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Convex Subtrellises of a Weakly Perfect Trellis

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Abstract

The concept of a trellis is introduced by E. Fried and H.L. Skala, independently, as a nonassociative generalization of a lattice. In this paper an attempt is made to extend some of the well-known results of lattices to trellises. We introduce a special class of trellises called weakly perfect trellises and study their algebraic properties. It is proved that a weakly $\vee$-perfect (weakly $\wedge$-perfect) trellis is a lattice if and only if it is acyclic. Convex subtrellis introduced in this paper differs from the notion of convex subtrellis due to H.L.Skala. Our approach is that of graph theoretic one. A partial order $\leqslant$ is defined on the set $CS(L)$ of all convex subtrellises of a trellis L and it is proved that the poset $\langle CS(L); \leqslant\rangle$ forms a lattice if L is weakly perfect. It is also proved that the set of all ideals of a weakly perfect trellis L is the principal ideal generated by L in $\langle CS(L); \leqslant\rangle$.

AMS subject classification: 06B05.
Keywords: psoset, trellis, convex subtrellis, weakly perfect trellis.

1. Introduction

Any reflexive and antisymmetric binary relation $\trianglelefteq$ on a set A is called a *pseudo-order* on A and $\langle A; \trianglelefteq\rangle$ is called a *pseudo-ordered set* or a *psoset*. By a *trellis* we mean a psoset, any two of whose elements have a greatest lower bound (g.l.b.) and a least upper bound (l.u.b.). The notion of a psoset and a trellis are due to Fried [1] and Skala [7]. In Bhatta [5], it is shown that any psoset can be regarded as a digraph (possibly infinite).

Define a relation $\sqsubseteq_B$ on a subset B of a psoset $\langle A; \trianglelefteq\rangle$ by setting $b \sqsubseteq_B b'$ for two elements b and b' of B if and only if there is a directed path in B from b to b' say

$b = b_0 \trianglelefteq b_1 \trianglelefteq ... \trianglelefteq b_n = b'$ for some $n \geq 0$. The relation $\sqsupseteq_B$ is defined dually. If for each pair of elements b and b' of B at least one of the relations $b \sqsubseteq_B b'$ or $b' \sqsubseteq_B b$, then B will be called a *pseudo-chain* or a *p-chain*. If for each pair of elements b and b' of B both the relations $b \sqsubseteq_B b'$ and $b' \sqsubseteq_B b$ hold, then B will be called a *cycle*. Empty set and single element set in a psoset are cycles. A nontrivial cycle contains at least three elements. A psoset is said to be *acyclic* if it does not contain any nontrivial cycle.

A *subtrellis* S of a trellis L is a nonempty subset of L such that $a, b \in S$ implies $a \wedge b$, $a \vee b$ belong to S where $\wedge$ and $\vee$ are considered in L. An *ideal* I of a trellis L is a subtrellis of L such that $i \in I$ and $a \in L$ imply that $a \wedge i \in I$ or equivalently $i \in I$, $a \in L$ and $a \trianglelefteq i$ imply that $a \in I$. A *dual ideal* or a *filter* of a trellis can be defined dually. Skala in [6] has included the empty set also as an ideal of a trellis. If B is a nonempty subset of a trellis L, then the *ideal generated by* B is defined to be the intersection of all ideals of L containing B and is denoted by $(B]$. An ideal generated by a single element a is called a *principal ideal generated by a* and is denoted by (a]. The dual notions are defined similarly. The set of all ideals of a trellis L forms a lattice with respect to set inclusion and it is denoted by $I(L)$. In fact, for $I, J \in I(L)$, $I \wedge J = I \cap J$, $I \vee J = (I \cup J]$.For the basic terminologies and definitions Gratzer [2], Bhatta [5] and Skala [7] may be referred.

2. Convex subtrellises of a trellis

By an *interval* $[a, b]$, $a \sqsubseteq_L b$ in a trellis L, we mean $\{x \in L | \; a \sqsubseteq_L x \sqsubseteq_L b\}$

Definition 2.1. A subtrellis C of a trellis L is said to be a convex subtrellis of L if whenever $a, b \in C$ with $a \sqsubseteq_L b$, then $[a, b] \in C$ or equivalently whenever $a, b \in C$ and $c \in L$ such that $a \sqsubseteq_L c \sqsubseteq_L b$, then $c \in C$.

