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# A Note on Hilbert-Kunz Multiplicity of Three-Dimensional Local Near- Rings

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**Abstract**

In this paper, we investigate a lower bound (say $s_{HK}(p, d)$) on Hilbert-Kunz multiplicities for non-regular unmixed local near - rings of Krull dimension ‘d’ with characteristic p > 0. Here wemainly focus three-dimensional Local Near-rings. In fact, as a main result, we will prove that $s_{HK}(p, 3) = 4/3$ and that a three-dimensional complete local ring of Hilbert-Kunz multiplicity 4/3 is isomorphic to the non-degenerate quadric hyperplanesk[[X, Y,Z,W]]/(X2 +Y 2 + Z2 +W2) under some existing conditions.

***Keywords*:** Ring, Near-ring, Local near-ring, Krull Dimension, Noetherian Local near-ring.

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## 1. PRELIMINARIES

In this section we give the existing definitions and examples which are used in next sections.

**1.1 Definition**: A **Ring** is non empty set *R* together with two binary operations’ +’ and ‘.’on*R* such that

(i) $(R, +)$ is an abelian group; (ii) $(R,.)$ is a semi-group; and

(iii) For any $a, b, c \in R$ we have $a(b + c) = ab + ac$, $(a + b)c = ac + bc$.

**1.2 Definition:** A nonempty set N is said to be **a Near-ring** with two binary operations '+'and '.' If

i) (N, +) is a group (not necessarily abelian); ii) (N, .) is a semi group and

iii) (x + y)z= xz + yz for all x ,y ,z∈N.

This is known as a right Near-ring because here we used only right distributive law.

Throughout this paper we consider Near-ring N as a right Near-ring.

**1.3 Example**: Let Z be the set of positive, negative integers with '0'.Then (Z, + , .) is a Near- ring with usual addition and multiplication.

**1.4 Definition:** A Near-ring N is said to be **Regular Near-ring** if for each element x∈N then there exists an element y ∈ N such that x= xyx.

**1.5 Example:** M(Γ) and $M_0(\Gamma)$ are Regular Near-rings (Beidleman(10)NR Text)

**1.6 Definition**: Let N be a Near-ring. By an **N-*group*** $_N$G, we mean an additively written group G (but not necessarily abelian), together with a mapping N × G →G (The image of (n, g) being denoted by ng) satisfying the following conditions:

(i) $(n_1 + n_2)g = n_1g + n_2g$; and

(ii) $n_1(n_2g) = (n_1n_2)g$ for all g ∈G and $n_1$, $n_2$∈ N.It is clear that $_N$N is an N-group.

**1.7 Definition:** A Near-ring N is called **Local** if and only if N has a unique maximal N-subgroup.

**1.8 Definition**: A Near-ringN is said to be **Regular Local Near-ring** if it satisfies both the conditions of Regular and Local Near- ring.

**1.9 Example:** $M_{aff}(v)$ is Local Near- ring.

**1.10 Definition**: The **dimension** of a Near-ringN, denoted by dim Nwill be taken as its Krull dimension, the maximum length n of a chain $P_0 \subset P_1 \subset \ldots \subset P_n$ of prime ideals of N. If there is no upper bound on the length of such a chain, we will take n = ∞ .

**1.11 Definition:** A Local Near-ring'N' is said to be **Noetherian Local Near-ring** if there exists ideals say $M_1, M_2, ...M_n$ such that they satisfies ascending chain condition.

**1.12 Definition:** Let (A, m, k) be a Local Near-ring of characteristic p>0. We say that A is weekly F-regular (resp. F-rational) if every ideal (resp.every parameter ideal) is tightly closed. Also, A is F- regular (resp. F- rational) if any Local Near-ring of A is weakly F-regular (resp. F-rational).

**1.13 Note:** An F-rational Local Near-ring is normal and Cohen Macaulay ring.

**1.14 Theorem (5):** Let (A, m, k) be an equi-dimensional local ring which is a homomorphic image of a Cohen-Macaulay local ring of characteristic p>0 then

1) If J is a parameter ideal of A, then (J) ≥$l_A$(A/J*).
2) Suppose that A is unmixed. If e(J)=$l_A$(A/J*).Then A is F-rational and is Cohen-Macaulay.

