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Lower Bounds for the Blow-Up Time in a Nonlocal Reaction-Diffusion Equation

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Abstract

We use a differential inequality technique to deal with the lower bounds for the blow-up time in this equation $u_t - \Delta u^m = \int_\Omega u^p\,dx - ku^p$ with two different boundary conditions if blow-up occurs.

Key words Differential inequality; Lower bounds; Blow-up time

1. INTRODUCTION

In this paper, we study the lower bounds for blow-up time of the following equations:

$$u_t - \Delta u^m = \int_\Omega u^p\,dx - ku^p, \qquad (x,t) \in \Omega \times (0,T) \qquad (1.1)$$

$$u(x,0) = u_0(x) \geq 0, \qquad x \in \overline{\Omega} \qquad (1.2)$$

$$u(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0,T) \qquad (1.3)$$

$$\text{or } \frac{\partial u}{\partial n} = 0, \qquad (x,t) \in \partial\Omega \times (0,T) \qquad (1.4),$$

where $\Omega \in R^3$ is a bounded smooth convex domain, $p > m > 1$, $0 < k < |\Omega|$, $\frac{\partial}{\partial n}$ is the outward normal derivative at the boundary $\partial\Omega$ which is sufficiently smooth, T is possible blow-up time.

There are some articles investigated the lower bounds for the blow-up time (see [3]-[4]). In [6] Song considered the lower bounds for the blow-up time of following problem:

$$\begin{aligned} &\frac{\partial u}{\partial t} = \Delta u + \int_\Omega u^p\, dx - ku^q, && \text{in } \Omega \times (0, t^*), \\ &u(x,0) = f(x) \geq 0, && \text{in } \Omega, \\ &u(x,t) = 0 \quad \text{or} \quad \frac{\partial u}{\partial n} = 0 && \text{on } \partial\Omega \times (0, t^*), \end{aligned} \tag{1.5}$$

where $p > q > 1$. [2] investigated the lower bounds for the blow-up time of a nonlinear nonlocal porous medium equation

$$\begin{aligned} &u_t = \Delta u^m + u^p \int_\Omega u^q\, dx, && (x,t) \in \Omega \times (0,T), \\ &u(x,0) = u_0(x) \geq 0, && x \in \overline{\Omega}, \\ &u(x,t) = 0 \quad \text{or} \quad \frac{\partial u}{\partial v} = 0, && (x,t) \in \partial\Omega \times (0,T), \end{aligned} \tag{1.6}$$

where $p,\ q \geq 0,\ p + q > m > 1$. Kinds of methods have been used to estimate the lower bounds for the blow-up time. And a differential inequality technique is used frequently if blow-up occurs.

Motivated by those works, we study the lower bounds for the blow-up time of equation (1.1)—(1.2) under the boundary conditions (1.3) or (1.4). In section 2, we obtain the lower bounds for the blow-up time under null the boundary condition (1.3). In section 3, we investigate it under the boundary condition (1.4)

Through this paper, we set $p = s + 1$

with $s > 0$. (1.7)

Rewrite (1.1) as $u_t - \Delta u^m = \int_\Omega u^{s+1}\, dx - ku^{s+1}$ (1.8)

2. THE LOWER BOUND FOR BLOW-UP TIME UNDER THE DIRICHLET BOUNDARY CONDITION

Define an auxiliary function

$$\phi(t) = \int_\Omega u^{ns}\, dx \tag{2.1}$$

with $n > \max\left\{\frac{1}{s}, \frac{4s-2(m-1)}{s}\right\}$. (2.2)

Taking the derivative of $\phi(t)$ with respect to t, then

$$\phi'(t) = ns \int_\Omega u^{ns-1} \cdot u_t \, dx$$

$$= ns \int_\Omega u^{ns-1} \left(\Delta u^m + \int_\Omega u^{s+1} \, dx - k u^{s+1}\right) dx$$

$$= -\frac{4nsm(ns-1)}{(ns+m-1)^2} \int_\Omega \left|\nabla u^{\frac{ns+m-1}{2}}\right|^2 dx + ns \int_\Omega u^{ns-1} \, dx \int_\Omega u^{s+1} \, dx - kns \int_\Omega u^{s(n+1)} \, dx$$

$$\leq -\frac{4nsm(ns-1)}{(ns+m-1)^2} \int_\Omega \left|\nabla u^{\frac{ns+m-1}{2}}\right|^2 dx + (ns|\Omega| - kns) \int_\Omega u^{s(n+1)} \, dx$$ (Using Hölder inequality).