Remark 2.2.

1. Every ideal of a trellis is a convex subtrellis of L. However, the converse is not true. For, in the trellis L of Figure 1, $\{a, b\}$ is a convex subtrellis of L that is not an ideal.

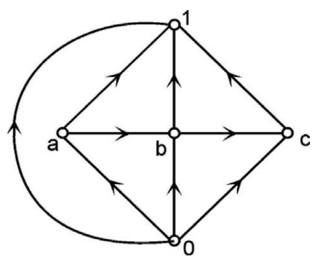

Figure 1:

2. The intersection of any number of convex subtrellises is a convex subtrellis unless empty.

3. The set of all convex subtrellises of a trellis L, denoted by $CS(L)$, forms a lattice with respect to set inclusion.

Theorem 2.3. Let $CS(L)$ be the set of all convex subtrellises of a trellis L. Then the binary relation $\leqslant$ on $CS(L)$ defined by for $A, B \in CS(L)$, $A \leqslant B$ if and only if for each $a \in A$, there exists an element $b \in B$ such that $a \sqsubseteq_L b$ and for each $b \in B$ there exists an $a \in A$ such that $b \sqsupseteq_L a$ is a partial order on $CS(L)$.

Proof. Clearly $\leqslant$ is reflexive. Let $A \leqslant B$ and $B \leqslant A$ where $A, B \in CS(L)$. Let $a \in A$. Since $A \leqslant B$, there exists some $b \in B$ such that $a \sqsubseteq_L b$. Since $B \leqslant A$, there exists some $b_1 \in B$ such that $a \sqsupseteq_L b_1$. Thus $b_1 \sqsubseteq_L a \sqsubseteq_L b$ where $b_1, b \in B$. By the convexity of B, $a \in B$. Thus $A \subseteq B$. Similarly $B \subseteq A$ and we get $A = B$ so that $\leq$ is antisymmetric. Let $A \leq B \leq C$, where $A, B, C \in CS(L)$. Let $a \in A$. Since $A \leqslant B$, there exists some $b \in B$ such that $a \sqsubseteq_L b$. Since $B \leqslant C$, there exists some $c \in C$ such that $b \sqsubseteq_L c$. This gives $a \sqsubseteq_L c$. Similarly it is true that for each $c \in C$ there exists some $a \in A$ such that $a \sqsubseteq_L c$. Thus $A \leqslant C$. Hence $\langle CS(L); \leqslant \rangle$ is a poset. ∎

Remark 2.4.

1. The poset $\langle CS(L); \leqslant \rangle$ of all convex subtrellises of a trellis L need not form a lattice in general. For example, set of all convex subtrellises of the trellis L of Figure 2 is $CS(L) = \{\{0\}, \{a\}, \{b\}, \{c\}, \{d\}, \{1\}, \{0, a\}, \{0, d\}, \{a, b\}, \{c, 1\}, \{b, c\}, \{c, d\}, \{0, a, b\}, \{b, c, 1\}, \{c, d, 1\}, \{L\}\}$.
   The poset $\langle CS(L); \leqslant \rangle$ is in Figure 3 and is not a lattice because the elements $\{0, a\}$ and $\{0, d\}$ of $CS(L)$ have two distinct minimal upper bounds, namely, $\{b, c\}$ and L.

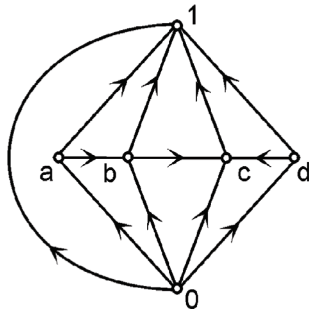

Figure 2:

2. If a trellis L is a cycle, then $\langle CS(L); \leqslant \rangle$ is a single element lattice.

3. If L is a finite bounded trellis, then $\langle CS(L); \leqslant \rangle$ is a finite bounded poset.

In the following the notion of a weakly perfect trellis is introduced as a generalization of a lattice and several interesting results are proved.