**1.15 Corollary (5):** Let (A, m, k) be an unmixed Local ring of characteristic p>o. Suppose that e(A)=2. Then Ais F-rational if and only if $e_{HK}(A) < 2$.When this is the case, A is ahypersurface.

**1.16 Theorem(5): L**et (A, m, k) be an unmixed Local ring of characteristic p>0 with dim A=4.

If $e(A) \geq 3, then\, e_{HK}(A) \geq \frac{5}{4} = \frac{30}{24}$.

Suppose that k=$\overline{k}$ and char k≠ 2put $A_{p,4} = \overline{\mathbb{F}_p}[[X_0, X_1, ...X_4]]/(X_0^2 + .......X_4^2)$

Then the followingstatement holds.

1) If A is not regular, then $e_{HK}(A) \geq e_{HK}(A_{p,4}) = \frac{29p^2+15}{24p^2+12}$.
2) The following conditions are equivalent
   a) Equality holds in (1).
   b) $e_{HK}(A) < \frac{5}{4}$
   c) The complementation of A is isomorphic to $A_{p,4}$.

**1.17 proposition [5]:** Let (A, m, k) be an unmixed Local ring of characteristic p>o with d = dimA. Let I be an m-primary ideal of A .Then $\frac{e(I)}{d\downarrow} \leq e_{HK}(I) \leq e(I)$ also

if d ≥2 then $\frac{e(I),}{d\downarrow} < e_{HK}(I) \leq e(I)$.

## 2. UNMIXED LOCAL NEAR-RINGS

In this section we derived some results on unmixed local near-rings.

**2.1 Theorem [8]:** Let (A,m,k) be an unmixed local near-ring of positive characteristic.

Then A is regular if and only if $e_{HK}(A) = 1$

**2.2 Theorem [8]:** Let (A.m)→(B,n) be a module - finite extension of local domains.

Then for every m-primary ideal I of A, we have $e_{HK}(I) = \frac{e_{HK}(IB)}{[Q(B):Q(A)]}.[B/n:A/m]$ where Q(A) denotes the fraction field of A.

Now we see some examples of Hilbert-Kunz multiplicities which are given by the above formula.

Consider the Veronese sub Near-ring A defined by

$$A = k\left\lfloor [x_1^{i_1}.....x_d^{i_d} \mid i_1,.....,i_d \geq 0, \sum i_j = r] \right\rfloor$$

By applying Theorem 2.2 to A→B = [[x,y]], then we get

$$e_{HK}(A) = \frac{1}{r}\binom{d+r-1}{r}$$

In particular, if d=2, r=e(A), then

$$e_{HK}(A) = \frac{e(A)+1}{2}$$

Let F be a field of characteristic p>2, and let N be the homogeneous coordinate Near-ring of the hyper quadric Q defined by q=q(X, Y, Z, W).

Put M= N+, unique homogeneous maximal ideal of N, and A = $R_{\mathfrak{M}} \otimes_k \bar{k}$.

By suitable coordinate transformation, we may assume that A is isomorphic to one of the following near-rings.

$$\begin{cases} k[[X,Y,Z,W]]/(X^2), & if\ rank(q) = 1, \\ k[[X,Y,Z,W]]/(X^2 - YZ), & if\ rank(q) = 2, \\ k[[X,Y,Z,W]]/(XY - ZW), & if\ rank(q) = 3. \end{cases}$$

**2.3 Theorem [8]:** Let $I \subseteq J$ be m-primary ideals of a local Near-ring (A,m,k) of characteristic p>0.

1. If $I^* = J^*$, then $e_{HK}(I) = e_{HK}(J)$.
2. Suppose that A is excellent, then by reducing toequi-dimensional, we getconverse of (1).

The following the theorem plays an important role in studying Hilbert-Kunz multiplicities for non-Cohen-Macaulay local near-rings.

**2.4 Theorem[8]:** Let (A, m,k) be an equi dimensional Local Near-ring which is a homomorphic image of a Cohen-Macaulay Local Near- ring of characteristic p>0 then

1) If J is a parameter ideal of A, then $e(J) \geq l_A(A/J^*)$.
2) Suppose that A is unmixed. If $e(J)=l_A(A/J^*)$.Then A is F-rational and is Cohen-Macaulay.

