots,\varphi_7)}\Big)^{-1}\Bigg[\frac{\partial^2}{\partial{\varphi_1}^2}\Big(\cos^2\varphi_2\ f\ \sqrt{G(\varphi_1,\cdots,\varphi_7)}\Big)\\
&+\frac{\partial^2}{\partial{\varphi_2}^2}\Big(\cot^2\varphi_1\ \sin^2\varphi_2\ f\ \sqrt{G(\varphi_1,\cdots,\varphi_7)}\Big)+\frac{\partial^2}{\partial{\varphi_3}^2}\Big(\cos^2\varphi_4\ f\ \sqrt{G(\varphi_1,\cdots,\varphi_7)}\Big)\\
&+\frac{\partial^2}{\partial{\varphi_4}^2}\Big(\cot^2\varphi_3\ \sin^2\varphi_4\ f\ \sqrt{G(\varphi_1,\cdots,\varphi_7)}\Big)+\frac{\partial^2}{\partial{\varphi_5}^2}\Big(\cos^2\varphi_6\ f\ \sqrt{G(\varphi_1,\cdots,\varphi_7)}\Big)\\
&+\frac{\partial^2}{\partial{\varphi_6}^2}\Big(\cot^2\varphi_5\ \sin^2\varphi_6\ f\ \sqrt{G(\varphi_1,\cdots,\varphi_7)}\Big)-\frac{\partial^2}{\partial{\varphi_7}^2}\Big(f\ \sqrt{G(\varphi_1,\cdots,\varphi_7)}\Big)\Bigg].\end{aligned} \tag{16}$$

By substituting $\sqrt{G(\varphi_1,\cdots,\varphi_7)} = \sin^6(\varphi_1)\sin^5(\varphi_2)\sin^4(\varphi_3)\sin^3(\varphi_4)\sin^2(\varphi_5)\sin(\varphi_6)$ to equation (16), we obtain the following Fokker-Planck equation

$$\begin{aligned}\frac{\partial f}{\partial t} = & \cot^2\varphi_1\Big((39-43\sin^2\varphi_2)f+\frac{15}{4}\sin 2\varphi_2\frac{\partial f}{\partial\varphi_2}+\frac{1}{2}\sin^2\varphi_2\frac{\partial^2 f}{\partial{\varphi_2}^2}\Big)\\
&+\cos^2\varphi_2\Big(-3f+6\cot\varphi_1\frac{\partial f}{\partial\varphi_1}+\frac{1}{2}\frac{\partial^2 f}{\partial{\varphi_1}^2}\Big)\\
&+\cot^2\varphi_3\Big((18-21\sin^2\varphi_4)f+\frac{11}{4}\sin 2\varphi_4\frac{\partial f}{\partial\varphi_4}+\frac{1}{2}\sin^2\varphi_4\frac{\partial^2 f}{\partial{\varphi_4}^2}\Big)\\
&+\cos^2\varphi_4\Big(-\frac{3}{2}f+4\cot\varphi_3\frac{\partial f}{\partial\varphi_3}+\frac{1}{2}\frac{\partial^2 f}{\partial{\varphi_3}^2}\Big)\\
&+\cot^2\varphi_5\Big((5-7\sin^2\varphi_6)f+\frac{7}{4}\sin 2\varphi_6\frac{\partial f}{\partial\varphi_6}+\frac{1}{2}\sin^2\varphi_6\frac{\partial^2 f}{\partial{\varphi_6}^2}\Big)\\
&+\cos^2\varphi_6\Big(-f+2\cot\varphi_5\frac{\partial f}{\partial\varphi_5}+\frac{1}{2}\frac{\partial^2 f}{\partial{\varphi_5}^2}\Big)-\frac{1}{2}\frac{\partial^2 f}{\partial{\varphi_7}^2}.\end{aligned} \tag{17}$$

