Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 14, Number 9 (2018), pp. 1177-1283

http://www.ripublication.com

Generalization of Contraction Principle on G-Metric Spaces

G Sudhaamsh Mohan Reddy

Department of Mathematics, Faculty of Science and Technology

ICFAI Foundation for Higher Education, Hyderabad-501203, India

Abstract

In this paper, we prove certain fixed point theorems of G-metric spaces using the generalized contraction principle on G-metric spaces.

Key words: G-metric space, Cauchy sequence.

Mathematics Subject Classification: 54H25, 47H10

1. INTRODUCTION AND PRELIMINARIES:

Metric fixed point theory is an important Mathematical discipline because of its applications in areas such as variation and linear inequalities, optimization and approximation theory, etc..,. The generalization of metric spaces were proposed by Gahler [5] (called 2-metric spaces) and Dhage [2, 3, 4] (called D-metric spaces). Hsiao [7] showed that every contractive definition, with $x_n = T^n x_0$, every orbit is linearly dependent, thus giving fixed point theorem in such spaces. However HA et. al. [6] have pointed out that the results obtained by Gahler for his 2-metric spaces are independent, rather than the generalizations of corresponding results in metric spaces. While Mustafa and Sims [8] have proved that the Dhage's notion of D-metric space is fundamentally incorrect and most of the results claimed by Dhage and others are invalid.

Mustafa and Sims [8] in 2003 have introduced a more appropriate and robust notion of generalized metric spaces as follows:

1.1 Definition: (See [8])

Let X be a non empty set and let $G: X \times X \times X \to [0,\infty)$ be a function satisfying the following axioms:

(G1) $G(x,y,z)=0$ if $x=y=z$.

(G2) $0<G(x,x,y),$ for all $x,y\in X$ with $x\neq y$

(G3) $G(x,x,y)<G(x,y,z)$ for all $x,y,z\in X$ with $y\neq z$

(G4) $G(x,y,z)=G(\sigma(x,y,z))$ for all $x,y,z\in X$, where σ is a permutation of the set $\{x,y,z\}$ (Symmetry in all three variables)

(G5) $G(x,y,z)\leq G(x,a,a)+G(a,y,z)$ for all $x,y,z,a\in X$ (Rectangular inequality)

Then the function G is called a *generalized metric* or more specifically a G-*metric* on X. The pair (X,G) is called a G*-metric space*.

1.2 Definition:

Let (X,G) be a G-metric space and let $\{x_n\}$ be a sequence of points of X, we say that $\{x_n\}$ is $G-$*convergent to* x if for every given $\varepsilon>0$, there exist $N\in N$ (set of all natural numbers) such that $G(x,x_n,x_m)<\varepsilon$ for all $m,n\geq N$. We denote it as $\lim\limits_{n,m\to\infty} G(x,x_n,x_m)=0$

1.3 Definition:

Let (X,G) be a metric space, a sequence $\{x_n\}$ in X is called $G-$*cauchy* if for every given $\varepsilon>0$, there exists $N\in N$ such that,

$G(x_n,x_m,x_l)<\varepsilon$ for all $n,m,l\geq N$, that is if, $\lim\limits_{n,m,l\to\infty} G(x_n,x_m,x_l)=0$

1.4 Definition:

A G-metric space (X,G) is said to be G*-complete* (or a *complete* G*-metric space*) if every G-Cauchy sequence in (X,G) is G-convergent to some point in (X,G).

1.5 Definition:

A G-metric space (X,G) is said to be *symmetric* if $G(x,y,y)=G(y,x,x)$ for all $x,y\in X$

1.6 Definition:

A function $\phi:[0,\infty)\to[0,\infty)$ is called an *Altering distance function*, if the following properties are satisfied.

(i) $\phi(0)=0$

(ii) ϕ is continuous and monotonically non-decreasing.