**2.5 Corollary:** Let (A, m, k) be an unmixed Local Near-ring of characteristic p>o. Suppose that e(A)=2. Then Ais F-rational if and only if $e_{HK}(A) < 2$ when this is the case, A is a hyper surface.

**Proof:** Since we know that every Cohen-Macaulay Local Near-ring of multiplicity 2 is a hyper surface, it is sufficient to prove the first statement.

Assume that A is complete and k is infinite. We can take a minimal reduction J of ***m***.

Suppose that $e_{HK}$ (A)<2. Then we show that A is Cohen-Macaulay and F-rational.

By Goto-Nakamura's theorem, we have $2= e(J) \geq i_A(A/J^*)$.

If equality does not hold $l_A(A/J^*)=1$ , then $e_{HK}(A)=e_{HK}(j^*)=e(j)=2$ by proposition 1.2 of [5]

This is a contradiction.

Hence $e(J)=l_A(A/J^*)$.

By Goto-Nakamura's theorem[10] again,we obtain that A is Cohen-Macaulay, and then F-rational.

Conversely suppose that A is complete F-rational.

Since A is Cohen-Macaulay and $J^*= J \neq m$ thenwe have

$e_{HK}(A) < e_{HK}(J)=e(J)=2$ by length criterion for tight closure.

**2.5 Notation:** For any positive real number s, we put

$$u_s := vol\left\{(x_1,.....x_d)\in[0,1]^d \middle| \sum_{i=1}^{n} x_i \le s\right\}, u_s^{'} := 1-u_s$$

Where vol(W) denotes the volume of $W \subseteq R^d$ .

**2.6 Remark:** When $1 \le s \le 2$,the right -hand side in equation $e_{HK}(A) \ge e(A)\left\{\frac{u_s - r.(s-1)^d}{d!}\right\}$ is equal to $e(A)(u_s - r.u_s - 1)$.

To prove the above theorem,we need the following lemma.

For any positive real number α, we define $I^{\alpha}=I^n$, where n is the minimum integer which does not exceed α.

To prove theorem 2.8, we need the following lemma.

**2.7 Lemma:** Let (A,m,k) be an unmixed Local Near- ring of characteristic p>0 with dim A= d ≥ 1.

Let J be a parameter ideal of A. Using the same notation as above, we have

$$\lim_{q\to\infty} \frac{l_A(A/J^{sq})}{q^d} = \frac{e(J)s^d}{d!}, \lim_{q\to\infty} l_A\left(\frac{j^{sq}+J^{(q)}}{J^{(q)}}\right) = e(J).u_s^1$$

**Proof:** Assume that A is complete.

Let $x_1,.........,x_d$ be asystem of parameters which generates Jand put

$N:=k[[x_1,.......,x_d]], n=(x_1,......x_d)N$.

Then N is a complete Regular Local Near- ring and A is a finitely generated N-module with A/m=N/n.

Since it is clear that in case of Regular Local Near- rings, it suffices to show the following claim.

Claim: *Let* $I=\{I_q\}_{q=p^e}$ be a set of ideals of A which satisfies the following conditions.

1) For each $q=p^e, I_q = J_q A$ holds for some ideal
2) There exists a positive integer t such that
   $n^{tq} \subseteq J_q \; for\, all\, q = p^e$
3) $\lim_{q\to\infty} l_R(R/J_q)/q^d$ exists.

Then

$$\lim_{x\to\infty}\frac{l_A(A/I_q)}{q^d}=e(J).\lim_{x\to\infty}\frac{l_R(R/J_q)}{q^d}.$$

Since A is unmixed, it is a torsion -free N-module of rank e = e(J).

Take a free N-module F of rank e such that $A_W \cong F_W, where W - A\{0\}$.

Since F and A are both torsion -free, there exist the following short exact sequences of finitely generated N-modules.

$$o\to F\to A\to C_1\to 0,\quad 0\to A\to F\to C_2\to 0,$$

Where $(C_1)_w=(c_2)_w=0.$

In particular , dim $C_1 < d \, and Him C_2 < d$ .

