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Taylor Method and Second Order Runge-Kutta Method for Solving a Two-Dimensional first order Dynamic Equations on Time Scales

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Abstract

This paper applies the Taylor method in time scales to solving a two dimensional dynamic equations on time scales. This involves giving a general representation of the Taylor method upto order 4 and then considering the set of real numbers and the set of integers as our Time scales to find the numerical approximation to some specific examples then comparing them with their analytical solutions. We finally use the derived Taylor method to derive second order Runge Kutta method of order two on time scales and carry out examples using it.

AMS subject classification:
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1. Introduction

Stefan Hilger introduced the idea of time scales in 1990 [1] with a purpose of unifying the continuous and discrete calculus. From that time a lot of research has been carried out in this topic that it has been found to have a lot of applications especially when it come to modelling natural phenomenon.

Examples of such articles is the paper by Agarwal and Bohner [2] who defined the rules of calculus on time scales and demonstrated some of its applications such as the Taylor polynomial and the l'Hospital's rule. Guseinov in the article [3] introduced the integration on time scales by applying the Riemann integration version of the time scale.

The article [4] considers the notion of dynamic equations on time scales. In this article, a dynamic equation on time scale is defined as a unification of the difference equation, differential equation, q-difference equations and many more. It therefore proposes a method of finding solutions that work for any time scale given a dynamic equation that has a regressive function.

The study of dynamic equations has gained prominence lately especially when it comes to finding its solutions, example of such article is [5] which study the existence of boundary value problems. The paper by Li [6] studies the dynamic inequalities on time scales.

Applications of dynamic equations on time scales have been shown in several fields, one such field is biology why papers such as [7] and [8] have studied population models and also the field of economics that can be found in the articles [9] and [10].

Most mathematical models that we get from natural phenomena are either difference equations, differential equations or q-difference equations and that is our motivation for studying dynamic equations. We also realize from numerous studies that in most cases we can not always get a compact analytical solutions to our dynamic equations and that is why we opt to approximate their solutions.

This paper is organized as follows: In section 2 we give an introduction to the basic concepts of time scales calculus. For the study of Taylor methods and Runge Kutta methods we will need the concept of partial differential equations on time scales which we study in section 3, and in section 4 we look at the numerical methods. Finally in section 5 and 6 we give some examples and discuss the results we get.

2. Preliminaries on Time Scales

A time scale $\mathbb{T}$ is defined as any arbitrary non-empty closed subset of the real numbers $\mathbb{R}$. Examples of time scales are the real numbers $\mathbb{R}$, natural numbers $\mathbb{N}_0$, integers $\mathbb{Z}$ and many more.

We assume that any given time scale $\mathbb{T}$ inherits the standard topology of the real numbers. To study the theory of calculus on time scales, we need the following definitions.

Definition 2.1. The forward jump operator is defined as the function $\sigma : \mathbb{T} \to \mathbb{T}$ such that

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \ \forall t \in \mathbb{T}$$

and the backward jump operator is defined as the function $\rho : \mathbb{T} \to \mathbb{T}$ such that

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\} \ \forall t \in \mathbb{T}.$$

Definition 2.2. We define t to be right scattered if $\sigma(t) > t$ and t is left scattered if $\rho(t) < t$. If t is both right scattered and left scattered, then it is referred to as isolated point.

Definition 2.3. We also define t to be right dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$ and t is left dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$. We term t to be dense if it is both right dense and left dense.

Definition 2.4. We define the grainless function $\mu, \nu : \mathbb{T} \to \mathbb{R}$ by

$$\mu(t) := \sigma(t) - t,$$
$$\nu(t) := t - \rho(t).$$

Definition 2.5. Let $f : \mathbb{T} \to \mathbb{R}$, then $f^\sigma : \mathbb{T} \to \mathbb{R}$ is defined as

$$f^\sigma(t) = f(\sigma(t)) \quad \forall t \in \mathbb{T}.$$

If time scale $\mathbb{T}$ has a left scattered maximum value m then we define $\mathbb{T}^k = \mathbb{T} - m$. Otherwise if $\mathbb{T}$ is left dense then $\mathbb{T}^k = \mathbb{T}$.

2.1. Differentiation on Time Scales

We say that the function $f : \mathbb{T} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{T}^k := \{t \in \mathbb{T} : t \text{ non-maximal or left dense}\}$ if a number $f^\Delta(t)$ exists such that for any $\epsilon > 0$, a neighbourhood U of t for some $\delta > 0$ exists and

$$|[f^\sigma(t) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s| \ \forall s \in U.$$

$f^\Delta(t)$ is called the delta derivative of $f(t)$ on $\mathbb{T}^k$.

The following theorem defines the general differentiation on time scales.

Theorem 2.6. If $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, then

1. f is continuous at t provided f is differentiable at t.

2. If a function f is continuous at t and t is right scattered, then f is differentiable at t with

$$f^\Delta(t) = \frac{f^\sigma(t) - f(t)}{\mu(t)}$$

3. If t is dense, then f is differentiable at t with

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

4. If f is differentiable at t, then

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t). \tag{2.1}$$

The last point unifies both the isolated and dense points.