1.7 Definition:

Let (X,G) be a G-metric space and let $T:X\to X$ be a mapping. T is called a *contraction of* X if

(1.7.1) $G(Tx,Ty,Tz)\le k\ \ G(x,y,z)$ for all $x,y,z\in X$

1.18 Definition:

A mapping $T:X\to X$, where (X,G) is a G-metric space, is said to be *weakly contractive* if $G(Tx,Ty,Tz)\le Gx,y,z)-\phi(G(x,y,z))$ for all $x,y,z\in X$,

where $\phi:[0,\infty)\to[0,\infty)$ is a continuous and non decreasing function such that $\phi(t)=0$ if and only if $t=0$.

1.19 Definition:

Let T be a self map of a complete G-metric space (X,G) with non empty fixed point set $F(T)$ (set of all fixed points of T). Then we say that T satisfies *property* P if $F(T)=F(T^n)$ for all $n\in N$ **.**

2. MAIN THEOREM

Very recently in 2008 P.N. Dutta et al [4] have obtained fixed point theorem of metric spaces using the concept of generalization of contraction principle. Here we state the theorem proved by Dutta et al [4].

2.1. Theorem:

Let (X,G) be a complete matrix space and $T:X\to X$ be a self mapping satisfying the inequality.

(2.1.1) $\psi(d(Tx,Ty))\le\ \psi(d(x,y))\ -\ \phi(d(x,y))$, for all $x,y\in X$,where $\psi,\phi:[0,\infty)\to[0,\infty)$ are both continuous and monotone non decreasing functions with $\psi(t)=0=\phi(t)$ if and only if $t=0.$ Then T has unique fixed point.

In this paper, we have establish and generalized the above theorem for G-metric spaces.

2.2. Theorem:

Let (X,G) be a complete G-metric space and Let $T:X\to X$ be a self mapping satisfying the inequality.

(2.2.1) $\psi(G(Tx,Ty,Tz))\le\psi(G(x,y,z))-\phi(G(x,y,z))$ for all $x,y,z\in X$ where $\psi,\phi:[0,\infty)\to[0,\infty)$ are both continuous and monotone non decreasing functions with $\psi(t)=0=\phi(t)$ if and only if $t=0.$ Then T has unique fixed point.

Proof:

Let $x_0 \in X$. We construct the sequence $\{x_n\}$ by $x_n = Tx_{n-1}$ for $n = 1,2,3,..........$ choosing $x_n = x_{n-1}$, $y = x_n$, $z = x_n$ in (2.2.1 obtain.), we obtain,

(2.2.2) $\psi(G(x_n, x_{n+1}, x_{n+1})) = \psi(G(Tx_{n-1}, Tx_n, Tx_n))$

$$\leq \psi(G(x_{n-1}, x_n, x_n)) - \phi(G(x_{n-1}, x_n, x_n)$$

Which implies

$\psi(G(x_n, x_{n+1}, x_{n+1})) \leq \psi(G(x_{n-1}, x_n, x_n))$ since $\phi(G(x_{n-1}, x_n, x_n) \geq 0$.

Now, using the monotone property of ψ, we get.

(2.2.3) $G(x_n, x_{n+1}, x_{n+1}) \leq G(x_{n-1}, x_n, x_n)$

This shows that the sequence $\{G(x_n, x_{n+1}, x_{n+1})\}$ is monotone decreasing and bounded below by " 0" in the complete G-metric space (X, G). Hence there exist
$r \geq 0$ such that $G(x_n, x_{n+1}, x_{n+1}) \to r$ as $n \to \infty$

Now, Letting $n \to \infty$ in (2.2.2), we get $\psi(r) \leq \psi(r) - \phi(r)$ it holds only when $r = 0$

Hence,

(2.2.4) $G(x_n, x_{n+1}, x_{n+1}) \to 0$ as $n \to \infty$.

