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Fixed Point Theorems for (ε, λ)-Uniformly Locally Generalized Contractions

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Abstract

In this paper, we define a class called (ε,λ)-uniformly locally generalized contractions and establish a fixed point theorem for such contractions.

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Introduction:

1. Definition: A selfmap f of a D^*-metric space (X, D^*) is called a (ε,λ) ***-uniformly locally generalized contraction***, if there is a number q with $0 \le q < 1$ and a positive constant ε, such that

(1.1)
$$D^*(fx, fy, fz) \le q.D^*(x,y,y) + r.D^*(x,fx,fx) + s.D^*(y,fy,fy)$$
$$t.\{D^*(x,fy,fy) + D^*(y,fx,fx)\}$$

for all $x, y \in X$ with $D^*(x,y,y) < \varepsilon$, where $\underset{x,y\in X}{Sup}\{q+r+s+2t\} = \lambda < 1$.

We now prove

Main Theorem:

2. Theorem: Suppose f is a (ε,λ)-uniformly locally generalized contraction of a D^*-metric space (X, D^*) and X is f-orbitally complete. Then for every $x \in X$,

either

(2.1) $D^*\left(f^s x, f^{s+1}x, f^{s+1}x\right) \geq \varepsilon$ for all integers $s \geq 0$

or

(2.2) the sequence $\left\{f^n x\right\}$ converges to u, which is a fixed point of f. Also there is no other fixed point $v \in X$ with $D^*(u,v,v) < \varepsilon$.

Proof: For any $x \in X$, consider $\left\{D^*\left(f^s x, f^{s+1}x, f^{s+1}x\right)\right\}_{s=0}^{\infty}$. Then we have either each of the term in this sequence is greater than or equal to ε or for some term in it is less than ε.

In the first case, the alternative of (2.1) of the hypothesis holds.

Let for some integer $s = s_0$, $D^*\left(f^{s_0} x, f^{s_0+1}x, f^{s_0+1}x\right) < \varepsilon$. Since f is a (ε,λ)-uniformly locally generalized contraction and $D^*\left(f^{s_0} x, f^{s_0+1}x, f^{s_0+1}x\right) < \varepsilon$, we get numbers *q*, *r*, *s*, and *t* (all depending on *x* and *y*) such that

$$D^*\left(f^{s_0+1}x, f^{s_0+2}x, f^{s_0+2}x\right) = D^*\left(ff^{s_0}x, ff^{s_0+1}x, ff^{s_0+1}x\right)$$

$$\leq q.D^*\left(f^{s_0}x, f^{s_0+1}x, f^{s_0+1}x\right) + r.D^*\left(f^{s_0}x, f^{s_0+1}x, f^{s_0+1}x\right)$$

$$+ s.D^*\left(f^{s_0+1}x, f^{s_0+2}x, f^{s_0+2}x\right)$$

$$+ t\left\{D^*\left(f^{s_0}x, f^{s_0+2}x, f^{s_0+2}x\right) + D^*\left(f^{s_0+1}x, f^{s_0+1}x, f^{s_0+1}x\right)\right\}$$

$$\leq q.D^*\left(f^{s_0}x, f^{s_0+1}x, f^{s_0+1}x\right) + r.D^*\left(f^{s_0}x, f^{s_0+1}x, f^{s_0+1}x\right)$$

$$+ s.D^*\left(f^{s_0+1}x, f^{s_0+2}x, f^{s_0+2}x\right)$$

$$+ t\left\{D^*\left(f^{s_0}x, f^{s_0+1}x, f^{s_0+1}x\right) + D^*\left(f^{s_0+1}x, f^{s_0+2}x, f^{s_0+2}x\right)\right\}$$

$$\leq \left(q + r + t\right).D^*\left(f^{s_0}x, f^{s_0+1}x, f^{s_0+1}x\right)$$

$$+ \left(s + t\right).D^*\left(f^{s_0+1}, f^{s_0+2}, f^{s_0+2}\right)$$

Therefore

$$(1 - s - t).D^*\left(f^{s_0+1}, f^{s_0+2}, f^{s_0+2}\right) \leq (q + r + t).D^*\left(f^{s_0}x, f^{s_0+1}x, f^{s_0+1}x\right)$$

This implies that

$$D*\left(f^{s_0+1}, f^{s_0+2}, f^{s_0+2}\right) \le \frac{(q+r+t)}{(1-s-t)}.D*\left(f^{s_0}x, f^{s_0+1}x, f^{s_0+1}x\right)$$

$$\le \lambda.D*\left(f^{s_0}x, f^{s_0+1}x, f^{s_0+1}x\right)$$

Also we get by repeated use of the above inequality that

$$D*\left(f^{s_0+p}, f^{s_0+p+1}, f^{s_0+p+1}\right) \le \lambda.D*\left(f^{s_0+p-1}x, f^{s_0+p}x, f^{s_0+p}x\right)$$

$$\le \lambda^2.D*\left(f^{s_0+p-2}x, f^{s_0+p-1}x, f^{s_0+p-1}x\right)$$

. . . .

