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Geometrical study of real hypersurfaces with differentials of structure tensor field in a Nonflat complex space form[1]

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Abstract

Let M be a real hypersurfaces in a complex space form $M_n(c)$, $c \neq 0$. In this paper we prove that if the structure tensor field is (i) $\nabla_\xi \phi + \mathcal{L}_\xi \phi = 0$ and (ii) ξ-parallel, then M is a Hopf hypersurface. We characterize such Hopf hypersurfaces of $M_n(c)$.

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1. Introduction

A complex n-dimensional Kaehlerian manifold of constant holomorphic sectional curvature c is called a *complex space form*, which is denoted by $M_n(c)$. It is well-known that a complete and simply connected complex space form is complex analytically isometric to a complex projective space $P_n\mathbf{C}$, a complex Euclidean space $\mathbf{C}^n$ or a complex hyperbolic space $H_n\mathbf{C}$, according to $c > 0$, $c = 0$ or $c < 0$.

In this paper we consider a real hypersurface M in a complex space form $M_n(c)$, $c \neq 0$. Then M has an almost contact metric structure (ϕ, g, ξ, η) induced from the Kaehler metric and complex structure J on $M_n(c)$. The Reeb vector field ξ is said to be *principal* if $A\xi = \alpha\xi$ is satisfied, where A is the shape operator of M and $\alpha = \eta(A\xi)$. In this case, it is known that α is locally constant ([4]) and that M is called a *Hopf hypersurface*.

Typical examples of Hopf hypersurfaces in $P_n\mathbf{C}$ are homogeneous ones, namely those real hypersurfaces are given as orbits under subgroup of the projective unitary groups $PU(n+1)$. R. Takagi ([10]) completely classified homogeneous real hypersurfaces in such hypersurfaces as six model spaces A_1, A_2, B, C, D and E. On the other hand, real hypersurfaces in $H_n\mathbf{C}$ have been investigated by Berndt [1], Montiel and Romero ([6]) and so on. Berndt ([1]) classified all homogeneous Hopf hyersurfaces in $H_n\mathbf{C}$ as four model spaces which are said to be A_0, A_1, A_2 and B.

Theorem 1.1. ([7],[10]) Let M be a homogeneous real hypersurfaces of $P_n\mathbf{C}$. Then M is tube of radius r over one of the following Kaehlerian submanifolds:

(A_1) A hyperplane $P_{n-1}\mathbf{C}$, where $0 < r < \frac{\pi}{\sqrt{c}}$;

(A_2) A totally geodesic $P_k\mathbf{C}$ $(1 \leq k \leq n-2)$, where $0 < r < \frac{\pi}{\sqrt{c}}$;

(B) A complex quadric $Q_n - 1$, where $0 < r < \frac{\pi}{2\sqrt{c}}$;

(C) $P_1\mathbf{C} \times P_{\frac{n-1}{2}}\mathbf{C}$ Where $0 < r < \frac{\pi}{2\sqrt{c}}$ and $n \geq 5$ is odd;

(D) A complex Grassmann $G_{2.5}\mathbf{C}$, Where $0 < r < \frac{\pi}{2\sqrt{c}}$ and $n = 9$;

(E) A Hermitian symmetric space SO(10)/U(5), Where $0 < r < \frac{\pi}{2\sqrt{c}}$ and $n = 15$.

Theorem 1.2. ([1],[7]) Let M be a real hypersurface in $H_n\mathbf{C}$. Then M has constant principal curvatures and ξ is principal if and only if M is locally congruent to one of the followings:

(A_0) A self-tube, that is, a holosphere;

(A_1) A geodesic hypersphere;

(A_2) A tube over a totally geodesics $H_k\mathbf{C}$ ($1 \leq k \leq n-1$);

(B) A tube over a totally real hyperbolic space $H_n\mathbf{C}$.

A real hypersurface of A_1 or A_2 in $P_n\mathbf{C}$ or A_0, A_1, A_2 in $H_n\mathbf{C}$, then M is said to be a type A for simplicity.

As a typical characterization of real hypersurfaces of type A, the following is due to Okumura [8] for $c > 0$ and Montiel and Romero [6] for $c < 0$.

Theorem 1.3. ([6,8]) Let M be a real hypersurface of $M_n(c)$, $c \neq 0$, $n \geq 2$. It satisfies $A\phi - \phi A = 0$ on M if and only if M is locally congruent to one of the model spaces of type A.