Applying the tensor product $-\otimes_R R/J_q$ to the above two exact sequences, respectively, we get

$$l_R(A/I_q)\le l_R(F/J_qF)+l_R(C_1/J_qC_1),$$
$$l_R(F/J_qF)\le l_R(A/I_q)+l_R(C_2/J_qC_2).$$

In general, if $\dim_R C<d$ then

$$\frac{l_R(C/J_qC)}{q^d}\le\frac{l_R(C/n^{tq}C)}{q^d}\to 0\ \ (q-\infty).$$

**2.8 Theorem:** Let (A.m,k) be an unmixed Local Near-ring of characteristic p>0.

Put d = dimA≥1. Let J be a minimal reduction of ***m*** and let r be an integer with $r\ge\mu_A(m/J^*),$

Where J* denotes the tight closure of J. Also let s ≥ 1 be a rational number then we have

$$e_{HK}(A)\ge e(A)\left\{u_s-r.\frac{(s-1)}{d!}\right\}.$$

**Proof:** For simplicity, we put L=J*. We will give an upper bound of $(l_A(m^{[q]}/J^{[q]})$. First we have the following inequality;

$$l_A(m^{[q]}/J^{[q]})\le l_A\left(\frac{m^{[q]}+m^{sq}}{J^{[q]}}\right)$$

$$l_A\left(\frac{m^{[q]}+m^{sq}}{L^{[q]}+m^{[q]}}\right)+l_A\left(L\frac{m^{[q]}+m^{sq}}{L^{[q]}+J^{sq}}\right)+l_A\left(\frac{L^{[q]}+J^{sq}}{J^{[q]}+J^{sq}}\right)+l_A\left(\frac{J^{[q]}+J^{sq}}{J^{[q]}}\right)=:l_1+l_2+l_3+l_4$$

Next, we see that

$$l_1\le r.l_A(A/J^{(a-1)q})+o(q^{d-1}).$$

By our assumption, we can write as

$$m = L + Aa_1 + ....... + Aa_r.$$

Since $m^{(s-1)q}a_i^q \subseteq m^{sq} + L^{[q]}$,

we have

$$l_1 = l_A\left(\frac{m^{[q]} + m^{sq}}{L^{[q]} + m^{sq}}\right) \le \sum_{i=1}^{n} l_A\left(\frac{Aa_i^q + l^{[q]} + M^{sq}}{L^{[q]} + m^{sq}}\right) = \sum_{i=1}^{n} l_A\left(A / L^{[q]} + m^{sq}) : a_i^q\right) \le r.l_A(A / m^{(s-1)q}).$$

Since J is a minimal reduction of ***m***, we have

Thus we have the required in equality. Similarly, we get

$$l_2 = l_A\left(\frac{L^{[q]} + m^{sq}}{L^{[q]} + J^{sq}}\right) \le l_A(m^{sq} / J^{sq}) = o(q^{d-1}).$$

Also we have $l_A(L^{[q]} / J^{[q]}) = o(q^{d-1})$ by length criterion for tight closure. Hence

$$l_A(m^{[q]} / J^{[q]}) \le l_A(A / J^{(s-1)q}) + l_A\left(\frac{J^{[q]} + J^{sq}}{J^{[q]}}\right) + o(q^{d-1})$$

It follows from the above argument that

$$e_{HK}(J) = e_{HK}(m) \le r.\lim_{x\to\infty} \frac{l_A(A / J^{(s-1)q})}{q^d} + \lim_{x\to\infty} \frac{1}{q^d} l_A\left(\frac{J^{[q]}J^{sq}}{J^{[q]}}\right)$$

$$= r.e\frac{(s-1)^d}{d!} + e.u_s^{'}$$

Since $e_{HK}(J) = e(J) = e, e_{HK}(m)\, and\, u_s^1 = 1 - u_s$

Then we get the required inequality.

**2.9 Theorem:** Let (A,m,k) be a three -dimensional unmixed local near- ring of characteristic p>0. Then

1) If A is not regular, then $e_{HK}(A) \ge \frac{4}{3}$
3) suppose that $k = \bar{k}$ and char k≠2then the following the conditions are equivalent:
   a) $e_{HK}(A) = \frac{4}{3}$
   b) $\hat{A} \cong k[[X,Y,ZW]] / (X^2 + Y^2 + Z^2 + W^2).$
   c) $gr_m(A) \cong k[X,Y,ZW] / (X^2 + Y^2 + Z^2 + W^2).$

That is $gr_m(A) \cong k[X,Y,Z,W] / (XY - ZW).$

**2.10 Proposition:** Let (A, m, k) be a three dimensional unmixed Local Near- ring of characteristic p>0. If $e_{HK}(A) < 2,$ then A is F-rational.