. . . .

$$\le \lambda^p.D*\left(f^{s_0}x, f^{s_0+1}x, f^{s_0+1}x\right)$$

That is, $D*\left(f^{s_0+p}x, f^{s_0+p+1}x, f^{s_0+p+1}x\right) < \varepsilon$ for every integer $p = 0, 1, 2, 3, \ldots$ and hence for $n \ge s_0$, we have

$$D*\left(f^n x, f^{n+p}x, f^{n+p}x\right) \le D*\left(f^n x, f^{n+1}x, f^{n+1}x\right) + D*\left(f^{n+1}x, f^{n+2}x, f^{n+2}x\right)$$

$$+ \ . \ . \ . \ + D*\left(f^{n+p-1}x, f^{n+p}x, f^{n+p}x\right)$$

$$\le \left(\lambda^{n-s_0} + \lambda^{n-s_0+1} + \ldots + \lambda^{n-s_0+p-1}\right) D*\left(f^{s_0}x, f^{s_0+1}x, f^{s_{0+1}}x\right)$$

$$\le \left(\lambda^{n-s_0} + \lambda^{n-s_0+1} + \ldots + \lambda^{n-s_0+p-1} + \ldots\right) D*\left(f^{s_0}x, f^{s_0+1}x, f^{s_0+1}x\right)$$

$$D*\left(f^n x, f^{n+p}x, f^{n+p}x\right) \le \frac{\lambda^{n-s_0}}{1-\lambda} D*\left(f^{s_0}x, f^{s_0+1}x, f^{s_0+1}x\right)$$

$$\to 0 \text{ as } n \to \infty$$

Thus the sequence $\{f^n x\}$ is a Cauchy sequence in a *f*-orbitally complete *D**-metric space (*X*, *D**) and hence there exists $u \in X$ such that

$$u = \lim_{n \to \infty} f^n x = \lim_{n \to \infty} f^{s_0+p}$$

Therefore there is an integer $n_0 > s_0$ such that

$$D*\left(f^n x,u,u\right)< \varepsilon \text{ for all } n \geq n_0$$

Now

$$D*\left(fu, ff^n x, ff^n x\right) \leq qD*\left(u, f^n x, f^n x\right) + rD*\left(u, fu, fu\right) + sD*\left(f^n x, f^{n+1}x, f^{n+1}x\right)$$

$$+ t\left\{D*\left(u, f^{n+1}x, f^{n+1}x\right) + D*\left(f^n x, fu, fu\right)\right\}$$

$$D*\left(fu, f^{n+1}x, f^{n+1}x\right) \leq qD*\left(u, f^n x, f^n x\right) + rD*\left(u, f^{n+1}x, f^{n+1}x\right)$$

$$+ rD*\left(f^{n+1}x, fu, fu\right) + sD*\left(f^n x, f^{n+1}x, f^{n+1}x\right)$$

$$+ tD*\left(u, f^{n+1}x, f^{n+1}x\right) + tD*\left(f^n x, f^{n+1}x, f^{n+1}x\right)$$

$$+ tD*\left(f^{n+1}x, fu, fu\right)$$

$$\leq qD*\left(u, f^n x, f^n x\right) + (r+t)D*\left(u, f^{n+1}x, f^{n+1}x\right)$$

$$+ (s+t)D*\left(f^n x, f^{n+1}x, f^{n+1}x\right)$$

$$+ (r+t)D*\left(fu, f^{n+1}x, f^{n+1}x\right)$$

$$\leq \lambda D*\left(u, f^n x, f^n x\right) + \lambda D*\left(u, f^{n+1}x, f^{n+1}x\right)$$

$$+ \lambda D*\left(f^n x, f^{n+1}x, f^{n+1}x\right) + \lambda D*\left(fu, f^{n+1}x, f^{n+1}x\right)$$

which gives

$$(1-\lambda)D*\left(fu, f^{n+1}x, f^{n+1}x\right) \leq \lambda\left\{ D*\left(u, f^n x, f^n x\right) + D*\left(u, f^{n+1}x, f^{n+1}x\right)\right.$$

$$\left. + D*\left(f^n x, f^{n+1}x, f^{n+1}x\right) \right\}$$

Therefore

$$D^*\left(fu, f^{n+1}x, f^{n+1}x\right) \le \frac{\lambda}{(1-\lambda)}\left\{D^*\left(u, f^n x, f^n x\right) + D^*\left(u, f^{n+1}x, f^{n+1}x\right)\right.$$

$$\left. + D^*\left(f^n x, f^{n+1}x, f^{n+1}x\right)\right\}$$

Now letting $n \to \infty$, it follows that $D^*(fu, u, u) = 0$ which implies that $fu = u$, showing that the sequence $\{f^n x\}$ converges to some point of *X*.

To prove the uniqueness of fixed point of f, suppose that $fv = v$ for some $v \in X$ and $D^*(u, v, v) < \varepsilon$. Then

$$D^*(u,v,v) = D^*(fu, fv, fv)$$

$$\le qD^*(u,v,v) + rD^*(u, fu, fu) + sD^*(v, fv, fv)$$

$$+ t\{D^*(u, fv, fv) + D^*(v, fu, fu)\}$$

$$= qD^*(u,v,v) + rD^*(u,u,u) + sD^*(v,v,v)$$

$$+ t\{D^*(u,v,v) + D^*(v,u,u)\}$$

$$= (q+2t)D^*(u,v,v) = \lambda . D^*(u,v,v)$$

which implies that $D^*(u,v,v) = 0$, since $\lambda < 1$ and hence $u = v$, proving the second part of (2.2).

2.2 **Corollary:** Suppose f is a (ε, λ)-uniformly locally generalized contraction of a *D**-metric space (*X*, *D**) and *X* is *f*-orbitally complete. If for every $x \in X$, there is an integer $n(x)$ such that

(2.2.1) $D^*\left(f^{n(x)}x, f^{n(x)+1}x, f^{n(x)+1}x\right) < \varepsilon$

Then f has a unique fixed point, provided any two fixed points u, v of f are such that $D^*(u,v,v) < \varepsilon$.

Also the sequence $\{f^n x\}$ for any $x \in X$ converges to the unique fixed point of f.

Proof: Follows from Theorem 1.1.

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