Theorem 2.7. Let $f, g : \mathbb{T} \to \mathbb{R}$ be differentiable at $t \in \mathbb{T}^k$, then

1. $(fg)^{\Delta}(t)$ is differentiable at t since

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) \tag{2.2}$$
$$= f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)). \tag{2.3}$$

2. $\dfrac{f}{g}$ is differentiable at t since

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}$$

provided $g(t)g(\sigma(t)) \neq 0$.

Definition 2.8. We define $f : \mathbb{T} \to \mathbb{R}$ as regulated if its right-sided limits exist at all right-dense points in $\mathbb{T}$ and its left-sided limits exist at all left-dense points in $\mathbb{T}$.

Definition 2.9. A function $f : \mathbb{T} \to \mathbb{R}$ is defined as rd-continuous if it is continuous at right-dense points in $\mathbb{T}$ and its left sided limits exist at all left dense point in $\mathbb{T}$. In this paper, we will denote rd-continuous functions by C_{rd}.

2.2. Dynamic Equations on time scales

Let $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$, then the ordinary dynamic equation given by

$$y^{\Delta}(t) = f(t, y(t)), \quad y(t_0) = y_0 \tag{2.4}$$

is referred to as an initial value problem.

A special type of initial value problems that has been extensively explored by researchers is those of the type

$$y^{\Delta}(t) = p(t)y(t), \quad y(t_0) = 1 \tag{2.5}$$

where $t_0 \in \mathbb{T}$.

The function $p : \mathbb{T} \to \mathbb{R}$ in (2.5) is called a regressive function provided

$$1 + \mu(t)p(t) \neq 0 \;\; \forall t \in \mathbb{T}.$$

The set of regressive and rd-continuous functions will be denoted in this paper by $\mathfrak{R}$.

Theorem 2.10. Let $p \in \mathfrak{R}$, then (2.5) has a unique solution.

The solution to (2.5) is referred to as the *exponential* function and it is defined by

$$e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right) \text{ for } s,t \in \mathbb{T} \tag{2.6}$$

where

$$\xi_h(z) = \begin{cases} \dfrac{\log(1+hz)}{h}, & \text{if } h \neq 0 \\ z, & \text{if } h = 0 \end{cases}.$$

2.3. Taylor Method on Time Scales

Ravi P. Agarwal and Martin Bohner stated and proved the Taylor formula in [2] and they found that

$$f(t) = \sum_{k=0}^{n-1} h_k(t,a) f^{\Delta^k}(a) + \int_a^{\rho^{n-1}(t)} h_{n-1}(t,\sigma(t)) f^{\Delta^n}(\tau)\Delta\tau \tag{2.7}$$

where

$$h_0(r,s) \equiv 1 \text{ and } h_{k+1}(r,s) = \int_s^r h_k(\tau,s)\Delta\tau \text{ for } k \in \mathbb{N}_0. \tag{2.8}$$

3. Partial Differentiation

Given the initial value problem (2.4), we can define the partial derivative with respect to t of the function $f(x, y(t))$ as

$$\lim_{s\to t} \frac{f(\sigma(t), y(t)) - f(s, y(t))}{\sigma(t) - s}$$

provided the above limit exist and throughout this paper we will denote it as

$$\frac{\partial f(t, y(t))}{\Delta t}.$$

Similarly, the partial derivative of $f(t, y)$ with respect to y is defined as

$$\lim_{s\to t} \frac{f(t, y(\sigma(t))) - f(t, y(s))}{\sigma(t) - s}$$

We now define the concepts of complete delta differentiation of a function at a point.

Definition 3.1. A function $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is said to be completely delta differentiable at a point $(t_0, y_0) \in \mathbb{T} \times \mathbb{R}$ if the numbers A and B exists and are independent of $(t, y) \in \mathbb{T} \times \mathbb{R}$ such that for all t in the neighborhood of U we have

$$f(t_0, y_0) - f(t, y) = A(t_0 - t) + B(y_0 - y) \tag{3.1}$$

$$+\alpha(t_0 - t) + \beta(y_0 - y) \tag{3.2}$$

and

$$\begin{aligned} f(t_0, y(\sigma(t_0))) - f(t, y) &= A[t_0 - t] + B[y(\sigma(t_0)) - y] + \\ &\quad \alpha(t_0 - t) + \beta(y(\sigma(t_0)) - y), \\ f(\sigma(t_0), y_0) - f(t, y) &= A[\sigma(t_0) - t] + B[y_0 - y] + \\ &\quad \alpha(\sigma(t_0) - t) + \beta(y_0 - y). \end{aligned}$$

The numbers $\alpha(t_0, t)$ and $\beta(y_0, y)$ are defined on the given neighborhood such that

$$\lim_{t \to t_0} \alpha(t, t_0) = 0 \text{ and} \tag{3.3}$$

$$\lim_{t \to t_0} \beta(t, t_0) = 0. \tag{3.4}$$

We can therefore get from the above definition that if $f(t, y(t))$ is completely delta differentiable at (t_0, y_0), then it is continuous at that point and

$$\frac{\partial f(t_0, y_0)}{\Delta t} = A \text{ and } \frac{\partial f(t_0, y_0)}{\Delta y} = B.$$

Definition 3.2. We say $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is σ-completely delta differentiable at (t_0, y_0) if

$$\begin{aligned} f(\sigma(t_0), y(\sigma(t_0))) - f(t, y) &= A[\sigma(t_0) - t] + B[y\sigma(t_0) - y] + \\ &\quad \alpha(\sigma(t_0) - t) + \beta(y(\sigma(t_0)) - y) \end{aligned}$$

for all (t, s) in the neighborhood of U.