**2.11 Lemma:** Let (A, m, k) be an unmixed Local Near-ring of positive characteristic , and let J be a minimal reduction of m. Then

1) $\mu_A(m/J^*) \leq e(A) - 1$

2) If A is not F-rational,then $\mu_A(m/J^*) \leq e(A) - 2$.

**2.12 Theorem:** Let (A, m, k) be a hyper surface Local Near-ringof characteristic p>0 with d= dimA ≥1 then $e_{HK}(A) \geq \beta_{d+1}.e(A)$

where $\beta_{d+1} = vol\{x \in [0,1]^d / \frac{d-1}{2} \leq \sum x_i \leq \frac{d+1}{2}\} = 1 - u_{\frac{d-1}{2}} - v^1_{\frac{d+1}{2}}$

## 3. GENERALIZED RESULTS

In this section we derived the generalization for theorem2.1and 4.1,4.3 of (5) in case of dim A ≥4 and let d ≥ 1 be an integer and p>2, a prime number.If we put

$$A_{p,d} := \overline{\mathbb{F}_p}[[X_0, X_1, ...., X_d]]/(X_0^2 + ... + X_d^2),$$

Then we take that $e_{HK}$(A, p,d)=$s_{HK}$(p,d) holds according to the observations till now . In the following, let us formulate this as a conjecture and prove that it is also true in case of dim A=4

**3.1 Theorem** :Under the above notation, we have

(3.1) $\lim_{x\to\infty} e_{HK}(A_{p,d}) = 1 + \frac{c_d}{d!}$ , where

(3.2) $\sec x + \tan x = \sum_{d=0}^{\infty} \frac{c_d}{d} \left( /x/ < \frac{\Pi}{2} \right)$

It is known that by the Taylor expansion of sec x (resp. tan x) at origin can be written as follows:

$$\text{secx} = \sum_{i=0}^{\infty} \frac{E_{2i}}{(2i)!} x^{2i}$$

$$\tan x = \sum_{i=0}^{n} (-1)^{i-1} \frac{2^{2i}(2^{2i}-1)B_{2i}}{(2i)!} x^{2i-1},$$

where $E_{2i}$(resp.$B_{2i}$ ) is said to be Euler number (resp. Bernoulli number).

Also, $c_d$ appeared in Eq. (3.1) is a positive integer since cost is an unitelement in a ring H $=\left\{\sum_{i=1} {}_{n=0}^{\infty} a_n \frac{t^n}{n!} \middle| a_n \in \mathbb{Z}\, for\, all\, n \geq 0\right\}$.

Based on the above observation, we establish the following conjecturefor dim 4.

**3.2. Note:** Let d≥1 be an integer and p>2 ,a prime number. Put $A_{p,d} := \overline{\mathbb{F}_p}[[X_O, X_1, ......X_d]]/(X_0^2 + ... + X_d^2)$.

Let (A,m,k) be a d- dimensional unmixed Local Near-ring with k= $\overline{\mathbb{F}_p}$

Then the following statements hold.

1) If A is not regular , then $e_{HK}(A) \geq e_{HK}(A, p, d) \geq 1 + \frac{c_d}{d!}$. inparticular $s_{HK}(p,d) = e_{HK}(A_{p,d})$

2) If $e_{HK}(A) = e_{HK}(A_{p,d}), then\, \hat{A} \cong A_{p,d}$ as Local Near-rings.

In the following, we prove that this is in case of dim A=4. Note that

$$\lim_{p\to\infty} e_{HK}(A_{p,4}) = \lim_{p\to\infty} \frac{29p^2+15}{24p^2+12} = \frac{29}{24} = 1 + \frac{c_4}{4!}$$

**3.3 Theorem: L**et (A,m,k) be an unmixed Local Near-ring of characteristic p>0 with dimA=4.

If $e(A) \geq 3, rthen\, e_{HK}(A) \geq \frac{5}{4} = \frac{30}{24}$.

Suppose that k=$\overline{k}$ and char k≠2put $A_{p,4} = \overline{\mathbb{F}_p}[[X_0, X_1, ...X_4]]/(X_0^2 + .......X_4^2)$

Then the followingstatement holds.