We get

$$\phi'(t) \leq -\frac{4nsm(ns-1)}{(ns+m-1)^2} \int_\Omega \left|\nabla u^{\frac{ns+m-1}{2}}\right|^2 dx + (ns|\Omega| - kns) \int_\Omega u^{s(n+1)} \, dx. \quad (2.3)$$

For simplicity, by denoting

$$v = u^s \text{ and } \delta = \frac{ns+m-1}{2s},$$

Inequality (2.3) can be written as

$$\phi'(t) \leq -C_1 \int_\Omega \left|\nabla v^\delta\right|^2 dx + C_2 \int_\Omega v^{n+1} \, dx, \quad (2.4)$$

where $C_1 = \frac{4nsm(ns-1)}{(ns+m-1)^2}$ and $C_2 = ns|\Omega| - kns$.

Next, our aim is to estimate $\int_\Omega v^{n+1}$ dx. Using Hölder inequality, we have

$$\int_\Omega v^{n+1} \, \mathrm{dx} \leq \left(\int_\Omega v^{4\delta} \, dx\right)^{\frac{1}{4}} \left(\int_\Omega v^{\frac{2(ns+2s-m+1)}{3s}} \, dx\right)^{\frac{3}{4}}$$

$$\leq |\Omega|^{\frac{(n-4)s+2(m-1)}{4ns}} \left(\int_\Omega v^{2\delta} \, dx\right)^{\frac{1}{8}} \left(\int_\Omega v^{6\delta} \, dx\right)^{\frac{1}{8}} \left(\int_\Omega v^n \, dx\right)^{\frac{(n+2)s-m+1}{2ns}} \quad (2.5)$$

By the flowing Sobolev inequality [7]

$$\left(\int_\Omega |\phi|^\beta \, dx\right)^{\frac{1}{\beta}} \le 4^{\frac{1}{3}} 3^{-\frac{1}{2}} \pi^{-\frac{2}{3}} \left(\int_\Omega |\nabla\phi|^\gamma \, dx\right)^{\frac{1}{\gamma}}, \tag{2.6}$$

with $\beta = 6$ and $\gamma = 2$, from (2.5) we know that

$$\int_\Omega v^{n+1} \, \mathrm{dx} \le C_3 \left(\int_\Omega v^{2\delta} \, dx\right)^{\frac{1}{8}} \left(\int_\Omega \left|\nabla v^\delta\right|^2 dx\right)^{\frac{3}{8}} \left(\int_\Omega v^n \, dx\right)^{\frac{(n+2)s-m+1}{2ns}}, \tag{2.7}$$

where $C_3 = 4^{\frac{1}{4}} 3^{-\frac{3}{8}} \pi^{-\frac{1}{2}} |\Omega|^{\frac{(n-4)s+2(m-1)}{4ns}}$.

From the Rayleigh principle, we know that

$$\lambda_1 \int_\Omega v^{2\delta} \, dx \le \int_\Omega \left|\nabla v^\delta\right|^2 dx \tag{2.8}$$

where λ_1 is the first eigenvalue of the eigenvalue problem

$$\begin{cases} -\Delta\psi(x) = \lambda\psi(x), & x \in \Omega \\ \psi(x) = 0, & x \in \partial\Omega \end{cases}.$$

Substituting inequality (2.8) into (2.7), we have

$$\int_\Omega v^{n+1} \, \mathrm{dx} \le C_4 \left(\int_\Omega \left|\nabla v^\delta\right|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega v^n \, dx\right)^{\frac{(n+2)s-m+1}{2ns}}, \tag{2.9}$$

where $C_4 = {\lambda_1}^{-\frac{1}{8}} C_3$.