In this case we get

$$B = \frac{\partial f(\sigma(t_0), y_0)}{\Delta y}.$$

From the above discussions on delta differentiability, we now state and prove a theorem that will help in finding derivative of a function of the form $f(t, y(t))$.

Theorem 3.3. Let the function $f(t, y(t))$ be σ-completely delta differentiable at the point (t_0, y_0). If $y(t)$ has a delta derivative at t_0, then

$$F(t) = f(t, y(t)) \text{ for } t \in \mathbb{T}$$

has a delta derivative at the point which is expressed by the formula

$$F^{\Delta}(t_0) = \frac{\partial f(t_0, y_0)}{\Delta t} + \frac{\partial f(\sigma(t_0), y_0)}{\Delta y} y^{\Delta}(t_0). \tag{3.5}$$

Proof. To prove the theorem, we consider the definition of σ-complete differentiability as shown below

$$f(\sigma(t_0), y(\sigma(t_0))) - f(t, y) = A[\sigma(t_0) - t] + B[y\sigma(t_0) - y] + \\ \alpha(\sigma(t_0) - t) + \beta(y(\sigma(t_0)) - y).$$

Using the above equation with the fact that

$$A = \frac{\partial f(t_0, y_0)}{\Delta t} \text{ and } B = \frac{\partial f(\sigma(t_0), y_0)}{\Delta y},$$

we have

$$\begin{aligned} F(\sigma(t_0)) - F(t) &= f(\sigma(t_0), y(\sigma(t_0))) - f(t, y(t)) \\ &= \frac{\partial f(t_0, y(t_0))}{\Delta t}[\sigma(t_0) - t_0] + \frac{\partial f(\sigma(t_0), y(t_0))}{\Delta y} \times \\ &\quad (y(\sigma(t_0)) - y(t_0)) + \alpha[\sigma(t_0) - t] + \beta[y(\sigma(t_0)) - y(t)]. \end{aligned}$$

To complete the proof we divide both sides of the equation by $\sigma(t_0) - t$ and pass the limit as t tends to t_0.

$$\lim_{t \to t_0} \frac{F(\sigma(t_0)) - F(t)}{\sigma(t_0) - t} = \lim_{t \to t_0} \frac{\partial f(t_0, y(t_0))}{\Delta t} + \frac{\partial f(\sigma(t_0), y(t_0))}{\Delta y} \times \\ \lim_{t \to t_0} \frac{y(\sigma(t_0)) - y(t_0)}{\sigma(t_0) - t} + \alpha + \beta \lim_{t \to t_0} \frac{y(\sigma(t_0)) - y(t)}{\sigma(t_0) - t}$$

Since the limit as $t \to t_0$ implies $\alpha \to 0$ and $\beta \to 0$ then

$$F^{\Delta}(t) = \frac{\partial f(t_0, y(t_0))}{\Delta t} + \frac{\partial f(\sigma(t_0), y(t_0))}{\Delta y} y^{\Delta}(t_0)$$

which completes the proof. ■

4. Numerical Methods

4.1. Taylor Method for Solving Dynamic Equations

We now solve the initial value problem (2.4) given by

$$y^{\Delta}(t) = f(t, y),$$
$$y(t_0) = y_0.$$

By applying and modifying the Taylor method given by the equation (2.7), then the solution of the initial value problem (2.4) is given by

$$y(t_0+h)=\sum_{k=0}^{n-1} h_k(t_0+h,t_0)y^{\Delta^k}(t_0). \tag{4.1}$$

Therefore, the only challenge left is to find the values of y^{Δ^i} for $i \in \mathbb{N}_0$.
Note that from the initial conditions of the dynamic equation given we have

$$y^{\Delta}(t_0)=f(t_0,y_0).$$

Taking the function f to be σ-completely differentiable at (t_0,y_0), then

$$y^{\Delta^2}(t_0)=\frac{\partial f(t_0,y(t_0))}{\Delta t}+\frac{\partial f(\sigma(t_0),y(t_0))}{\Delta y}f(t_0,y(t_0)).$$

