Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 15, Number 1 (2019), pp 1–5

Norm Inequalities on Fan Product of Matrices

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Abstract

Recently, a number of inequalities involving the norm of the Hadamard product of two square complex matrices and their conjugate transpose are shown for any unitarily invariant norm by Du [3]. In this paper, these inequalities for the Fan product of two square complex matrices are proven.

Keywords: Fan product; matrix inequalities; unitarily invariant norm; singular value.

AMS Subject Classifications: 15A18, 15A45, 15A60..

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1. INTRODUCTION

Let M_n denote the set of complex matrices of order n. For $A = (a_{ij})$, $B = (b_{ij}) \in M_n$, we denote by A^* the conjugate transpose, by A^T the transpose, and by $\bar{A}$ the entrywise complex conjugate of A.
The Fan product of A and B is defined by $A \star B = (c_{ij})$, where

$$c_{ij} = \begin{cases} -a_{ij}b_{ij} & , i \neq j \\ a_{ij}b_{ij} & , i = j \end{cases}$$

and the Hadamard product of $A = (a_{ij})$ and $B = (b_{ij})$ is defined as $A \circ B = (a_{ij}b_{ij})$. For $A \in M_n$, the singular values of A, denoted by $s_1(A) \geq s_2(A) \geq \ldots \geq s_n(A)$ in decreasing order. Denote by $\|A\|_{(1)}$ the spectral norm of $A \in M_n$, which equals the largest singular value. Furthermore, the trace norm and the Frobenius norm of A are

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given by $\|A\|_{tr} = \sum_{i=1}^{n} s_i(A)$ and $\|A\|_F = (\sum_{i,j=1}^{n} |a_{ij}|^2)^{1/2} = (\sum_{i=1}^{n} s_i^2(A))^{1/2}$, respectively [1].

It is well known [2] that if $A, B \in M_n$, then

$$\|A \circ B\|_{(1)}^2 \leq \|A \circ \bar{A}\|_{(1)} \|B \circ \bar{B}\|_{(1)}$$

and

$$\|A \circ B\|_{(1)}^2 \leq \|A\|_{(1)} \|B\|_{(1)}$$

as the submultiplicativity inequality [2]. Zhan [4] conjectured that

$$\|(A \circ B)(A \circ B)^*\| \leq \|(A \circ \bar{A})(B \circ \bar{B})^T\| \tag{1}$$

for $A, B \in M_n$ and any unitarily invariant norm. Then, Du [3] proved the above conjecture for the spectral norm, the trace norm, and the Frobenius norm. In this study, we prove this conjecture using the Fan product of two matrices $A, B \in M_n$ for aforementioned norms.

2. MAIN RESULTS

Let $A = (a_{ij})$ and $B = (b_{ij})$ and define $|A| = (|a_{ij}|)$ as entrywise absolute value. We write $A \geq B$ if $A - B \geq 0$ as nonnegative matrix. The spectral radius of A is denoted by $\rho(A)$. First, we give some lemmas which are useful for obtaining the main results.

Lemma 1. *[1] Let $A, B \in M_n$ and suppose that B is nonnegative. If $|A| \leq_e B$, then*

$$\rho(A) \leq \rho(|A|) \leq \rho(B).$$

Lemma 2. *[4] For any $A \in M_n$,*

$$|trA| \leq \sum_{i=1}^{n} s_i(A).$$

Lemma 3. *Let $A, B \in M_n$. Then*

$$A \star B = (2I_n - J_n) \circ (A \circ B) \tag{2}$$

where J_n is the $n \times n$ matrix whose all entries 1 and I_n is the $n \times n$ identity matrix.

Proof. Since $2I_n - J_n$ is a square matrix with ones on the diagonal and minus ones elsewhere, then the Hadamard product of $2I_n - J_n$ and $A \circ B$ give the Fan product of A and B. □

Theorem 4. *Let $A, B \in M_n$. Then, the Fan product satisfy the inequality*

$$\|(A \star B)(A \star B)^*\| \leq \|(A \star \bar{A})(B \star \bar{B})^T\|$$

for the spectral norm, the trace norm, and the Frobenius norm.