By substituting again $U_1^1, U_1^2, \cdots U_1^7$ to equation (15), the rate of information-theoretic entropy is obtained as follows:

$$\begin{aligned}\frac{dS}{dt} &= -\frac{1}{2}\int_{S_s^7}\Bigg[\cos^2\varphi_2\Big(-\frac{1}{f}\Big(\frac{\partial f}{\partial\varphi_1}\Big)^2+\frac{\partial^2 f}{\partial{\varphi_1}^2}\Big)+\cot^2\varphi_1\sin^2\varphi_2\Big(-\frac{1}{f}\Big(\frac{\partial f}{\partial\varphi_2}\Big)^2+\frac{\partial^2 f}{\partial{\varphi_2}^2}\Big)\\ &\quad+\cos^2\varphi_4\Big(-\frac{1}{f}\Big(\frac{\partial f}{\partial\varphi_3}\Big)^2+\frac{\partial^2 f}{\partial{\varphi_3}^2}\Big)+\cot^2\varphi_3\sin^2\varphi_4\Big(-\frac{1}{f}\Big(\frac{\partial f}{\partial\varphi_4}\Big)^2+\frac{\partial^2 f}{\partial{\varphi_4}^2}\Big)\\ &\quad+\cos^2\varphi_6\Big(-\frac{1}{f}\Big(\frac{\partial f}{\partial\varphi_5}\Big)^2+\frac{\partial^2 f}{\partial{\varphi_5}^2}\Big)+\cot^2\varphi_5\sin^2\varphi_6\Big(-\frac{1}{f}\Big(\frac{\partial f}{\partial\varphi_6}\Big)^2+\frac{\partial^2 f}{\partial{\varphi_6}^2}\Big)\\ &\quad+\Big(-\frac{1}{f}\Big(\frac{\partial f}{\partial\varphi_7}\Big)^2+\frac{\partial^2 f}{\partial{\varphi_7}^2}\Big)\Bigg]d_{S_s^7}V. \end{aligned}\tag{18}$$

3.1. η-Almost Everywhere Stochastic Flow on S_s^7

Let $\mathcal{M}$ be a differentiable compact d-dimensional Riemannian manifold, and let $V_0, V_1, V_2, \cdots, V_k$ be $k+1$-vector fields on $\mathcal{M}$. Consider now the following stochastic differential equation z_t on $\mathcal{M}$ given by

$$dz_t = V_0(z_t)dt+\sum_{i=1}^{k} V_i(z_t)\circ dW_t^i, \tag{19}$$

where $\circ dW_t^i$ denotes the Stratonovich stochastic differential equation and $(W_t^1)_{t\geq 0}$, $(W_t^2)_{t\geq 0}, \cdots, (W_t^k)_{t\geq 0}$ are independent k-dimensional Brownian motions on the classical Wiener space $(\Omega, \mathcal{F}, P; (\mathcal{F}_t)_{t\geq 0})$, where Ω is the space of all continuous function from $\mathbb{R}_+$ to $\mathbb{R}^d$ with locally uniform convergence topology, $\mathcal{F}$ is the Borel σ-field on Ω generated by the topology, P is the Wiener measure on $\mathcal{F}$, and $(\mathcal{F}_t)_{t\geq 0}$ is the filtration generated by $W_t(\omega) = \omega_t$.

There are at least two ways to solve the differential equation [4, 9]. It is well-known that the two above-mentioned ways work well only if the drift vector field V_0 and the other vector fields V_i $(i = 1, \cdots, k)$ appering in the above differential equation are differentiable (at least C^2). For the case of V_0 be a Sobolev vector fields with bounded divergence and V_i $(i = 1, \cdots, k)$ are C^2, Zhang [12] has shown the existence, uniqueness and stability of the so called η-almost everywhere stochastic invertible flows of stochastic differential equation, where η is the Riemannian measure (Lebesgue measure) on $\mathcal{M}$

Definition 3.1. **(Zhang [12])** Let $z_t(\omega, z)$ be an $\mathcal{M}$-valued measurable stochastic field on $\mathbb{R}_+\times\Omega\times\mathcal{M}$. The flow $z_t(z)$ is called a η-almost everywhere stochastic flow of (19) corresponding to vector fields $V_i(i = 1, \cdots, k)$ if

1. For η-almost all $z \in \mathcal{M}$, $t \mapsto z_t(z)$ is continuous and $(\mathcal{F}_t)$-adapted stochastic process and, satisfies that for any $t > 0$ and $\xi \in C^\infty(\mathcal{M})$

$$\xi(z_t(z)) = \xi(z)+\int_0^t V_0\, \xi(z_t(z))\, d\tau+\int_0^t V_i\, \xi(z_t(z))\circ dW_\tau^i \quad \forall t \geq 0. \tag{20}$$

2. For arbritary $t \geq 0$ and P-almost all $\omega \in \Omega$, $(\eta\circ z_t)(\omega, \cdot) \ll \eta$. For any $T > 0$, there is a positive constant K_{T,V_0,V_i} such that for all non-negative measurable function ξ on $\mathcal{M}$

$$\sup_{t\in[0,T]} \mathbb{E}\int_{\mathcal{M}} \xi(z_t(z))\eta(dz) \;\leq\; K_{T,V_0,V_i}\int_{\mathcal{M}} \xi(z)\eta(dz). \tag{21}$$