Now, using theorem (3.3) and applying it on $y^{\Delta^2(t_0)}$ we get $y^{\Delta^3(t_0)}$ to be

$$\begin{aligned}
y^{\Delta^3(t_0)} &= \frac{\partial^2 f(t_0,y(t_0))}{\Delta t^2}+\frac{\partial^2 f(\sigma(t_0),y(t_0))}{\Delta t\Delta y}f(t_0,y(t_0))\\
&+\frac{\partial^2 f(\sigma(t_0),y(t_0))}{\Delta t\Delta y}f(\sigma(t_0),y(\sigma(t_0)))+\\
&\frac{\partial^2 f(\sigma^2(t_0),y(t_0))}{\Delta y^2}f(t_0,y_0)f(\sigma(t_0),y(\sigma(t_0)))+\\
&\frac{\partial f(t,y)}{\Delta t}\frac{\partial f(\sigma(t_0),y_0)}{\Delta y}+\left(\frac{\partial f(\sigma(t_0),y_0)}{\Delta y}\right)^2 f(t_0,y_0).
\end{aligned}$$

Using theorem (3.3) and the chain rule given by equation (2.2) we get y^{Δ^4} as

$$\begin{aligned}
y^{\Delta^4}(t_0) &= \frac{\partial^3 f(t_0,y_0)}{\Delta t^3}+\frac{\partial^3 f(\sigma(t_0),y_0)}{\Delta t^2\Delta y}f(t_0,y_0)\\
&+\frac{\partial^3 f(\sigma(t_0),y_0)}{\Delta t^2\Delta y}f(\sigma(t_0),y(\sigma(t_0)))+\frac{\partial^3 f(\sigma^2(t_0),y_0)}{\Delta t\Delta y^2}\\
&\times f(t_0,y_0)f(\sigma(t_0),y(\sigma(t_0)))+\frac{\partial^2 f(\sigma(t_0),y_0)}{\Delta t\Delta y}\frac{\partial f(t_0,y_0)}{\Delta t}\\
&+\frac{\partial^2 f(\sigma(t_0),y_0)}{\Delta t\Delta y}\frac{\partial f(\sigma(t_0),y_0)}{\Delta y}f(t_0,y_0)
\end{aligned}$$

$$+\frac{\partial^3 f(\sigma(t_0), y_0)}{\Delta t^2 \Delta y} f(\sigma^2(t_0), y(\sigma^2(t_0))) + \frac{\partial^3 f(\sigma^2(t_0), y_0)}{\Delta t \Delta y^2}$$
$$\times f(t_0, y_0) f(\sigma^2(t_0), y(\sigma^2(t_0))) + \frac{\partial^2 f(\sigma(t_0), y_0)}{\Delta t \Delta y}$$
$$\times \frac{\partial f(\sigma(t_0), y(\sigma(t_0)))}{\Delta t} + \frac{\partial^2 f(\sigma(t_0), y_0)}{\Delta t \Delta y} \frac{\partial f(\sigma^2(t_0), y(\sigma(t_0)))}{\Delta y}$$
$$\times f(\sigma(t_0), y(\sigma(t_0))) + \frac{\partial^3 f(\sigma^2(t_0), y_0)}{\Delta t \Delta y^2} f(\sigma(t_0), y(\sigma(t_0)))$$
$$\times f(\sigma^2(t_0), y(\sigma^2(t_0))) + \frac{\partial^3 f(\sigma^3(t_0), y_0)}{\Delta y^3} f(t_0, y_0) \times$$
$$f(\sigma(t_0), y(\sigma(t_0))) f(\sigma^2(t_0), y(\sigma^2(t_0))) + \frac{\partial^2 f(\sigma^2(t_0), y_0)}{\Delta y^2}$$
$$\times \frac{\partial f(t_0, y_0)}{\Delta t} f(\sigma^2(t_0), y(\sigma^2(t_0))) + \frac{\partial f(\sigma^2(t_0), y_0)}{\Delta y^2}$$
$$\times \frac{\partial f(\sigma(t_0), y_0)}{\Delta y} f(t_0, y_0) f(\sigma^2(t_0), y(\sigma^2(t_0)))$$
$$+\frac{\partial^2 f(\sigma^2(t_0), y_0)}{\Delta y^2} \frac{\partial f(\sigma(t_0), y(\sigma(t_0)))}{\Delta t} f(t_0, y_0)$$
$$+\frac{\partial^2 f(\sigma^2(t_0), y_0)}{\Delta y^2} \frac{\partial f(\sigma^2(t_0), y(\sigma(t_0)))}{\Delta y} f(t_0, y_0) f(\sigma(t_0), y(\sigma(t_0)))$$
$$+\frac{\partial^2 f(\sigma(t_0), y_0)}{\Delta t \Delta y} \frac{\partial f(\sigma(t_0), y(\sigma(t_0)))}{\Delta t} + \frac{\partial^2 f(\sigma^2(t_0), y_0)}{\Delta y^2}$$
$$\times \frac{\partial f(\sigma(t_0), y(\sigma(t_0)))}{\Delta t} f(t_0, y_0) + \frac{\partial^2 f(t_0, y_0)}{\Delta t^2} \frac{\partial f(\sigma(t_0), y_0)}{\Delta y}$$
$$+\frac{\partial^2 f(\sigma(t_0), y_0)}{\Delta t \Delta y} \frac{\partial f(\sigma(t_0), y_0)}{\Delta y} f(t_0, y_0) + \frac{\partial^2 f(\sigma(t_0), y_0)}{\Delta t \Delta y}$$
$$\times \frac{\partial f(\sigma^2(t_0), y(\sigma(t_0)))}{\Delta y} + \frac{\partial^2 f(\sigma^2(t_0), y_0)}{\Delta y^2} \frac{\partial f(\sigma^2(t_0), y(\sigma(t_0)))}{\Delta y}$$
$$\times f(t_0, y_0) + \frac{\partial^2 f(\sigma(t_0), y_0)}{\Delta t \Delta y} \frac{\partial f(\sigma(t_0), y_0)}{\Delta y} f(\sigma(t_0), y(\sigma(t_0)))$$
$$+\frac{\partial^2 f(\sigma^2(t_0), y_0)}{\Delta y^2} \frac{\partial f(\sigma(t_0), y_0)}{\Delta y} f(t_0, y_0) f(\sigma(t_0), y(\sigma(t_0)))$$
$$+\left(\frac{\partial f(\sigma(t_0), y_0)}{\Delta y}\right)^2 \frac{\partial f(t_0, y_0)}{\Delta t} + \left(\frac{\partial f(\sigma(t_0), y_0)}{\Delta y}\right)^2$$
$$\times \frac{\partial f(\sigma(t_0), y_0)}{\Delta y} f(t_0, y_0).$$