Proof. Let $A = (a_{ij}), B = (b_{ij}) \in M_n$. Define three matrices

$$C = (A \star B)(A \star B)^* = [c_{ij}], \quad D = (A \star \bar{A})(B \star \bar{B})^T = [d_{ij}], \quad 2I_n - J_n = [m_{ij}].$$

By Lemma 3 and using $(A \circ B)^* = A^* \circ B^*$ [1], we have

$$C = [(2I_n - J_n) \circ (A \circ B)][(2I_n - J_n)(A \circ B)]^*,$$

$$c_{ij} = \sum_{k=1}^{n} m_{ik}a_{ik}b_{ik}\overline{m_{jk}a_{jk}b_{jk}},$$

$$D = [(2I_n - J_n) \circ (A \circ \bar{A})][(2I_n - J_n)^T(B \circ \bar{B})^T],$$

and

$$d_{ij} = \sum_{k=1}^{n} m_{ik}a_{ik}\overline{a_{ik}}m_{jk}b_{jk}\overline{b_{jk}}.$$

Let

$$x = (\overline{m_{i1}a_{i1}}b_{j1}, \overline{m_{i2}a_{i2}}b_{j2}, \ldots, \overline{m_{in}a_{in}}b_{jn})^T$$

and

$$y = (\overline{m_{j1}a_{j1}}b_{i1}, \overline{m_{j2}a_{j2}}b_{i2} \ldots, \overline{m_{jn}a_{jn}}b_{in})^T$$

are $n \times 1$ column vectors. We remark that C is positive semidefinite, D is nonnegative, m_{ij} is 1 or -1, so $\overline{m_{ik}} = m_{ik}$, for $k = 1, 2, \ldots, n$. We get

$$\begin{aligned} \mid c_{ij} \mid &= \mid \sum_{k=1}^{n} m_{ik}a_{ik}b_{ik}\overline{m_{jk}a_{jk}b_{jk}} \mid \\ &= \mid \sum_{k=1}^{n} m_{ik}a_{ik}\overline{b_{jk}}m_{jk}\overline{a_{jk}}b_{ik} \mid \\ &= \mid x^*y \mid \end{aligned}$$

□

By using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \mid c_{ij} \mid^2 = \mid x^*y \mid &\leq (x^*x)(y^*y) \\ &= (\sum_{k=1}^{n} m_{jk}a_{ik}\overline{b_{jk}}m_{ik}\overline{a_{ik}}b_{jk})(\sum_{k=1}^{n} m_{ik}a_{jk}\overline{b_{ik}}m_{jk}\overline{a_{ik}}b_{ik}) \\ &= (\sum_{k=1}^{n} m_{ik}m_{jk} \mid a_{ik} \mid^2 \mid b_{jk} \mid^2)(\sum_{k=1}^{n} m_{ik}m_{jk} \mid a_{jk} \mid^2 \mid b_{ik} \mid^2) \\ &= d_{ij}d_{ji}. \end{aligned} \tag{3}$$

Thus, the following the above inequality, we write

$$2|c_{ij}| \leq 2\sqrt{d_{ij}}\sqrt{d_{ji}} \leq d_{ij} + d_{ji}.$$

Accordingly,

$$|C| \leq_e D + D^T.$$

Using Lemma 1, we get

$$2\rho(C) \leq 2\rho(|C|) \leq \rho(D + D^T). \quad (4)$$

Because of the spectral radius is less than or equal to the spectral norm [4], we write

$$\rho(D + D^T) \leq \|D + D^T\|_{(1)} \leq \|D\|_{(1)} + \|D^T\|_{(1)} = 2\|D\|_{(1)}. \quad (5)$$