Now, using the fundamental inequality

$${a_1}^{r_1} {a_2}^{r_2} \le r_1 a_1 + r_2 a_2, \; a_1, \; a_2 > 0, \; r_1, \; r_2 \ge 0 \text{ and } r_1 + r_2 = 1, \tag{2.10}$$

with an undetermined positive weight factor θ to have

$$\int_\Omega v^{n+1} \, \mathrm{dx} \le \frac{\theta C_4}{2} \int_\Omega \left|\nabla v^\delta\right|^2 dx + \frac{C_4}{2\theta} \left(\int_\Omega v^n \, dx\right)^{\frac{(n+2)s-m+1}{ns}}, \tag{2.11}$$

taking $\theta = \frac{2C_1}{C_2 C_4}$.

Then, from (2.4) and (2.11) it follows that

$$\phi'(t) \le C_5 [\phi(t)]^{\frac{(n+2)s-m+1}{ns}}, \tag{2.12}$$

where $C_5 = \frac{C_2 C_4}{2\theta}$. Integrating (2.12) from 0 to t, we have

$$t \geq C_7 \left(\frac{1}{[\phi(0)]^{C_6}} - \frac{1}{[\phi(t)]^{C_6}} \right), \tag{2.13}$$

where $C_6 = \frac{2s-m+1}{ns}$ and $C_7 = \frac{1}{C_5 C_6}$.

We know that if u blows up in the measure ϕ from (2.13), then the lower bound for T is

$$T \geq \frac{C_7}{[\phi(0)]^{C_6}} = C_7 \left(\int_\Omega {u_0}^{ns}\, dx \right)^{-C_6}. \tag{2.14}$$

Now, we can establish the following theorem.

Theorem 2.1 Let u be the classical positive solution of (1.1)—(1.3) with $p > m > 1,\ 0 < k < |\Omega|$, then a lower bound of the blow-up time for the solution which blows up in L^{ns} norm is given by (2.14), where s and n respectively satisfy (1.7) and (2.2).

3 The lower bound for Blow-up time under the Neumann boundary condition

Similar to considering the auxiliary function (2.1) with (1.7) and

$$n > \max\left\{ \frac{1}{s}, \frac{4s-3(m-1)}{2s} \right\}. \tag{3.1}$$

Set $$\tau = \frac{n+2\delta}{2}.$$

Now, first we use the Hölder inequality to estimate $\int_\Omega v^{n+1}\,\mathrm{dx}$ in inequality (2.4), we get

$$\int_\Omega v^{n+1}\,\mathrm{dx} \leq |\Omega|^{\frac{3\tau-2(n+1)}{3\tau}} \left(\int_\Omega v^{\frac{3\tau}{2}}\, dx \right)^{\frac{2(n+1)}{3\tau}}, \tag{3.2}$$

Next, we have (see [1] and [2])

$$\int_\Omega v^{\frac{3\tau}{2}}\,\mathrm{dx} \leq \frac{1}{3^{\frac{3}{4}}} \left\{ \frac{3}{2\rho_0} \int_\Omega v^\tau\, dx + \frac{(d+\rho_0)\tau}{2\rho_0\delta} \left(\int_\Omega v^n\, dx \right)^{\frac{1}{2}} \left(\int_\Omega \left|\nabla v^\delta\right|^2 dx \right)^{\frac{1}{2}} \right\}^{\frac{3}{2}}, \tag{3.3}$$

So

$$\left(\int_\Omega v^{\frac{3\tau}{2}}\, dx \right)^{\frac{2(n+1)}{3\tau}} \leq \frac{1}{3^{\frac{n+1}{2\tau}}} \left\{ \frac{3}{2\rho_0} \int_\Omega v^\tau\, dx + \frac{(d+\rho_0)\tau}{2\rho_0\delta} \left(\int_\Omega v^n\, dx \right)^{\frac{1}{2}} \left(\int_\Omega \left|\nabla v^\delta\right|^2 dx \right)^{\frac{1}{2}} \right\}^{\frac{n+1}{\tau}} \tag{3.4}$$

where $\rho_0 = \min\limits_{\partial\Omega} x_i v_i$, $d^2 = \max\limits_{\bar{\Omega}} x_i x_i$, $i = 1$, 2 , 3 , and v_i denotes the i th component of the unit outer normal vector. By the Hölder inequality, we obtain

$$\int_\Omega v^\tau \, \mathrm{dx} \le \left(\int_\Omega v^n \, dx\right)^{\frac{1}{2}} \left(\int_\Omega v^{2\delta} \, dx\right)^{\frac{1}{2}}$$

