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Fixed point theorems of contractions of G-metric Spaces and property 'P' in G-Metric spaces

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Abstract

In this paper, we establish certain fixed point theorems of contractions of G-metric spaces and also showing property P in G-metric spaces.

Key words: G-metric space, G-Cauchy sequence, G-convergent, Contractive mapping.

Mathematics Subject Classification: 54H25, 47H10

1. INTRODUCTION AND PRELIMINARIES:

Metric fixed point theory is an important Mathematical discipline because of its applications in areas such as variation and linear inequalities, optimization and approximation theory, etc..,. The generalization of metric spaces were proposed by Gahler [5] (called 2-metric spaces) and Dhage [2, 3, 4] (called D-metric spaces). Hsiao [7] showed that every contractive definition, with $x_n = T^n x_0$, every orbit is linearly dependent, thus giving fixed point theorem in such spaces. However HA et. al. [6] have pointed out that the results obtained by Gahler for his 2-metric spaces are independent, rather than the generalizations of corresponding results in metric spaces. While Mustafa and Sims [8] have proved that the Dhage's notion of D-metric space is fundamentally incorrect and most of the results claimed by Dhage and others are invalid.

1.1 Definition: (See [8])

Let X be a non empty set and let $G : X \times X \times X \to [0,\infty)$ be a function satisfying the following axioms:

(G_1) $G(x, y, z) = 0$ if $x = y = z$.

(G_2) $0 < G(x,x,y)$, for all $x, y \in X$ with $x \neq y$

(G_3) $G(x,x,y) < G(x,y,z)$ for all $x, y, z \in X$ with $y \neq z$

(G_4) $G(x,y,z) = G(\sigma(x,y,z))$ for all $x, y, z \in X$, where σ is a permutation of the set $\{x, y, z\}$ (Symmetry in all three variables)

(G_5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x, y, z, a \in X$ (Rectangular inequality)

Then the function G is called a *generalized metric* or more specifically a G-*metric* on X. The pair (X,G) is called a G-*metric space*.

1.2 Definition:

Let (X,G) be a G-metric space and let $\{x_n\}$ be a sequence of points of X, we say that $\{x_n\}$ is $G-$*convergent to* x if for every given $\varepsilon > 0$, there exist $N \in N$ (set of all natural numbers) such that $G(x, x_n, x_m) < \varepsilon$ for all $m, n \geq N$. We denote it as $\displaystyle\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$

1.3 Definition:

Let (X,G) be a metric space, a sequence $\{x_n\}$ in X is called $G-$*cauchy* if for every given $\varepsilon > 0$, there exists $N \in N$ such that,

$G(x_n, x_m, x_l) < \varepsilon$ for all $n, m, l \geq N$, that is if, $\displaystyle\lim_{n,m,l\to\infty} G(x_n, x_m, x_l) = 0$

1.4 Definition:

A G-metric space (X,G) is said to be G-*complete* (or a *complete* G-*metric space*) if every G-Cauchy sequence in (X,G) is G-convergent to some point in (X,G).

1.5 Definition:

A G-metric space (X,G) is said to be *symmetric* if $G(x,y,y) = G(y,x,x)$ for all $x, y \in X$

1.6 Definition:

Let (X,G) be a G-metric space and let $T : X \to X$ be a mapping. T is called a *contraction of* X if

(1.6.1) $G(Tx,Ty,Tz) \leq k\ G(x,y,z)$ for all $x, y, z \in X$

1.7 Definition:

Let T be a self map of a complete G-metric space (X,G) with non empty fixed point set $F(T)$ (set of all fixed points of T). Then we say that T satisfies *property* P if $F(T) = F(T^n)$ for all $n \in N$.

2. MAIN THEOREM

In 1974 Lj. B. Ciric [1] was generalized the Banach contraction principle in metric spaces which states as follows.

2.1. Theorem

Let T be a self map of a metric space X such that X is T orbitally complete. Suppose that T satisfies.

(2.1.1) $$d(Tx,Ty) \le k.\max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$$

Where k is a real number with $0 \le k < 1$. Then T has unique fixed point $u \in X$. Moreover for each $x \in X$,

(2.1.2) $$\lim_{n \to \infty} T^n x = u \text{ and}$$

(2.1.3) $$d(T^n x, u) \le \frac{k^n}{1-k} d(x,Tx)$$

In this paper, we establish a fixed point theorem for complete G-metric spaces similar to that of Ciric's [1] result stated above.