Therefore, the first order, second order, third order and fourth order Taylor methods are given by

$$y(t_0 + h) = y_0 + h_1 y^{\Delta}(t_0), \tag{4.2}$$

$$y(t_0 + h) = y_0 + h_1 y^{\Delta}(t_0) + h_2 y^{\Delta^2}(t_0), \tag{4.3}$$

$$y(t_0 + h) = y_0 + h_1 y^{\Delta}(t_0) + h_2 y^{\Delta^2}(t_0) + h_3 y^{\Delta^3}(t_0) \text{ and} \tag{4.4}$$

$$y(t_0 + h) = y_0 + h_1 y^{\Delta}(t_0) + h_2 y^{\Delta^2}(t_0) + h_3 y^{\Delta^3}(t_0) \tag{4.5}$$

$$+ h_4 y^{\Delta^4}(t_0). \tag{4.6}$$

4.2. Runge Kutta Method for solving Ordinary Dynamic Equations on Time Scales

To find a second order Runge Kutta method for solving ordinary dynamic equations on time scales, we follow the methodology in texts when the function f is continuous and modifying it to a time scales version.

For a function $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ where $t \in \mathbb{T}$ and $y \in \mathbb{R}$, define

$$y(t_{n+1}) = y_n + h_1(w_1 k_1 + w_2 k_2) \tag{4.7}$$

with

$$k_1 = f(t_n, y_n),$$
$$k_2 = f(t_n + ah, y + hbk_1)$$

where w_1, w_2, a and b are arbitrary constants that we need to find.

Assuming that f is σ-completely delta differentiable, we can expand k_2 to get

$$k_2 = f(t_n, y_n) + ah_1 \frac{\partial f(t_n, y_n)}{\Delta t} + bh_1 k_1 \frac{\partial f(\sigma(t_n), y_n)}{\Delta y}$$
$$+ a^2 h_2 \frac{\partial^2 f(t_n, y_n)}{\Delta t^2} + 2abk_1 h_2 \frac{\partial^2 f(\sigma(t_n), y_n)}{\Delta t \Delta y} + b^2 k_1^2 h_2$$
$$\times \frac{\partial^2 f(\sigma^2(t_n), y_n)}{\Delta y}.$$

Substituting this expansion of k_2 back into (4.7) we have

$$y_{n+1} = y_n + h_1(w_1 + w_2) f(t_n, y_n) +$$
$$h_1^2 w_2 \left(a \frac{\partial f(t_n, y_n)}{\Delta t} + bk_1 \frac{\partial f(\sigma(t_n), y_n)}{\Delta y} \right) + \dots$$

We now compare the above value of y_{n+1} with the second order Taylor method and we get the following equations.

$$w_1 + w_2 = 1, \tag{4.8}$$

$$h_1^2 a w_2 = h_2, \tag{4.9}$$

$$h_1^2 b w_2 = h_2. \tag{4.10}$$

This implies that $a = b$.

5. Numerical Examples

In this section we solve several examples of ordinary dynamical equations on time scales using the methods discussed in the previous section. We will consider instances when the time scale is either a set of real numbers, $\mathbb{R}$ or a set of integers, $\mathbb{Z}$.

The examples are worked out below:

1. The first example we consider is the problem given by (2.5). The Taylor method for this problem is given by the equation (4.1). We therefore get

$$\begin{aligned} y^{\Delta}(t_n) &= p(t_n)y(t_n), \\ y^{\Delta^2}(t_n) &= p^{\Delta}(t_n)y(t_n) + p(\sigma(t_n))p(t_n)y(t_n) \\ &= p^{\Delta}(t_n)y(t_n) + p(t_n + h_1)p(t_n)y(t_n), \\ y^{\Delta^3}(t_n) &= p^{\Delta^2}(t_n)y(t_n) + p^{\Delta}(\sigma(t_n))p(t_n)y(t_n) \\ &\quad + p^{\Delta}(\sigma(t_n))p(\sigma(t_n))(y(t_n) + \mu p(t_n)y(t_n)) \\ &\quad + p^{\Delta}(t_n)p(\sigma(t_n))y(t_n) + (p(\sigma(t_n)))^2 p(t_n)y(t_n). \end{aligned}$$

   Substituting the above derivatives back to (4.1) we get a third order Taylor approximation to the equation (2.5).