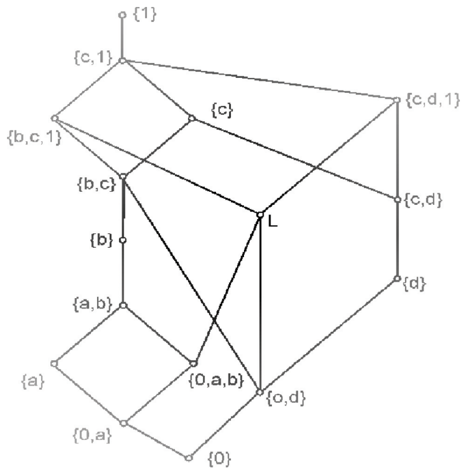

Figure 3:

3. Weakly perfect trellis

Definition 3.1. A trellis L is said to be

(i) weakly $\vee$-perfect if $a \sqsubseteq_L c$ and $b \sqsubseteq_L c$ imply $a \vee b \sqsubseteq_L c$ for $a, b, c \in L$, or equivalently $a \sqsubseteq_L b$ and $c \sqsubseteq_L d$ imply $a \vee c \sqsubseteq_L b \vee d$ for $a, b, c, d \in L$.

(ii) weakly $\wedge$-perfect if $a \sqsupseteq_L c$ and $b \sqsupseteq_L c$ imply $a \wedge b \sqsupseteq_L c$ for $a, b, c \in L$, or equivalently $a \sqsupseteq_L b$ and $c \sqsupseteq_L d$ imply $a \wedge c \sqsupseteq_L b \wedge d$ for $a, b, c, d \in L$.

(iii) weakly perfect if it is both weakly $\vee$-perfect and weakly $\wedge$-perfect.

Remark 3.2. (i) and (ii) in the above definition are independent. For example, consider the trellis of Figure 4. It is weakly $\vee$-perfect. For, the only noncomparable pairs of this trellis are (a, d) and (b, d) with $a \wedge d = 0$, $a \vee d = e$, $b \wedge d = c$, $b \vee d = e$ and we observe that $e \sqsubseteq_L a, b, c, d$. Thus, if there is a directed path from any two points x and y to some other point z, then there is a directed path from $x \vee y$ to z and hence the trellis is weakly $\vee$-perfect. However, it is not weakly $\wedge$-perfect as $a \sqsupseteq_L c$ and $d \sqsupseteq_L c$ do not imply $a \wedge d \sqsupseteq_L c$. In fact, $a \wedge d = 0 \sqsubseteq_L c$.

Remark 3.3. If a trellis L is a lattice, a tournament or a cycle, then it is weakly perfect. However, the converse is not true. For, the trellis given in Figure 5 is weakly perfect but not a lattice or a tournament. The trellis of Figure 6 is weakly perfect but not a cycle.
Theorem 3.4. A weakly $\vee$-perfect (weakly $\wedge$-perfect) trellis is a lattice if and only if it is acyclic.

Proof. Necessity is trivial. Conversely, let L be an acyclic weakly $\vee$-perfect trellis. Let $a, b, c \in L$ such that $a \trianglelefteq b \trianglelefteq c$. Since $a \sqsubseteq_L c$ and $c \sqsubseteq_L c$, by the weak $\vee$-perfectness of L, $a \vee c \sqsubseteq_L c$. Also $c \trianglelefteq a \vee c$. If $c \neq a \vee c$, then $c \triangleleft a \vee c \sqsubseteq_L c$ which imply that L contains a nontrivial cycle, a contradiction. Thus $c = a \vee c$ and therefore $a \trianglelefteq c$.

Thus $\trianglelefteq$ is transitive so that L is a lattice [7]. The result holds good for weakly $\wedge$-perfect trellises by duality. ∎

Definition 3.5. A nonempty subset A of a trellis L is said to be a

(i) $\vee$-subtrellis of L if $a \vee b \in A$ whenever $a, b \in A$.

(ii) $\wedge$-subtrellis of L if $a \wedge b \in A$ whenever $a, b \in A$.

Lemma 3.6. If A is a $\vee$-subtrellis of a weakly $\vee$-perfect trellis L, then $(A] = \{x \in L | x \sqsubseteq_L a$ for some $a \in A\}$. In particular, if $a \in L$, then
$(a] = \{x \in L | x \sqsubseteq_L a\}$.