2.2 Theorem:

Let (X,G) be a complete G-metric space and let T be a self map of X satisfying for all $x,y,z \in X$

(2.2.1) $$G(Tx,Ty,Tz) \le k \max\left\{ \begin{array}{l} G(x,y,z), G(x,Tx,Tx), G(y,Ty,Ty), G(z,Tz,Tz), \\ \frac{G(x,Ty,Ty)+G(z,Tx,Tx)}{2}, \frac{G(x,TyTy),+G(y,Tx,Tx)}{2}, \\ \frac{G(y,Tz,Tz)+G(z,Ty,Ty)}{2}, \frac{G(x,Tz,Tz)+G(z,Tx,Tx)}{2} \end{array} \right\}$$

Proof: Let $x_0 \in X$. Define the sequence $\{x_n\}$ by $x_n = T^n x_0$.

That is $x_1 = Tx_0,\ x_2 = Tx_1,\ x_3 = Tx_2, \ldots$

If for some N, $x_N = x_{N+1}$, then $Tx_N = x_N$, Showing x_N is a fixed point of T.

Now assume that $x_n \neq x_{n+1}$ for each $n = 0, 1, 2, \ldots$

Choosing $x = x_{n-1}, y = x_n, z = x_n$ in (2.2.1), we get

$$G(x_n, x_{n+1}, x_{n+1}) = G(Tx_{n-1}, Tx_n, Tx_n)$$

$$\leq k \max \left\{ G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), \right.$$

$$G(x_n, x_{n+1}, x_{n+1}), \frac{G(x_{n-1}, x_{n+1}, x_{n+1}) + 0}{2}, \frac{G(x_{n-1}, x_{n+1}, x_{n+1)} + 0}{2},$$

$$\left. G(x_n, x_{n+1}, x_{n+1}), \frac{G(x_{n-1}, x_{n+1}, x_{n+1}) + 0}{2} \right\}$$

which implies,

(2.2.2) $$G(x_n, x_{n+1}, x_{n+1}) \leq k.\max \left\{ G(x_{n-1}, x_n, x_n), \frac{G(x_{n-1}, x_{n+1}, x_{n+1})}{2} \right\}$$

Case (i): Suppose $\max \left\{ G(x_{n-1}, x_n, x_n), \frac{G(x_{n-1}, x_{n+1}, x_{n+1})}{2} \right\} = G(x_{n-1}, x_n, x_n)$

Then (2.2.2) becomes

(2.2.3) $$G(x_n, x_{n+1}, x_{n+1}) \leq kG(x_{n-1}, x_n, x_n)$$

Thus we get,

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &\leq k.G(x_{n-1}, x_n, x_n) \\ &\leq k^2.G(x_{n-2}, x_{n-1}, x_{n-1}) \\ &\vdots \\ &\leq k^n.G(x_0, x_1, x_1) \end{aligned}$$

That is,

(2.2.4) $$G(x_n, x_{n+1}, x_{n+1}) \leq k^n G(x_0, x_1, x_1) \text{ for all } n \in N.$$

Now, for every $m, n \in N$ $m > n$ with $m > n$, using rectangular inequality (G_5), we get

$$G(x_n, x_m, x_m) \leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + G(x_{n+2}, x_{n+3}, x_{n+3}) + \ldots$$

$$+ G(x_{m-1}, x_m, x_m).$$

$$\leq k^n.G(x_0, x_1, x_1) + k^{n+1}G(x_0, x_1, x_1) + k^{n+2}G(x_0, x_1, x_1) + \ldots$$

$$+ k^{m-1}G(x_0, x_1, x_1).$$

$$= (k^n + k^{n+1} + \ldots + k^{m-1})G(x_0, x_1, x_1).$$

$$\leq (k^n + k^{n+1} + \ldots + k^{m-1} + \ldots)G(x_0, x_1, x_1).$$

$$G(x_n, x_m, x_m) \leq \frac{k^n}{1-k} G(x_0, x_1, x_1)$$

Taking $m, n \to \infty$ on the both the sides, we get,

$$\lim_{n,m \to \infty} G(x_n, x_m, x_m) = 0 \qquad \text{(Since } k < 1\text{)}$$

Therefore $\{x_n\}$ is a G – Cauchy sequence in complete G -metric space (X, G).