Now, we prove that $\{x_n\}$ is not a Cauchy sequence then there exist some $\in > 0$ for which we can find the sub sequences $\{x_{m_k}\}$, $\{x_{n_k}\}$ of $\{x_n\}$ with $n_k > m_k > k$ such that

(2.2.5) $G(x_{m_k}, x_{n_k}, x_{n_k}) \geq \in$

Further, corresponding to m_k we can choose n_k in such a way that it is the smallest integer satisfying (2.2.5). Then,

(2.2.6) $G(x_{m_k}, x_{n_k-1}, x_{n_k-1}) < \in$, now, we have

$$\in \leq G(x_{m_k}, x_{n_k}, x_{n_k}) \leq G(x_{m_k}, x_{n_k-1}, x_{n_k-1}) + G(x_{n_{k-1}}, x_{n_k}, x_{n_k})$$

$$< \in + G(x_{n_{k-1}}, x_{n_k}, x_{n_k})$$

Taking $k \to \infty$ on both the sides and using (2.2.4), we have

(2.2.7) $$\underset{k \to \infty}{\text{Lt}} \; G(x_{m_k}, x_{n_k}, x_{n_k}) = \in$$

Again,

(2.2.8) $G(x_{n_k}, x_{m_k}, x_{m_k}) \leq G(x_{n_k}, x_{n_k -1}, x_{n_k -1}) + G(x_{n_{k-1}}, x_{m_k}, x_{m_{k-1}})$

$+ G(x_{m_{k-1}}, x_{m_k}, x_{m_k})$

$G(x_{m_{k-1}}, x_{n_k -1}, x_{n_k -1}) \leq G(x_{m_{k-1}}, x_{m_k}, x_{m_k}) + G(x_{m_k}, x_{n_k}, x_{n_k})$

$+ G(x_{n_k}, x_{n_k -1}, x_{n_k -1})$

Letting $k \to \infty$ in the above two inequalities and using (2.2.4) and (2.2.7).

We get,

(2.2.9) $\underset{k \to \infty}{\text{Lt}} \quad G(x_{m_{k-1}}, x_{n_k -1}, x_{n_k -1}) = \in$

Choosing $x = x_{m_{k-1}}, y = x_{n_{k-1}}, z = x_{n_{k-1}}$

in (2.2.2) and using (2.2.5)

We obtain

$\in \leq \psi(G(x_{m_k}, x_{n_k}, x_{n_k})) \leq \psi(G(x_{m_{k-1}}, x_{n_k -1}, x_{n_k -1}))$ - $\phi(G(x_{m_{k-1}}, x_{n_k -1}, x_{n_k -1}))$

Taking $k \to \infty$ on both the sides.

$\in \leq \psi(\in) \leq \psi(\in) - \phi(\in)$

Which is a contradiction if $\in > 0$

Therefore,

$\in = 0$.

This shows that $\{x_n\}$ is a Cauchy sequence in complete G-metric space (X, G) and hence is convergent to some $u \in X$.

That is,

(2.2.10) $\{x_n\}$ convergent to x (say) as $n \to \infty$.

Now we claim that u is a fixed point of T

Consider $x = x_{n-1}, y = u, z = u$ in (2.2.1). We obtain

(2.2.11) $\psi(G(x_n, Tu, Tu)) \leq \psi(G(x_{n-1}, u, u))$ - $\phi(G(x_{n-1}, u, u))$

Letting $n \to \infty$ and using (2.2.10), we get $\psi(G(u, Tu, Tu)) \leq \psi(G(u, u, u))$ - $\phi(G(u, u, u))$

This is

$\psi(G(u, Tu, Tu)) \leq \psi(0)$ - $\phi(0) = 0$

Hence $G(u,Tu,Tu) = 0$ which gives $Tu = u$.

To prove the uniquencess of the fixed point, Let us suppose that u_1, u_2 are two fixed points of T.