2. We now solve a specific problem of the form given by equation (2.5). Consider an initial condition

$$\begin{aligned} y^{\Delta}(t) &= 2ty(t) \\ y(t_0) &= 1. \end{aligned}$$

   We need to solve the equation given that $\mathbb{T} = \mathbb{R}$. When solving this equation numerically, note that the time scale will take a discrete form. Assuming that we are evaluating in the time scale $[0, 1] \subseteq \mathbb{R}$ with a step-size, $h = 0.1$. Then,

$$\begin{aligned} \sigma(t_n) &= t_n + h = t_n + 0.1, \\ \mu(t_n) &= h = 0.1. \end{aligned}$$

   From the definition of differentiation given by (2.1), we have

$$y(\sigma(t_n)) = y(t_n) + \mu(t_n)f(t_n, y(t_n)).$$

   The derivatives of the Taylor method are given by

$$\begin{aligned} y^{\Delta}(t_n) &= 2t_n y(t_n), \\ y^{\Delta^2}(t_n) &= 2y(t_n) + 4(t_n + 0.1)y(t_n)t_n, \\ y^{\Delta^3}(t_n) &= 4t_n y(t_n) + 4(t_n + 0.1)(y(t_n) + 2\mu t_n y_n) \\ &\quad * 4y(t_n)(t_n + 0.1) + 2(2(t_n + 0.1))^2 t_n y_n. \end{aligned}$$

Therefore, the third order Taylor method is given by

$$y(t_{n+1}) = y(t_n) + h_1 y^{\Delta}(t_n) + h_2 y^{\Delta^2}(t_n) + h_3 y^{\Delta^3}(t_n).$$

For the case when $\mathbb{T} = \mathbb{R}$, it has been shown in [4], that

$$h_k(t_0 + h, t_0) = \frac{h^k}{k!}. \tag{5.1}$$

Note also that the exact solution of the dynamic equation we are solving is given by the exponential function shown in (2.6) and the solution is

$$y(t) = \exp(t^2).$$

In the table below we compare the actual results, the solution as a result of the normal Taylor method when a function f is continuous and our Taylor method.

Table 1: Taylor Method for time scale $\mathbb{R}$

t	Exact Solution	Taylor order 4	Time Scale order 3
0.0	1.000000	1.000000	1.000000
0.1	1.010050	1.010000	1.010133
0.2	1.040811	1.040909	1.041388
0.3	1.094174	1.094642	1.095718
0.4	1.173511	1.174629	1.176623
0.5	1.284025	1.286185	1.289529
0.6	1.433329	1.437097	1.442398
0.7	1.632316	1.638544	1.646672
0.8	1.896481	1.906470	1.918698
0.9	2.247908	2.263676	2.281890
1.0	2.718282	2.742996	2.770026

3. We now solve an equation of the form (2.5) when $\mathbb{T} = \mathbb{Z}$. Consider the initial value problem below

$$y^{\Delta}(t) = ty$$
$$y(0) = 1.$$

We evaluate the values in the interval $[0, 10] \subseteq \mathbb{T}$. In the case $\mathbb{T} = \mathbb{Z}$, note that

$$\sigma(t_n) = t_n + 1,$$
$$\mu(t_n) = 1 \text{ and}$$
$$y(\sigma(t_n)) = y(t_n) + f(t_n, y(t_n)).$$

In this Time Scale, the derivatives are given by

$$
\begin{aligned}
y(t_0) &= 1,\\
y^{\Delta}(t) &= f(t,y) = ty,\\
y^{\Delta^2}(t) &= y + (t+1)ty,\\
y^{\Delta^3}(t) &= ty + (t+1)(y+ty) + (t+1)y + (t+1)^2 ty.
\end{aligned}
$$

For the case when $\mathbb{T} = \mathbb{Z}$, then it has been shown in [4] that

$$h_k(t,t_0) = \binom{t-t_0}{k}. \tag{5.2}$$

Therefore, the first, second and third order Taylor method are given by

$$
\begin{aligned}
y(t_n+1) &= y(t_n) + t_n y_n,\\
y(t_n+2) &= y(t_n) + 2t_n y_n + (y_n + (t_n+1)t_n y_n),\\
y(t_n+3) &= y(t_n) + 3t_n y_n + 3(y_n + (t_n+1)t_n y_n)\\
&\quad + t_n y_n + (t_n+1)(y_n + t_n y_n) + (t_n+1)y_n\\
&\quad + (t_n+1)^2 t_n y_n.
\end{aligned}
$$

Note that the solution of the given dynamic equation is given by

$$y(t) = \prod_{\tau=0}^{t-1} (1+\tau).$$

When calculating the k-th order Taylor approximation when $\mathbb{T} = Z$, we note that the difference between t_n and t_{n+1} is a multiple of k. We now present the results in the table (2).