Hence $\{x_n\}$ G-convergent to some point $u \in X$ (say).

$$\lim_{n,m \to \infty} x_n = u$$

Case (ii): Suppose that

$$Max\left\{G(x_{n-1}, x_n, x_n), \frac{G(x_{n-1}, x_{n+1}, x_{n+1})}{2}\right\} = \frac{G(x_{n-1}, x_{n+1}, x_{n+1})}{2}$$

Then (2.2.2) becomes

(2.2.5) $$G(x_n, x_{n+1}, x_{n+1}) \leq \frac{k}{2} G(x_{n-1}, x_{n+1}, x_{n+1})$$

Using rectangular equality (G_5) with $a = x_n$ in the above, we obtain,

$$G(x_n, x_{n+1}, x_{n+1}) \leq \frac{k}{2}\{G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1})\}$$

This implies, $\left(1 - \frac{k}{2}\right) G(x_n, x_{n+1}, x_{n+1}) \leq \frac{k}{2} G(x_{n-1}, x_n, x_n)$

$$G(x_n, x_{n+1}, x_{n+1}) \leq \frac{k}{2-k} G(x_{n-1}, x_n, x_n) < k\, G(x_{n-1}, x_n, x_n)$$

That is, $G(x_n, x_{n+1}, x_{n+1}) \leq k\, G(x_{n-1}, x_n, x_n)$

Which is same as (2.2.3) and hence $\{x_n\}$ is a G –Cauchy sequence and

$$\lim_{n \to \infty} x_n = u, \quad u \in X.$$

To show that u is a fixed point of T . Take $x = x_n, y = u, z = u$ in (2.2.1)

$$G(x_{n+1}, Tu, Tu) \le$$

$$k.Max\left\{ G(x_{n+1}, u, u), G(x_n, x_{n+1}, x_{n+1}), G(u, Tu, Tu), G(u, Tu, Tu), \frac{G(x_n, Tu, Tu) + G(u, x_{n+1}, x_{n+1})}{2}, \frac{G(x_n, Tu, Tu) + G(u, x_{n+1}, x_{n+1})}{2}, G(u, Tu, Tu), \frac{G(x_n, Tu, Tu) + G(u, x_{n+1}, x_{n+1})}{2} \right\}$$

Taking $n \to \infty$ on both the sides and using the fact that G is continuous in all the three variables, we get $G(u, Tu, Tu) \le k\, G(u, Tu, Tu)$. This is true only when $G(u, Tu, Tu) = 0$ (Since $k < 1$). This implies $u = Tu$.

To show that T has unique fixed point u. If possible assume that there exist $v \in X$, such that $Tv = v$ and $v \ne u$. Then,

$$G(u, v, v) = G(u, Tv, Tv)$$

$$\le k.Max\left\{ G(u, v, v), 0, 0, 0, \frac{G(u, v, v) + G(v, u, u)}{2}, \frac{G(u, v, v) + G(v, u, u)}{2}, 0, \frac{G(u, v, v) + G(v, u, u)}{2} \right\}$$

That is,

$$G(u, v, v) \le \frac{k}{2} G(u, v, v) + \frac{k}{2} G(v, u, u).$$

(2.2.6) $$G(u, v, v) \le \frac{k}{2-k} G(v, u, u)$$

Now consider, $G(v, u, u) = G(Tv, Tu, Tu)$

$$\le k.Max\left\{ G(v, u, u),\ 0,\ 0,\ 0,\ \frac{G(v, u, u) + G(u, v, v)}{2}, \frac{G(v, u, u) + G(u, v, v)}{2},\ 0, \frac{G(v, u, u) + G(u, v, v)}{2} \right\}$$

which implies that $G(v, u, u) \le \frac{k}{2} G(v, u, u) + \frac{k}{2} G(u, v, v)$

(2.2.7) $$G(v,u,u) \le \frac{k}{2-k} G(u,v,v)$$

Combining (2.2.6) and (2.2.7), we have

$G(u,v,v) \le \left(\frac{k}{2-k}\right)^2 G(u,v,v)$, which is impossible, since $\frac{k}{2-k} < 1$. Therefore $u = v$.

In 1974, Lj. B. Ciric [1] was established the following fixed point theorem for complete metric spaces.