$$\le |\Omega|^{\frac{s-m+1}{2(n+1)s}} \left(\int_\Omega v^{n+1} \, dx\right)^{\frac{\delta}{n+1}} \left(\int_\Omega v^n \, dx\right)^{\frac{1}{2}}, \tag{3.5}$$

form (3.4) and (3.5), we have

$$\left(\int_\Omega v^{\frac{3\tau}{2}} \, dx\right)^{\frac{2(n+1)}{3\tau}} \le \frac{1}{3^{\frac{n+1}{2\tau}}} \left\{ \frac{3}{2\rho_0} |\Omega|^{\frac{s-m+1}{2(n+1)s}} \left(\int_\Omega v^{n+1} \, dx\right)^{\frac{\delta}{n+1}} \left(\int_\Omega v^n \, dx\right)^{\frac{1}{2}} \right.$$

$$\left. + \frac{(d+\rho_0)\tau}{2\rho_0\delta} \left(\int_\Omega v^n \, dx\right)^{\frac{1}{2}} \left(\int_\Omega \left|\nabla v^\delta\right|^2 dx\right)^{\frac{1}{2}} \right\}^{\frac{n+1}{\tau}}. \tag{3.6}$$

Using the following inequality

$$(a+b)^\beta \le 2^\beta \left(a^\beta + b^\beta\right), \ a, \ b > 0 \ \text{ and } \ \beta \ge 1,$$

we compute

$$\left(\int_\Omega v^{\frac{3\tau}{2}} \, dx\right)^{\frac{2(n+1)}{3\tau}} \le \left(\frac{2}{\sqrt{3}}\right)^{\frac{n+1}{\tau}} \left\{ |\Omega|^{\frac{s-m+1}{2\tau s}} \left(\frac{3}{2\rho_0}\right)^{\frac{n+1}{\tau}} \left(\int_\Omega v^{n+1} \, dx\right)^{\frac{\delta}{\tau}} \left(\int_\Omega v^n \, dx\right)^{\frac{n+1}{2\tau}} + \right.$$

$$\left. \left(\frac{(d+\rho_0)\tau}{2\rho_0\delta}\right)^{\frac{n+1}{\tau}} \left(\int_\Omega v^n \, dx\right)^{\frac{n+1}{2\tau}} \left(\int_\Omega \left|\nabla v^\delta\right|^2 dx\right)^{\frac{n+1}{2\tau}} \right\}$$

$$\le \left(\frac{2}{\sqrt{3}}\right)^{\frac{n+1}{\tau}} \left\{ |\Omega|^{\frac{s-m+1}{2\tau s}} \left(\frac{3}{2\rho_0}\right)^{\frac{n+1}{\tau}} \left[\frac{\delta k_1}{\tau} \int_\Omega v^{n+1} \, dx + \frac{n}{2\tau} k_1^{-\frac{2\tau}{n}} \left(\int_\Omega v^n \, dx\right)^{\frac{n+1}{n}} \right] \right.$$

$$+ \left(\frac{(d+\rho_0)\tau}{2\rho_0\delta}\right)^{\frac{n+1}{\tau}} \left[\frac{ns-(s-m+1)}{2\tau s} k_2^{-\frac{2\tau s}{ns-(s-m+1)}} \left(\int_\Omega v^n \, dx\right)^{\frac{(n+1)s}{ns-(s-m+1)}} \right.$$

$$\left. \left. + \frac{(n+1)k_2}{2\tau} \int_\Omega \left|\nabla v^\delta\right|^2 dx \right] \right\}, \tag{3.7}$$

where $k_1, \ k_2 > 0$.

Then

$$\int_\Omega v^{n+1}\,\mathrm{dx} \le |\Omega|^{\frac{3\tau-2(n+1)}{3\tau}\left(\frac{2}{\sqrt{3}}\right)^{\frac{n+1}{\tau}}} \left\{ |\Omega|^{\frac{s-m+1}{2\tau s}} \left(\frac{3}{2\rho_0}\right)^{\frac{n+1}{\tau}} \left[\frac{\delta k_1}{\tau}\int_\Omega v^{n+1}\,dx \right.\right.$$

$$\left. + \frac{n}{2\tau} k_1^{-\frac{2\tau}{n}} \left(\int_\Omega v^n\,dx\right)^{\frac{n+1}{n}}\right] + \left(\frac{(d+\rho_0)\tau}{2\rho_0\delta}\right)^{\frac{n+1}{\tau}} \left[\frac{ns-(s-m+1)}{2\tau s}\right.$$