4. We now consider the problem given in book [11] chapter 2 exercise 2.79 problem (ii). The problem requires the solution of the initial value problem

$$y^{\Delta}(t) = 2y + 3^t \tag{5.3}$$

$$y(0) = 0 \tag{5.4}$$

given that the time scale $\mathbb{T} = \mathbb{Z}$.

It has been proved in [11] that given a first order initial value problem dynamic equation of the form

$$y^{\Delta}(t) = p(t)y(t) + f(t), \;\; y(t_0) = y_0.$$

Table 2: Taylor Method for solving $y^{\Delta}(t) = ty$ when $\mathbb{T} = \mathbb{Z}$

t	Exact Solution	Taylor order 1	Taylor order 2	Taylor order 3
0	1	1	1	1
1	1	1		
2	2	2	2	
3	6	6		6
4	24	24	24	
5	120	120		
6	720	720	720	720
7	5040	5040		
8	40320	40320	40320	
9	362880	362880		362880
10	3628800	3628800	3628800	

then its solution is given by

$$y(t) = e_p(t, t_0)y_0 + \int_{t_0}^{t} e_p(t, \sigma(\tau))f(\tau)\Delta\tau.$$

Therefore, using the above expression, then the solution of (5.3) is given by

$$y(t) = t \times 3^{t-1}.$$

We now seek to find the Taylor approximation of the above dynamic equation and compare it to the analytic solution.

Therefore, in this case

$$\begin{aligned} y(0) &= 0, \\ y^{\Delta}(t_n) &= f(t_n, y_n) = 2y_n + 3^{t_n}, \\ y^{\Delta^2}(t_n) &= 4(y + 3^t), \\ y^{\Delta^3}(t_n) &= 4(2y + 3 \times 3^t). \end{aligned}$$

A table that compare the exact solution and the numerical results is given below

Table 3: Taylor Method for solving $y^{\Delta}(t) = 2y + 3^t$ when $\mathbb{Z}$

t	Exact Solution	order 1	order 2	order 3
0	0	0	0	0
1	1	1		
2	6	6	6	
3	27	27		27
4	108	108	108	
5	405	405		
6	1458	1458	1458	1458
7	5103	5103		
8	17496	17496	17496	
9	59049	59049		59049
10	196830	196830	196830	

5. To end the discussion on Taylor methods, we consider the example below when $\mathbb{T} = \mathbb{R}$.

$$ty^{\Delta}(t) + 2y = t^2 - t + 1, \;\; y(1) = 0.5.$$

The solution of this dynamic equation is given by

$$y(t) = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{1}{12}t^2.$$

We now find the numerical approximation using the Taylor method by taking $h = 0.1$ we have devised.

In this example, the derivatives are given by

$$\begin{aligned} y(1) &= 0, \\ y^{\Delta}(t_n) &= f(t_n, y_n) = \frac{-2}{t_n}y_n + t_n - 1 + \frac{1}{t_n}, \\ y^{\Delta^2} &= \frac{2}{t_n^2}y_n + 1 - \frac{1}{t_n^2} - \frac{2}{t_n + 0.1}\left(\frac{-2}{t_n}y_n + t_n - 1 + \frac{1}{t_n}\right), \\ y^{\Delta^3}(t_n) &= \frac{-4}{t_n^3}y_n + \frac{2}{t_n^3} + \frac{2}{(t_n+1)^2}\left(\frac{-2}{t_n}y_n + t_n - 1 + \frac{1}{t_n}\right) \\ &\quad + \frac{2}{(t_n+0.1)^2}\left(\frac{-2}{t_n+0.1}\left(y_n + 0.1f(t_n, y_n)\right) + (t_n + 0.1) - 1 + \frac{1}{t_n + 0.1}\right) \\ &\quad + \left(\frac{-2}{t_n+0.1}\right)\left(\frac{2}{t_n^2}y_n + 1 - \frac{1}{t_n^2}\right) + \left(\frac{-2}{t_n+0.1}\right)^2\left(\frac{-2}{t_n}y_n + t_n - 1 + \frac{1}{t_n}\right). \end{aligned}$$

The numerical results is as shown in the table below

Table 4: Taylor Method when $\mathbb{T} = \mathbb{R}$

t	Exact Solution	Time Scale Order 3	Taylor order 4
1.0	0.500000	0.500000	0.500000
1.1	0.536667	0.504878	0.504708
1.2	0.580000	0.518124	0.517876
1.3	0.630000	0.538760	0.538483
1.4	0.686667	0.566135	0.565857
1.5	0.750000	0.599804	0.599543
1.6	0.820000	0.639456	0.639224
1.7	0.896667	0.684868	0.684673
1.8	0.980000	0.735874	0.735725
1.9	1.070000	0.792353	0.792255
2.0	1.166667	0.854213	0.854171

6. We now consider an example evaluated using Runge-Kutta method of order two. When $\mathbb{T} = \mathbb{R}$, consider the problem

$$\begin{aligned} y^{\Delta}(t) &= 2y + t, \\ y(0) &= 0. \end{aligned}$$

The exact solution of the problem is given by

$$y(t) = \frac{1}{4}\exp(2t) - \frac{t}{2} - \frac{1}{4}.$$

To evaluate the dynamic equations using the Runge Kutta method, we first find the coefficients w_1, w_2 and a as defined by the conditions (4.8). We first note that when $\mathbb{T} = \mathbb{R}$ then

$$h_2 = \frac{h_1^2}{2}$$

as defined in (5.1).