2.3 Theorem: (See [1])

Let T be a self map of a complete metric space (X,d) such that T satisfies

(2.3.1) $$d(Tx,Ty) \le k.Max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$$

Then T has unique fixed point $u \in X$. Moreover for each $x \in X$,

(2.3.2) $$\lim_{n\to\infty} T^n x = u$$ and

(2.3.3) $$d(T^n x, u) \le \frac{q^n}{1-q} d(x,Tx)$$

2.4 Theorem:

Let (X,G) be a complete G-metric space and let T be a self map of X satisfying for all $x,y,z \in X$

(2.4.1) $$G(Tx,Ty,Tz) \le k.\ Max\{G(x,y,z),\ G(x,Tx,Tx),\ G(y,Ty,Ty),\ G(z,Tz,Tz),\ G(x,Ty,Ty),\ G(y,Tx,Tx)\}$$

or

(2.4.2) $$G(Tx,Ty,Tz) \le k.\ Max\{G(x,y,z),\ G(x,x,Tx),\ G(y,y,Ty),\ G(z,z,Tz),\ G(x,x,Ty),\ G(y,y,Tx)\},$$

where k is a constant satisfying $0 \le k < 1$

Then T has unique fixed point $u \in X$ (say) and T is G–continuous at u.

Proof: Suppose that T satisfies (2.4.1). Now taking $z = y$ in (2.4.1), we get

(2.4.3) $$G(Tx,Ty,Ty) \le k.Max\{G(x,y,y), G(x,Tx,Tx), G(y,Ty,Ty), G(x,Ty,Ty), G(y,Tx,Tx)\}$$

Suppose that (X,G) is symmetric then by the definition of d_G and the property $d_G(x,y) = 2G(x,y,y)$ in symmetric G-metric spaces, we have

$$d_G(Tx,Ty) \le k.Max\{d_G(x,y),\ d_G(x,Tx),\ d_G(y,Ty),\ d_G(x,Ty),\ d_G(y,Tx)\}$$

Now, by theorem 2.3 T has unique fixed point. Suppose that (X,G) is not symmetric. Define

(2.4.4) $$A_n = \left\{ G(Tx^i, Tx^j, Tx^j) : 0 \le i, j \le n \right\} \text{ and } \delta_n = Max\ A_n,$$

where $\delta_n = G(Tx^i, Tx^m, Tx^m)$ for some i, m satisfying $0 \le i, m \le n$ and for $x_0 \in X$. Define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for $n = 0, 1, 2, \ldots$

Suppose that $i > 0$. Then by (2.4.1) we have

$$\delta_n = G(x_i, x_m, x_m) \le k.\ Max\{G(x_{i-1}, x_{m-1}, x_{m-1}), G(x_{i-1}, x_i, x_i), G(x_{m-1}, x_m, x_m),$$
$$G(x_{i-1}, x_m, x_m), G(x_{m-1}, x_i, x_i)\}$$
$$\le k.\delta_n$$

Therefore,

(2.4.5) $$G(x_n, x_m, x_m) \le \frac{k^n}{1-k}\delta$$

Letting $m, n \to \infty$ in (2.4.5), we get

$\lim_{m,n \to \infty} G(x_n, x_m, x_m) = 0$, showing $\{x_n\}$ is G - Cauchy sequence in complete G-metric space (X,G). Therefore there exist some $u \in X$ such that $\{x_n\}$ is $G-$ Convergent to u as $n \to \infty$. Now, we claim u is a fixed point of T. For this consider,

$$G(x_n, Tu, Tu) = G(Tx_{n-1}, Tu, Tu)$$
$$\le k.Max\{G(x_{n-1}, u, u), G(x_{n-1}, u, u), G(u, Tu, Tu), G(u, Tu, Tu)$$
$$G(x_{n-1}, Tu, Tu), G(u, x_n x_n)\}$$

That is,

(2.4.6) $$G(x_n, Tu, Tu) \le k.Max\{G(x_{n-1}, u, u), G(x_{n-1}, x_n, x_n), G(u, Tu, Tu), G(u, x_n, x_n)\}$$

Letting $n \to \infty$ on both the sides, we get

$$G(u, Tu, Tu) \le k.\ Max\{G(u,u,u), G(u,u,u), G(u,Tu,Tu), G(u,u,u)\}$$

This gives

$G(u,Tu,Tu) \le k.\ G(u,Tu,Tu)$, a contradiction if $Tu \ne u$ showing u is a fixed point of T. Therefore $i = 0$, Thus for some m satisfying $0 \le m \le n$ using the property (G_5) and (2.4.1), we get

$$\delta_n = G(x_0, x_m, x_m) \le G(x_0, x_1, x_1) + G(x_1, x_m, x_m)$$
$$\le G(x_0, x_1, x_1) + k.\ \delta_n$$

$$\delta_n \le \frac{1}{1-k} G(x_0, x_1, x_1)$$

Showing δ_n is bounded, let δ be the upper bound of $\{\delta_n\}$.