$$\left.\left. k_2^{-\frac{2\tau s}{ns-(s-m+1)}} \left(\int_\Omega v^n\,dx\right)^{\frac{(n+1)s}{ns-(s-m+1)}} + \frac{(n+1)k_2}{2\tau}\int_\Omega \left|\nabla v^\delta\right|^2 dx\right]\right\} \tag{3.8}$$

taking $k_1 < \frac{\tau}{\delta}\left(\frac{\rho_0}{\sqrt{3}}\right)^{\frac{n+1}{\tau}} |\Omega|^{\frac{2(n+1)-3\tau}{3\tau}-\frac{s-m+1}{2\tau s}}$.

For convenience

$$A_1\int_\Omega v^{n+1}\,dx \le \quad A_2\left(\int_\Omega v^n\,dx\right)^{\frac{n+1}{n}} + A_3\left(\int_\Omega v^n\,dx\right)^{\frac{(n+1)s}{ns-(s-m+1)}} + k_2A_4\int_\Omega \left|\nabla v^\delta\right|^2 dx \tag{3.9}$$

where $A_1 = 1 - \frac{\delta k_1}{\tau}\left(\frac{\sqrt{3}}{\rho_0}\right)^{\frac{n+1}{\tau}} |\Omega|^{\frac{3\tau-2(n+1)}{3\tau}+\frac{s-m+1}{2\tau s}}$,

$A_2 = \frac{n}{2\tau} k_1^{-\frac{2\tau}{n}} \left(\frac{\sqrt{3}}{\rho_0}\right)^{\frac{n+1}{\tau}} |\Omega|^{\frac{3\tau-2(n+1)}{3\tau}+\frac{s-m+1}{2\tau s}}$,

$A_3 = \frac{ns-(s-m+1)}{2\tau s} k_2^{-\frac{2\tau s}{ns-(s-m+1)}} \left(\frac{(d+\rho_0)\tau}{\sqrt{3}\rho_0\delta}\right)^{\frac{n+1}{\tau}} |\Omega|^{\frac{3\tau-2(n+1)}{3\tau}}$,

$A_4 = \frac{n+1}{2\tau}\left(\frac{(d+\rho_0)\tau}{\sqrt{3}\rho_0\delta}\right)^{\frac{n+1}{\tau}} |\Omega|^{\frac{3\tau-2(n+1)}{3\tau}}$,

and taking $k_2 = \frac{C_1A_1}{C_2A_4}$.

From (2.4) and (3.9), we have

$$\phi'(t) \le A_5\left(\int_\Omega v^n\,dx\right)^{\frac{n+1}{n}} + A_6\left(\int_\Omega v^n\,dx\right)^{\frac{(n+1)s}{ns-(s-m+1)}}, \tag{3.10}$$

namely,

$$\phi'(t) \le A_5[\phi(t)]^{\frac{n+1}{n}} + A_6[\phi(t)]^{\frac{(n+1)s}{ns-(s-m+1)}}, \tag{3.11}$$

where $A_5 = \frac{C_2A_2}{A_1}$, $A_6 = \frac{C_2A_3}{A_1}$.

From (3.11), it follows that

$$t \geq \int_{\phi(0)}^{\phi(t)} \frac{d\zeta}{A_5 \zeta^{\frac{n+1}{n}} + A_6 \zeta^{\frac{(n+1)s}{ns-(s-m+1)}}}, \tag{3.12}$$

that is

$$T \geq \int_{\phi(0)}^{\infty} \frac{d\zeta}{A_5 \zeta^{\frac{n+1}{n}} + A_6 \zeta^{\frac{(n+1)s}{ns-(s-m+1)}}}, \tag{3.13}$$

where $\phi(0) = \int_\Omega {u_0}^{ns}\, dx$.

Now, we have the following theorem.

Theorem 3.1 Let u be the classical positive solution of (1.1)—(1.2) and (1.4) with $p > m > 1,\ 0 < k < |\Omega|$, then a lower bound of the blow-up time for the solution which blows up in L^{ns} norm is given by (3.13), where s and n respectively satisfy (1.7) and (3.1).

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