Therefore

$$\begin{aligned} w_1 + w_2 &= 1, \\ a w_2 &= \frac{1}{2}. \end{aligned}$$

From the above given equations, we see that we have two equations but three unknown variables and therefore to solve it, we have to set some values. In our case, set $w_1 = w_2 = \frac{1}{2}$, then $a = 1$.

Therefore, Runge Kutta method in this case will be given by

$$\begin{aligned} y(t_{n+1}) &= h_1(w_1k_1 + w_2k_2), \\ &= h_1(\frac{1}{2}k_1 + \frac{1}{2}k_2), \\ k_1 &= f(t_n, y_n), \\ k_2 &= f(t_n + ah_1, y_n + ah_1k_1). \end{aligned}$$

Take $h_1 = 0.1$, then the following table show the numerical result and the exact solution at each step t.

Table 5: Runge Kutta Method for time scale $\mathbb{R}$

t	Exact Solution	Runge Kutta order 2
0.0	0.000000	0.000000
0.1	0.005351	0.005000
0.2	0.022956	0.022100
0.3	0.055530	0.053962
0.4	0.106385	0.103834
0.5	0.179570	0.175677
0.6	0.280029	0.274326
0.7	0.413800	0.405678
0.8	0.588258	0.576927
0.9	0.812412	0.796851
1.0	1.097264	1.076158

7. We now consider the case when $\mathbb{T} = \mathbb{Z}$ and solve the initial value problem

$$\begin{aligned} y^{\Delta}(t) &= ty, \\ y(0) &= 1. \end{aligned}$$

For this case, the solution is given by

$$y(t) = \prod_{\tau=0}^{t-1}(1 + \tau).$$

If $\mathbb{T} = \mathbb{Z}$, then the coefficients w_1, w_2 and a defined by the conditions will be calculated. Firstly, we note that

$$h_1 = 2 \text{ and } h_2 = 1.$$

We therefore have

$$\begin{aligned} w_1 + w_2 &= \frac{1}{2}, \\ 4aw_1 &= 1. \end{aligned}$$

Using the same arguments as in the previous problem, we set $w_1 = w_2 = \frac{1}{2}$ and then $a = \frac{1}{2}$.
Then Runge Kutta equation takes the form

$$\begin{aligned} y_{n+1} &= y_n + \frac{h_1}{2}(k_1 + k_2), \\ &= y_n + k_1 + k_2, \\ k_1 &= f(t_n, y_n), \\ k_2 &= f(t_n + 1, y + f(t_n, y_n)). \end{aligned}$$

The following table gives the exact solution and the Runge Kutta method approximate to the same problem.

Table 6: Runge Kutta Method for time scale $\mathbb{Z}$

t	Exact Solution	y order 2
0.0	1	1
2.0	2	2
4.0	24	24
6.0	720	720
8.0	40320	40320
10.0	3628800	3628800
12.0	479001600	479001600
14.0	87178291200	87178291200
16.0	20922789888000	20922789888000
18.0	6402373705728000	6402373705728000
20.0	$2.43290200817664 \times 10^{18}$	$2.43290200817664 \times 10^{18}$

6. Discussion of the Results

In the examples we have considered in the previous section, we have noted from tables (2) and (3) that the Taylor method we have come up with gives us an exact solution of the initial value problem when $\mathbb{T} = \mathbb{Z}$.

We have also noted that in the case when $\mathbb{T} = \mathbb{Z}$, then first, second and third order Taylor method gives us the solutions of y_{n+1}, y_{n+2} and y_{n+3} all from the initial value at y_n as opposed to the case when $\mathbb{T} = \mathbb{R}$ that we only get y_{n+1} from y_n no matter the order. This therefore implies that if we want a value of y_{n+k} when $\mathbb{T} = \mathbb{Z}$ we do not need to iterate all the steps to get there especially if we have a k-th order Taylor formula and therefore saves us the computational cost.

From tables (1) and (4) where we considered $\mathbb{T} = \mathbb{R}$, we compared the normal Taylor method and our Taylor version and as we can see that the difference between their approximations are nearly negligible.

For Runge Kutta method of order two when $\mathbb{T} = \mathbb{Z}$, table (6) shows that the solution of the Runge Kutta method is similar to the exact solution.

Finally, when $\mathbb{T} = \mathbb{R}$ then our Runge Kutta method is similar to the classical Runge Kutta method of order 2 and it gives an approximation with minimal error to the initial value problem.

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