Without loss of generality, we may assume that $x_n \neq x_{n+1}$ for each n

For, if there exist $N \in N$ such that $x_N = x_{N+1}$,

Then, $Tx_N = x_{N+1} = x_N$, showing x_N is a fixed point of T. From (2.4.1)

$$G(x_n, x_{n+1}, x_{n+1}) \le k.\ Max\{\ G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), G(x_{n-1}, x_{n+1}, x_{n+1}), 0\ \}$$
$$= k.Max\{G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_{n+1}, x_{n+1})\}$$
$$\le k.Max\{G(x_{n-1}, x_n, x_n), \delta\}$$
$$\vdots$$
$$\le k^n.Max\{G(x_0, x_1, x_1), \delta\}$$
$$\le k^n.\ \delta$$

For any $m, n \in N;\ m > n,$

$$G(x_n, x_m, x_m) \le G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + G(x_{m-1}, x_m, x_m)$$
$$\le (k^n + k^{n+1} + \ . \ . \ . \ + k^{m-1})\ \delta$$
$$\le (k^n + k^{n+1} + \ . \ . \ . \ + k^{m-1} + \ . \ . \ .)\delta$$
$$= \frac{k^n}{1-k}.\delta$$

To prove T has unique fixed point. Assume that v is another fixed point of T with $v \neq u$ then,

$$G(u, v, v) = G(Tu,, Tv, Tv)$$
$$\le k.Max\ \{\ G(u, v, v), 0, 0,\ G(u, v, v),\ G(v, u, u)\ \}$$

(2.4.7) $\quad G(u, v, v) = k.\ G(v, u, u)$

Again using (2.4.1) we have

$$G(v,u,u) = G(Tv,Tu,Tu)$$

$$\leq k.Max\{G(v,u,u),\ 0,\ 0,\ G(v,u,u),\ G(u,v,v)\}$$

(2.4.8) $G(v,u,u) \leq k.\ G(u,v,v)$

Now from (2.4.7), (2.4.8), we have $G(u,v,v) \leq k^2\, G(u,v,v)$, a contradiction. Hence $u = v$. Let $\{y_n\} \subseteq X$ be a sequence with $\lim\limits_{n\to\infty} y_n = u$.

Then, (2.4.1) gives

$$G(Ty_n,u,Ty_n) = G(Ty_n,Tu,Ty_n)$$

$$\leq k.Max\{G(y_n,u,y_n),\ G(y_n,Ty_n,Ty_n),\ 0,\ G(y_n,u,u),\ G(u,Ty_n,Ty_n),\ G(y_n,Ty_n,Ty_n)\}$$

$$G(Ty_n,u,Ty_n) \leq k.Max\{G(y_n,u,y_n)\ G(y_n,u,u) + G(u,Ty_n,Ty_n),\ G(u,Ty_n,Ty_n)\}$$

This implies

$$G(Ty_n,u,Ty_n) \leq k.Max\{G(y_n,u,y_n),\ G(y_n,u,u),\ G(u,Ty_n,Ty_n)\}$$

This upon simplifying, we have

$$G(Ty_n,u,Ty_n) \leq Max\{k.\ G(y_n,u,y_n),\ \frac{k}{1-k}\ G(y_n,u,u)\}$$

Letting $n\to\infty$ on both the sides of above, we have $G(Ty_n,u,Ty_n) \to 0$ as $n\to\infty$, showing $\lim\limits_{n\to\infty} Ty_n = u = Tu$ and T is G–continuous at u.

2.5 Theorem:

Let T be a self map of a complete G-metric space (X,G) and satisfies (2.4.1) then T has property P.

Proof: From theorem 2.4, T has a fixed point $u \in X$. That is, $Tu = u$, $u \in F(T)$ that is $F(T) \neq \phi$.