For the structure tensor field ϕ on M, we define the Lie derivative $\mathcal{L}_\xi$ by $(\mathcal{L}_\xi\phi)X = [\xi, \phi X] - \phi[\xi, X]$, and $(\nabla_\xi\phi)X$ with respect to a unit vector field X. We call the Lie derivative and covariant derivative in the Reeb vector field ξ direction of the structure tensor field as ξ-Lie parallel and ξ-parallel. Many Geometrician have studied real hypersurfaces from certain conditions and obtains some results on the classification of real hypersurfaces in complex space forms $M_n(c)$.

As for the derivatives of structure tensor field, Lim ([4]) have proved the following Theorems.

Theorem 1.4. ([4]) Let M be a real hypersurface of $M_n(c)$, $c \neq 0$. It satisfies $\mathcal{L}_\xi\phi = \nabla_\xi\phi$ if and only if M is a locally congruent to one of the model space of type A.

In this paper we shall study a real hypersurface in a nonflat complex space form $M_n(c)$, with sum operators $\mathcal{L}_\xi$ and ∇_ξ of structure tensor field, and give some characterizations of such a real hypersurface in $M_n(c)$.

All manifolds in the present paper are assumed to be connected and of class C^∞ and the real hypersurfaces supposed to be orientable.

2. Preliminaries

Let M be a real hypersurface immersed in a complex space form $M_n(c)$, and N be a unit normal vector field of M. By $\widetilde{\nabla}$ we denote the Levi-Civita connection with respect to the Fubini-Study metric tensor $\widetilde{g}$ of $M_n(c)$. Then the Gauss and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \qquad \widetilde{\nabla}_X N = -AX$$

for any vector fields X and Y tangent to M, where g denotes the Riemannian metric tensor of M induced from $\widetilde{g}$, and A is the shape operator of M in $M_n(c)$. For any vector field X on M we put

$$JX = \phi X + \eta(X)N, \qquad JN = -\xi,$$

where J is the almost complex structure of $M_n(c)$. Then we see that M induces an almost contact metric structure (ϕ, g, ξ, η), that is,

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\xi) = 1,$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi) \tag{1}$$

for any vector fields X and Y on M. Since the almost complex structure J is parallel, we can verify from the Gauss and Weingarten formulas the followings:

$$\nabla_X \xi = \phi AX, \tag{2}$$

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi. \tag{3}$$

Since the ambient manifold is of constant holomorphic sectional curvature c, we have the following Gauss, Codazzi equations and operator of Lie derivative respectively:

$$\begin{aligned} R(X,Y)Z = \ & \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ & -2g(\phi X, Y)\phi Z\} + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{4}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}, \tag{5}$$

for any vector fields X, Y and Z on M, where R denotes the Riemannian curvature tensor of M.

Let Ω be the open subset of M defined by

$$\Omega = \{p \in M \mid A\xi - \alpha\xi \neq 0\} \tag{6}$$

where $\alpha = \eta(A\xi)$. We put

$$A\xi = \alpha\xi + \mu W, \tag{7}$$

where W be a unit vector field orthogonal to ξ and μ does not vanish on Ω.

3. Some Lemmas

In this section, we assume that Ω is not empty, then we shall prove Theorem 1 and 2. If the vector field ξ is a principal curvature vector in a nonflat complex space form i.e. $A\xi$=$\alpha\xi$ then M is called a Hopf hypersurface of $M_n(c)$. For such a Hopf hypersurface, we now recall some well known results which will be used to prove our results (see [7]).

Lemma 3.1. ([7]) Let be a Hopf hypersurface in a nonflat complex space form $M_n(c)$. If X is a unit vector such that $AX = \lambda X$, Then

$$(\lambda - \frac{\alpha}{2})A\phi X = \frac{1}{2}(\alpha\lambda + \frac{c}{2})\phi X. \tag{8}$$

Lemma 3.2. L et M be a real hypersurface of $M_n(c), c \neq 0$. If M satisfies $\nabla_\xi \phi + \mathcal{L}_\xi \phi = 0$, Then M is a Hopf hypersurface in $M_n(c)$.

Proof. We assume that $\nabla_\xi \phi + \mathcal{L}_\xi \phi = 0$ for any vector field X. Then we have

$$\begin{aligned}(\nabla_\xi \phi)X + (\mathcal{L}_\xi \phi)X =& (\nabla_\xi \phi)X + [\xi, \phi X] - \phi[\xi, X] \\ =& (\nabla_\xi \phi)X + \nabla_\xi(\phi X) - \nabla_{\phi X}\xi - \phi(\nabla_\xi X - \nabla_X \xi) \\ =& 2(\nabla_\xi \phi)X - \phi A \phi X - AX + \eta(AX)\xi,\end{aligned}$$

Under the our assumption, it follows from the above equation that

$$2(\nabla_\xi \phi)X = \phi A \phi X + AX - \eta(AX)\xi \tag{9}$$

If we substitute (3) into (9) and using the forth equation (1), then we obtain

$$\phi A \phi X - AX + \{\alpha\eta(X) - \mu w(x)\}\xi + 2\mu\eta(X)W = 0. \tag{10}$$

for any vector field X, where w is the dual 1-form of the unit vector field W.