That is $T(u_1) = u_1$, $T(u_2) = u_2$

Taking $x = u_1$, $y = u_2$, $z = u_2$ in (2.2.1).

$$\psi(G(Tu_1, Tu_2, Tu_2)) \leq \psi(G(u_1, u_2, u_2)) - \phi(G(u_1, u_2, u_2))$$

This is

$$\psi(G(u_1, u_2, u_2)) \leq \psi(G(u_1, u_2, u_2)) - \phi(G(u_1, u_2, u_2))$$

This gives,

$\phi(G(u_1, u_2, u_2)) \leq 0$, it holds only when $G(u_1, u_2, u_2) = 0$

That is $u_1 = u_2$.

Showing T has unique fixed point.

2.3. Corollary:

Let (X,G) be a complete G-metric space, $T : X \to X$ be a self mapping which satisfying the following inequality.

(2.3.1) $\psi(G(Tx,Ty,Tz)) \leq k \ \psi(G(x,y,z))$ for all $x,y,z \in X$ where $0 \leq k \leq 1$ $\psi : [0,\infty) \to [0,\infty)$ is a continuous and monotone non decreasing functions with $\psi(t) = 0$ if and only if $t = 0$.

Then T has unique fixed point.

Proof of the corollary follows by taking

$\phi(t) = (1-k)\psi(t)$ in theorem 2.2.

2.4. Corollary:

Let $T : X \to X$ be a weakly contractive mapping of a complete G -metric space (X,G), then T has unique fixed point.

Proof: Given T is weakly contractive mapping that is $G(Tx,Ty,Tz) \leq G(x,y,z) - \phi(G(x,y,z))$ for all $x,y,z \in X$ where $\phi : [0,\infty) \to [0,\infty)$ is a continuous and non decreasing functions.

Taking $\psi(t) = t$ in theorem 2.3 corollary follows.

REFERENCES

[1] Circic. Lj. B,. "A generalization of Banach's contraction principle," Proceedings of the American Mathematical Society, Vol. 45, pp. 267 – 273, 1974.

[2] Dhage, B.C., "Generalized metric space and mapping with fixed points", Bulletin of the Calcutta Mathematical Society, Vol. 84, pp. 329 – 336, 1992.

[3] Dhage. B.C., "Generalized metric spaces and topological structure – I", Analete Stiintifice ale Universitatii." Al. I. Cuza" dina Iasi. Serie Nova, Mathametical, Vol. 46, no. 1, pp . 3 – 24, 2000.

[4] Dutta. P.N. and Choudhaury, Binayak S, "A generalization of contraction principle in metric spaces," Journal of Fixed Point Theory and Applications, Vol. 2008.

[5] Gahler. S, "2–metriche Raume undihre topologische struktur," Mathematische Nachrichten. Vol. 26, no. 1-4, pp. 115 – 148, 1963.

[6] Ha. R.S., Cho. Y.J. and White A, "Strictly Convex and Strictly 2 – convex 2 – normed spaces," Mathematica Japanica, vol. 33, no. 3, pp. 375 – 384, 1988.

[7] Hsio. C.R., "A property of contractive type mappings in 2-metric spaces," Inanabha, Vol. 16, pp. 223-239, 1986.

[8] Mustafa. Z and Sims. B, "Some remarks concerning D-metric spaces," in proceedings of the International Conference on Fixed Point Theorey and Applications, pp. 189 – 198, Valencica, Spain, July 2003.

[9] Mustafa. Z, A new structure for generalized metric spaces – with applications to fixed point theory, Ph.D. thesis, the University of New Castle, Callaghan, Australia, 2005.

[10] Mustafa. Z and Sims. B, "A new approach to generalized metric spaces," Journal of Nonlinear and Convex Analysis, Vol. 7, No. 2, pp. 289 – 297, 2006.

[11] Rashwan. R. A. and A.M. Sadeek, "A common fixed point theorem in complete metric spaces," Southwest Journal of Pure and Applied Mathematics, Vol. 01, 1996, pp. 6-10.