$Tu = u$ implies $T^n u = u$ for any $n \in N$ and hence $u \in F(T^n)$ that is, $F(T^n) \neq \phi$ for all $n \in N$. That is,

(2.5.1) $F(T) \subseteq F(T^n)$

Assume that $v \in F(T^n)$, we shall show that $v \in F(T)$.

Suppose $v \neq Tv$, now consider,

$$G(v,Tv,Tv) = G(T^n v, T^{n+1}v, T^{n+1}v) = G(TT^{n-1}v, TT^n v, TT^n v)$$

$$\leq k.Max\{G(T^{n-1}v,T^n v,T^n v),\ G(T^{n-1}v,T^n v,T^n v),\ G(T^n v,T^{n+1}v,T^{n+1}v),\ G(T^{n-1}v,T^{n+1}v,T^{n+1}v),\ 0,\ 0\}$$

Define, $B_n = \{\ G(T^i v, T^j v, T^j v) : 0 \leq i, j \leq n\ \}$

Then, $\delta_n = \underset{i,j}{Max}\ B_n$. Then,

(2.5.2) $\delta_n = G(T^i v, T^m v, T^m v)$ for some $0 \leq i, m \leq n$. Assume that δ_n >0. Then from (2.4.1),

$$\delta_n = G(T^i v, T^m v, T^m v)$$

(2.5.3) $$\leq k.Max\{G(T^{i-1}v,T^{m-1}v,T^{m-1}v), G(T^{i-1}v,T^i v,T^i v), G(T^{m-1}v,T^m v,T^m v),, G(T^{i-1}v,T^m v,T^m v), G(T^{m-1}v,T^i v,T^i v), G(T^{m-1}v,T^m v,T^m v)\}$$

$\delta_n \leq k\delta_n$, a contradiction, therefore $\delta_n = 0$

In particular, $G(v,Tv,Tv) = 0$ and $v = Tv$. That is,

(2.5.4) $$F(T^n) \subseteq F(T)$$

From (2.5.1) & (2.5.4) we have $F(T) = F(T^n)$ for all $n \in N$,

Showing T has property P.

REFERENCES

[1] Circic. Lj. B,. "A generalization of Banach's contraction principle," Proceedings of the American Mathematical Society, Vol. 45, pp. 267 – 273, 1974.

[2] Dhage, B.C., "Generalized metric space and mapping with fixed points", Bulletin of the Calcutta Mathematical Society, Vol. 84, pp. 329 – 336, 1992.

[3] Dhage. B.C., "Generalized metric spaces and topological structure – I", Analete Stiintifice ale Universitatii." Al. I. Cuza" dina Iasi. Serie Nova, Mathametical, Vol. 46, no. 1, pp . 3 – 24, 2000.

[4] Dutta. P.N. and Choudhaury, Binayak S, "A generalization of contraction principle in metric spaces," Journal of Fixed Point Theory and Applications, Vol. 2008.

[5] Gahler. S, "2–metriche Raume undihre topologische struktur," Mathematische Nachrichten. Vol. 26, no. 1-4, pp. 115 – 148, 1963.

[6] Ha. R.S., Cho. Y.J. and White A, "Strictly Convex and Strictly 2 – convex 2 – normed spaces," Mathematica Japanica, vol. 33, no. 3, pp. 375 – 384, 1988.

[7] Hsio. C.R., "A property of contractive type mappings in 2-metric spaces," Inanabha, Vol. 16, pp. 223-239, 1986.

[8] Mustafa. Z and Sims. B, "Some remarks concerning D-metric spaces," in proceedings of the International Conference on Fixed Point Theorey and Applications, pp. 189 – 198, Valencica, Spain, July 2003.

[9] Mustafa. Z, A new structure for generalized metric spaces – with applications to fixed point theory, Ph.D. thesis, the University of New Castle, Callaghan, Australia, 2005.

[10] Mustafa. Z and Sims. B, "A new approach to generalized metric spaces," Journal of Nonlinear and Convex Analysis, Vol. 7, No. 2, pp. 289 – 297, 2006.

[11] Rashwan. R. A. and A.M. Sadeek, "A common fixed point theorem in complete metric spaces," Southwest Journal of Pure and Applied Mathematics, Vol. 01, 1996, pp. 6-10.