Furthermore, the η-almost everywhere stochastic flow $z_t(z)$ of (19) is said to be invertible if for all $t \geq 0$ and P-almost all $\omega \in \Omega$, there exists a measurable invers ${z_t}^{-1}(\omega, \cdot)$ of $z_t(\omega, \cdot)$ so that $\eta\circ {z_t}^{-1}(\omega, \cdot) = \rho_t(\omega, \cdot)\eta$, where the density $\rho_t(z)$ is given by

$$\rho_t(z) = \exp\left[\int_0^t \mathrm{div}V_0(z_t)d\tau+\int_0^t \mathrm{div}V_i(z_t)\circ dW_\tau^i\right] \tag{22}$$

Let $C^l(T\mathcal{M})$ be the set of all C^l-differentiable vector fields on $\mathcal{M}$, for every $l \in \mathbb{N}\cup\{\infty\}$. For every $p \geq 1$, we define

$$\|V\|_p := \left[\int_{\mathcal{M}} |V(z)|^p \eta(dz)\right]^{1/p} \tag{23}$$

and

$$\|V\|_{1,p} := \|V\|_p + \left[\int_{\mathcal{M}} |\nabla V(z)|^p \eta(dz)\right]^{1/p} \tag{24}$$

for every $V \in C^\infty(T\mathcal{M})$, where ∇ is the Levi-Civita connection associated to the metric tensor g on $\mathcal{M}$. The completions of $C^\infty(T\mathcal{M})$ with respect to $\|\cdot\|_p$ and $\|\cdot\|_{1,p}$ will be denoted by $L^p(T\mathcal{M})$ and $\mathbb{H}_1^p(T\mathcal{M})$ respectively. In this case, let $L^\infty(T\mathcal{M})$ to denote the set of all bounded measurable vector fields. Then Zhang found the following fact.

Theorem 3.2. ***(Zhang [12])*** *Let $V_0 \in L^\infty(T\mathcal{M})\cap\mathbb{H}_1^p(T\mathcal{M})$ for some $p \geq 1$, div $V_0 \in L^\infty(\mathcal{M})$, and for each $i = 1, 2, \cdots, k$, $V_i \in C^2(T\mathcal{M})$. Then there exists a unique η-almost everywhere stochastic invertible flow $z_t(z)$ of* (19) *in the sense of Definition 3.1*

Now consider the case where $\mathcal{M}$ is the standard sphere S_s^7 provided with the natural Riemannian metric tensor g induced from Euclidean metric in $\mathbb{R}^8$. Here we study the stochastic differential equation (19) and its η-almost everywhere stochastic flow where the drifting vector fields are the frame vector fields $U_\mu(\mu = 1, \cdots, 7)$ as being discussed in section 2.2. However, since $U_\mu(\mu = 1, \cdots, 7)$ is a differentiable vector field on S_s^7, there exists a unique η-almost everywhere stochastic invertible flow $z_t(z)$ of (19) in the sense of Definition 3.1.

Consider now for instance the vector field U_1 on S_s^7,

$$\begin{aligned} U_1 \;=\; & \cos\varphi_2 \frac{\partial}{\partial\varphi_1} - \cot\varphi_1 \sin\varphi_2 \frac{\partial}{\partial\varphi_2} + \cos\varphi_4 \frac{\partial}{\partial\varphi_3} - \cot\varphi_3 \sin\varphi_4 \frac{\partial}{\partial\varphi_4} + \cos\varphi_6 \frac{\partial}{\partial\varphi_5} \\ & - \cot\varphi_5 \sin\varphi_6 \frac{\partial}{\partial\varphi_6} - \frac{\partial}{\partial\varphi_7}. \end{aligned} \tag{25}$$