1) If A is not regular, then $e_{HK}(A) \geq e_{HK}(A_{p,4}) = \frac{29p^2+15}{24p^2+12}$

2) The following conditions are equivalent

a) Equality holds in (1).

b) $e_{HK}(A) < \frac{5}{4}$

c) The completion of A is isomorphic to A $_{p,4.}$

**Proof:** Put e=e(A), the multiplicity of A. We mayassume that A is complete with e≥2 and k is infinite. In particular, A is a homomorphic image of aCohen - Macaulay local near-ring, and there exists aminimal reduction J of m. Then $\mu_A(m/J^*) \leq e-1$

by lemma 2.11 we first show that $e_{HK}(A) \geq \frac{5}{4} if\, e \geq 3.$

**Claim 1:** If $3 \leq e \leq 10$, then $e_{HK}(A) \geq 5/4$.
Put r=e -1 and s=2 in theorem 2.8,

We have $v_2 = \frac{1}{2}$

And $e_{HK}(A) \geq \left\{ v_{3/2} - \frac{(e-1)1^4}{4!} \right\} = \frac{(13-e)e}{24} \geq \frac{30}{24}$

**Claim 2:** If
$11 \leq e \leq 29, then\, e_{HK}(A) \geq \frac{737}{384} (> \frac{5}{4})$

By 2.12 we have $v_{3/2} = \frac{1-\beta_{4+1}}{2} = \frac{77}{384}. Putting\, r = e-1 and\, s = \frac{3}{2}$ in Theorem 2.8, we have

$e_{HK}(A) \geq e\left\{ v_{3/2} - \frac{e-1}{24} . \left(\frac{1}{2}\right)^4 \right\} = \frac{(78-e)e}{384} \geq \frac{11(78-11)}{384} = \frac{737}{384},$

**Claim 3:** If e $\geq$ 30 then $e_{HK}(A) \geq \frac{5}{4}$

By proposition 1.17**,** we have $e_{HK}(A) \geq \frac{e}{4!} \geq \frac{30}{24}$

We assume that $k = \overline{k}, char\, k \neq 2\, and\, e \geq 2.$

To see (1),(2) we may assume that e=2 by the above argument. Then since $e_{HK}(A) = 2$ ,if A is not f-rational ,we may also assume that A is F-rational and thus a hyper surface.

A can be written as $A = k[[X_0, X_1, ....X_4]]/(X_0^2 - \varphi(X_1, X_2, X_3, X_4))$

A is isomorphic to$A_{p,4}$it is known that $e_{HK}(A) = \frac{29p^2+15}{24p^2+12}$

Suppose that A is not isomorphic to $A_{p,4}$ . Then one can take a minimal numbers of generators x,y,z,w,u of *m* and one can define a function ord: $A \to \mathbb{Q}$ such that

Ord(x)=ord (y)=ord (z)= $\frac{1}{2}$ ,ord (u)=$\frac{1}{3}$

If we put J = (y,z,w,u) A and, $f_n = \{\alpha \in A / ord(\alpha) \geq n\}$ then by the similar argument as in the proof of proposition 2.10, we have
$l_A(m^{[q]}/J^{[q]}) \leq 2.l_A(A/J^{[q]} + F_{2q/3}).$

Dividing the both sides by $q^d$ and taking a limit $q \to \infty$ , we get $e(A) - e_{HK}(A) \leq 2.e(A).vol\left\{(y,z,w,u) \in [0,1]^4\right\}\left|\frac{y}{2} + \frac{z}{2} + \frac{w}{2} + \frac{u}{2} \leq \frac{2}{3}\right|.$

To calculate the volume in the right - hand side, we put

$$F_u = \begin{cases} \left\{ \frac{1}{6}\left(\frac{4}{3}-\frac{2}{3}u\right)^3 - \frac{1}{6}\left(\frac{1}{3}-\frac{2}{3}u\right)^3 \right\} & \left(0 \le u \le \frac{1}{2}\right) \\ \frac{1}{6}\left(\frac{4}{3}-\frac{2}{3}u\right)^3 & \left(\frac{1}{2} \le u \le 1\right) \end{cases}$$

Then one can easily calculatethe above volume $= \int_0^1 f_u du = \frac{237}{2^4 3^4}$

It follows that $e_{HK}(A) \ge 2 - 4X\frac{237}{2^4 3^4} = \frac{411}{324} > \frac{5}{4}$

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