Joining the last two inequalities Eq.4 and Eq.5 we obtain

$$\rho(C) \leq \|D\|_{(1).} \quad (6)$$

Then, using Eq.6 and the fact that C is positive semidefinite we have

$$\|C\|_{(1)} = s_1(C) = \rho(C) \leq \|D\|_{(1)},$$

namely,

$$\|(A \star B)(A \star B)^*\|_{(1)} \leq \|(A \star \bar{A})(B \star \bar{B})^T\|_{(1)}.$$

This is the first part of the proof as the spectral norm case. Since the matrices C and D have the same diagonal entries $c_{ii} = d_{ii}, i = 1, 2, \ldots, n$, and C is positive semidefinite, using Lemma 2 we obtain

$$\|C\|_{(n)} = trC = trD \leq \|D\|_{(n)}.$$

In this way, the trace norm case of proof is shown. For the Frobenius case by Eq.3 we get

$$|c_{ij}|^2 + |c_{ji}|^2 = 2|c_{ij}|^2 \leq 2d_{ij}d_{ji} \leq d_{ij}^2 + d_{ji}^2.$$

Finally, we write

$$\|C\|_F \leq \|D\|_F.$$

□

3. EXAMPLES

In this section, we will consider two examples for validating our results.

Example 5. *First, we will use the same two matrices as in [3]. Let consider the following matrices*

$$A = \begin{bmatrix} 0.3152 & 0.7909 & 2.9887 & -0.4885 & -0.4386 \\ 0.8525 & -0.3830 & 0.3892 & 1.6355 & -0.3320 \\ -0.0405 & 1.5986 & -0.3436 & -0.5866 & 0.3810 \\ -0.4454 & -0.9488 & -0.5812 & 0.4093 & -1.1284 \\ -0.2793 & 0.1359 & 0.3513 & 0.1095 & 0.5030 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.4905 & -0.6168 & -1.7193 & 0.9559 & 0.5181 \\ -0.1844 & -0.1877 & -0.0369 & -0.8642 & -0.7574 \\ -3.0149 & 0.6638 & 0.5690 & -0.1921 & 0.8942 \\ -0.9555 & -0.3928 & -0.2499 & 0.2702 & -0.2462 \\ 0.0770 & 1.1954 & -0.3655 & 0.1311 & -1.2339 \end{bmatrix}.$$

Then, we have the following upper bounds for $\|A\star B\|$ with the spectral norm, the trace norm, and the Frobenius norm, respectively.

$$\|A\star B\|_{(1)}^2 = 27.1128 \leq \|(A\star\bar{A})(B\star\bar{B})\|_{(1)} = 27.3243$$

$$\|A\star B\|_{(n)} = 31.1937 \leq \|(A\star\bar{A})(B\star\bar{B})\|_{(n)} = 39.5191$$

$$\|A\star B\|_{(F)}^2 = 27.2420 \leq \|(A\star\bar{A})(B\star\bar{B})\|_{(F)} = 28.7183.$$

Example 6. *Now,we give the second example and look the following matrices*

$$A = \begin{bmatrix} 2 & 1+i & -i \\ 3-i & 5+2i & 1 \\ -6+i & 4 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1-i & 2i & 3 \\ 4 & 3 & -2i \\ 2-3i & 4+7i & -i \end{bmatrix}.$$

Then, the upper bounds for $\|A\star B\|$ are obtained as,

$$\|A\star B\|_{(1)}^2 = 1805.9 \leq \|(A\star\bar{A})(B\star\bar{B})\|_{(1)} = 2331.2$$

$$\|A\star B\|_{(n)} = 58.0036 \leq \|(A\star\bar{A})(B\star\bar{B})\|_{(n)} = 2949.3$$

$$\|A\star B\|_{(F)}^2 = 44.4971 \leq \|(A\star\bar{A})(B\star\bar{B})\|_{(F)} = 2410.73.$$

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