Divergence of a vector field $U = U_1$ on S_s^7 with metric g that

$$\begin{aligned} \text{div}U &= \sum_i (\partial_i U_i) + \sum_j \Gamma_{ij}^i U_j \\ &= \frac{1}{\sqrt{G(\varphi_1, \cdots, \varphi_7)}} \sum_i \partial_i\big(\sqrt{G(\varphi_1 \cdots, \varphi_7)} U_i\big) \\ &= 0. \end{aligned} \tag{26}$$

where

$$\begin{aligned} \frac{\partial}{\partial\varphi_1}\big(\sqrt{G(\varphi_1 \cdots, \varphi_7)} \cos\varphi_2\big) &= 6\sqrt{G(\varphi_1 \cdots, \varphi_7)} \cot\varphi_1 \cos\varphi_2 \\ \frac{\partial}{\partial\varphi_2}\big(-\sqrt{G(\varphi_1 \cdots, \varphi_7)} \cot\varphi_1 \sin\varphi_2\big) &= -6\sqrt{G(\varphi_1 \cdots, \varphi_7)} \cot\varphi_1 \cos\varphi_2 \\ \frac{\partial}{\partial\varphi_3}\big(\sqrt{G(\varphi_1 \cdots, \varphi_7)} \cos\varphi_4\big) &= 4\sqrt{G(\varphi_1 \cdots, \varphi_7)} \cot\varphi_3 \cos\varphi_4 \\ \frac{\partial}{\partial\varphi_4}\big(-\sqrt{G(\varphi_1 \cdots, \varphi_7)} \cot\varphi_3 \sin\varphi_4\big) &= -4\sqrt{G(\varphi_1 \cdots, \varphi_7)} \cot\varphi_3 \cos\varphi_4 \\ \frac{\partial}{\partial\varphi_5}\big(\sqrt{G(\varphi_1 \cdots, \varphi_7)} \cos\varphi_6\big) &= \sqrt{G(\varphi_1 \cdots, \varphi_7)} \cot\varphi_5 \cos\varphi_6 \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial \varphi_6}\big(-\sqrt{G(\varphi_1 \cdots, \varphi_7)} \cot\varphi_5 \sin\varphi_6\big) &= -\sqrt{G(\varphi_1 \cdots, \varphi_7)} \cot\varphi_5 \cos\varphi_6 \\ \frac{\partial}{\partial \varphi_7}\big(-\sqrt{G(\varphi_1 \cdots, \varphi_7)}\big) &= 0. \end{aligned} \tag{27}$$

For the case of U_μ be a differentiable vector field, it can be proved that $\|\text{div } U_\mu(\mu = 1, \cdots, 7)\| \leq 1$. Proved that div $U_\mu(\mu = 1, \cdots, 7) \in L^\infty(S_s^7)$.

REFERENCES

[1] Arnold, L., 2003, Random Dynamical Systems: Springer Monographs in Mathematics, Springer-Verlag, Heidelberg.

[2] Belopolskaya, Ya. L., and Daletsky, Yu. L., 1990, Stochastic Differential Equation and Differential Geometry, Mathematics and its Applications (Soviet Series), Kluwer Academic Publishers, Dordrecht-Boston-London.

[3] Chirikjian, G. S., 2009, Stochastic Models, Information Theory, and Lie Groups: Classical Results and Geometric Methods, Applied and Numerical Harmonic Analysis, Volume 1, Birkhauser, Boston.

[4] Elworthy, K. D., 1982, Stochastic Differential Equations on Manifolds, Cambridge University Press, Cambridge.

[5] Émery, M., 1989, Stochastic Calculus in Manifolds with an Appendix by P.A. Meyer, Springer, Berlin.

[6] Englert, F., Roman, M., and Spindel, P., 1983, ”Supersymmetry Breaking by Torsion and Ricci-flat Squeshed Seven-Sphere,” Physics Letters, 127(1-2), pp. 47–50.

[7] Helgason, S., 1978, Differential Geometry, Lie Groups, and Symmetric Spaces, Academic Press, New York.

[8] Hsu, E. P., 2002, Stochastic Analysis on Manifolds, Graduate Studies in Mathematics, American Mathematical Society, Illinois.

[9] Ikeda, N., and Watanabe, S., 1989, Stochastic Differential Equations and Diffusion Processes, Second edition, North-Holland Mathematical Library, North-Holland Publishing Company, Nort-Holland.

[10] Risken, H., 1989, The Fokker-Planck Equation, Methods of Solution and Applications, 2nd ed., Springer Series in Synergetics, Volume 18, Springer-Verlag, Berlin.

[11] Witten, E., 1985, ”Global Gravitational Anomalies,” Comm. Math. Phys., 100(2), pp. 197–229.

[12] Zang, X., 2011, ”Quasi-invariant stochastic flows of SDEs with non-smooth drifts on compact manifolds,” Stochastic Processes and their Application, 121, pp. 1373–1388.