If we put $X = \xi$ into (10), then we have

$$A\xi = \alpha\xi + 2\mu W. \tag{11}$$

If we compare (7) with (11), then we have $\mu = 0$ on Ω and hence a contradiction. ∎

Lemma 3.3. Let M be a real hypersurface of $M_n(c), c \neq 0$. If M satisfies $\nabla_\xi \phi + \mathcal{L}_\xi \phi = 0$ and $\nabla_\xi \phi = 0$, Then M is a Hopf hypersurface in $M_n(c)$.

Proof. By virtue of the hypothesis and (9), the foregoing equation yields

$$\phi A \phi X + AX = \eta(AX)\xi. \tag{12}$$

If we put $X = \xi$ into (12) and make use of (7), then we get $\mu = 0$ and hence it is a contradiction. ∎

4. Characterizations of real hypersurfaces

In this section, we will discuss the characterization of the real hypersurface for Lemma 3.1 and 3.2 in the complex space form, that is, we shall prove Theorem 4.1 and 4.2.

Theorem 4.1. Let M be a real hypersurface of $M_n(c), c \neq 0$. It satisfies $\nabla_\xi \phi + \mathcal{L}_\xi \phi = 0$ if and only if M is a locally congruent to one of the model space of type A.

Proof. By Lemma 3.2, M is a Hopf hypersurface in $M_n(c)$. Since ξ is a Reeb vector field, we obtain $(\nabla_\xi \phi)X = 0$ by the virtue of (3). Thus, the assumption $\nabla_\xi \phi + \mathcal{L}_\xi \phi = 0$ is given by

$$\phi A \phi X + AX = \alpha\eta(X)\xi. \tag{13}$$

For any vector field $X \perp \xi$ on M such that $AX = \lambda X$, it follows from (13) that

$$\phi A\phi X + \lambda X = 0. \tag{14}$$

We can choose an orthonormal frame field $\{X_0 = \xi, X_1, X_2, \dots, X_2(n-1)\}$ on M such that $AX_i = \lambda_i X_i$ for $1 \le i \le 2(n-1)$. By the virtue of the equation (8), if $\lambda_i \ne \frac{\alpha}{2}$, then it follows from Lemma 3.1 that ϕX_i is also a principal direction, say $A\phi X_i = \mu\phi X_i$. Applying all these we at once obtain from (14)

$$\lambda_i - \mu_i = 0. \tag{15}$$

For the case $\lambda_i = \mu_i$, it follows from (8) that both λ_i and μ_i are constants and therefore $\phi AX_i = A\phi X_i$. If $\lambda_i \ne \frac{\alpha}{2}$ and $\lambda_j = \frac{\alpha}{2}$ for $1 \le i \le p$ and $p+1 \le j \le 2(n-1)$ respectively, then it follows from (15) that

$$\phi A\phi X_j + \frac{\alpha}{2} X_j = 0. \tag{16}$$

If we take inner product of (16) that ϕX_i and by using $\lambda_i \ne \frac{\alpha}{2}$, then we obtain $g(\phi X_j, X_i) = 0$ for $1 \le i \le p$. Thus the vector field ϕX_j is expressed by a linear combination of $X_j{}'$ only, which implies $A\phi X_j = \frac{\alpha}{2}\phi X_j = \phi AX_j$. If $\lambda_j = \frac{\alpha}{2}$ for $1 \le j \le 2(n-1)$, then it is easily seen that $A\phi X_j = \phi AX_j$ for all j. Therefore we have $\phi A - A\phi = 0$ on M. Theorem 4.1 follow from Theorem A. ∎

Corollary 4.2. Let M be a real hypersurface of $M_n(c)$, $c \ne 0$. It satisfies $\nabla_\xi \phi + \mathcal{L}_\xi \phi = 0$ and $\nabla_\xi \phi = 0$ if and only if M is a locally congruent to one of the model space of type A.

Proof. From Lemma 3.3, M is a Hopf hypersurface in $M_n(c)$. The hypothesis $\nabla_\xi \phi + \mathcal{L}_\xi \phi = 0$ and $\nabla_\xi \phi = 0$ is equivalent to

$$\phi A\phi X + AX = \alpha\eta(X)\xi. \tag{17}$$

Therefore, from the results of the Theorem 4.1 or Theorem B, we conclude that M is a locally congruent to one of the model space of type A. ∎

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