Proof. Let $I = \{x \in L | x \sqsubseteq_L a$ for some $a \in A\}$. Let $x_1, x_2 \in I$. Then $x_1 \sqsubseteq_L a$ and $x_2 \sqsubseteq_L b$ for some $a, b \in A$. By the weak $\vee$-perfectness of L, $x_1 \vee x_2 \sqsubseteq_L a \vee b$. Since $a \vee b \in A$, we have $x_1 \vee x_2 \in I$. Let $x \in I$ and $y \in L$ such that $y \trianglelefteq x$. Then $y \trianglelefteq x \sqsubseteq_L a$ for some $a \in A$ so that $y \in I$. Thus I is an ideal containing A. If J is any ideal containing A, then $I \subseteq J$. For, let $x \in I$. Then $x \sqsubseteq_L a$ for some $a \in A$. Then $a \in J$ and since J is an ideal, $x \in J$. Thus I is the smallest ideal containing A. i.e. $I = (A]$.∎

Proposition 3.7. Let L be a trellis. Then the following are equivalent.

(i) L is weakly $\vee$-perfect.

(ii) $(a] = \{x \in L | x \sqsubseteq_L a\}$.

Proof. (i)$\Rightarrow$ (ii): Follows from Lemma 3.6.
(ii)$\Rightarrow$ (i): Let $a \sqsubseteq_L b$ and $c \sqsubseteq_L d$ where $a, b, c, d \in L$. Now $a, c \in (b \vee d]$. Therefore $a \vee c \in (b \vee d]$ as $(b \vee d]$ is an ideal. Thus $a \vee c \sqsubseteq_L b \vee d$ so that (i) holds. ∎

Lemma 3.8. Let A and B be $\vee$-subtrellises of a weakly $\vee$-perfect trellis L. Then $(A \cup B]$ = $\{x \in L | x \sqsubseteq_L a \vee b$ for some $a \in A, b \in B\}$.

Proof. Let $I = \{x \in L | x \sqsubseteq_L a \vee b$ for some $a \in A, b \in B\}$. Let $x_1, x_2 \in I$. Then $x_1 \sqsubseteq_L a_1 \vee b_1$ and $x_2 \sqsubseteq_L a_2 \vee b_2$ for some $a_1, a_2 \in A$ and $b_1, b_2 \in B$. Since

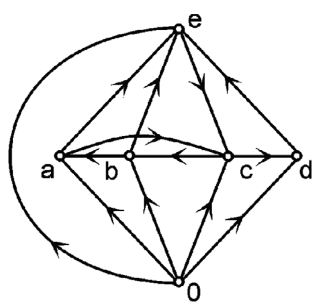

Figure 4:

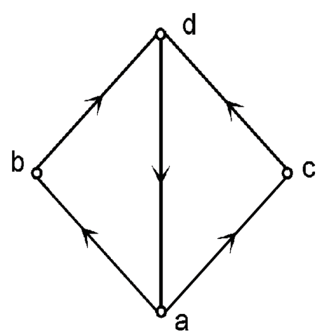

Figure 5:

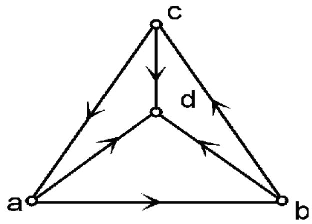

Figure 6:

$a_1, a_2 \trianglelefteq a_1 \vee a_2$ and $b_1, b_2 \trianglelefteq b_1 \vee b_2$, we have $x_1 \sqsubseteq_L a_1 \vee b_1 \sqsubseteq_L (a_1 \vee a_2) \vee (b_1 \vee b_2)$ and $x_2 \sqsubseteq_L a_2 \vee b_2 \sqsubseteq_L (a_1 \vee a_2) \vee (b_1 \vee b_2)$. Therefore $x_1 \vee x_2 \sqsubseteq_L (a_1 \vee a_2) \vee (b_1 \vee b_2) = a \vee b$, where $a = a_1 \vee a_2 \in A$ and $b = b_1 \vee b_2 \in B$ as A and B are $\vee$-subtrellises. Thus $x_1 \vee x_2 \in I$. Let $y \in L$ and $x \in I$ such that $y \trianglelefteq x$. Then $y \trianglelefteq x \sqsubseteq_L a \vee b$ for some $a \in A, b \in B$. Therefore $y \in I$. Thus I is an ideal containing $A \cup B$. If J is any ideal containing $A \cup B$, then $I \subseteq J$. For, if $x \in I$, then $x \sqsubseteq_L a \vee b$ for some $a \in A, b \in B$. Since J contains $A \cup B$, we have $a, b \in J$. Therefore $a \vee b \in J$ so that $x \in J$. Thus I is the smallest ideal containing $A \cup B$. That is, $I = (A \cup B]$. By duality, if A and B are $\wedge$-subtrellises of a weakly $\wedge$-perfect trellis L, then $[A \cap B) = \{x \in L | x \sqsupseteq_L a \wedge b\}$ for some $a \in A, b \in B\}$. ■

Remark 3.9. By Lemma 3.8, it follows that in a weakly $\vee$-perfect trellis L, $(a \vee b] = (a] \vee (b]$ in the ideal lattice $I(L)$ for every $a, b \in L$. The converse is not true. For, in the trellis L of Figure 1, $I(L) = \{\{0\}, \{0, a\}, \{0, a, b\}, \{L\}\}$. and $(a \vee b] = (a] \vee (b]$ in the ideal lattice $I(L)$ for every $a, b \in L$. Also, $I(L)$ forms a chain with respect to set inclusion. But L is not weakly $\vee$-perfect as $a \sqsubseteq_L c, c \sqsubseteq_L c$ do not imply $a \vee c \sqsubseteq_L c$.

Lemma 3.10. If A and B are $\vee$-subtrellises of a weakly perfect trellis L, then $(A] \cap (B] = \{x \in L | x \sqsubseteq_L a \wedge b$ for some $a \in A, b \in B\}$.

Proof. Let $I = \{x \in L | x \sqsubseteq_L a \wedge b$ for some $a \in A, b \in B\}$.Let $x \in I$. Then $x \sqsubseteq_L a \wedge b \trianglelefteq a, b$ for some $a \in A$ and $b \in B$ so that $x \in (A]$ and $(B]$ by Lemma 3.6 This implies $x \in (A] \cap (B]$ so that $I \subseteq (A] \cap (B]$. Let $x \in (A] \cap (B]$. Then $x \sqsubseteq_L a$ for some $a \in A$ and $x \sqsubseteq_L b$ for some $b \in B$ which in turn gives $x \sqsubseteq_L a \wedge b$ as L is weakly $\wedge$-perfect. Therefore $x \in I$ and $(A] \cap (B] \subseteq I$. Thus $(A] \cap (B] = I$. ■

Theorem 3.11. Let L be a weakly perfect trellis and $CS(L)$ be the set of all convex subtrellises of L. Then $\langle CS(L); \leqslant \rangle$ is a lattice where, for $A, B \in CS(L)$, $A \wedge B = \{x \in L | a \wedge b \sqsubseteq_L x \sqsubseteq_L a_1 \wedge b_1$ for some $a, a_1 \in A$ and $b, b_1 \in B\}$ and $A \vee B = \{x \in L | a \vee b \sqsubseteq_L x \sqsubseteq_L a_1 \vee b_1$ for some $a, a_1 \in A$ and $b, b_1 \in B\}$.

Proof. Let $X = \{x \in L | a \wedge b \sqsubseteq_L x \sqsubseteq_L a_1 \wedge b_1$ for some $a, a_1 \in A$ and $b, b_1 \in B\}$. Clearly $X = I \cap D$ where $I = (A] \cap (B]$ and $D = [A \cup B)$ by Lemma 3.10 and by the dual of Lemma 3.8. Also X is a convex subtrellis of L. If $x \in X$, then $x \sqsubseteq_L a_1 \wedge b_1 \trianglelefteq a_1$ for some $a_1 \in A$ and $b_1 \in B$. Also, if $a \in A$, then for any $b \in B, a \wedge b \in X$ and $a \trianglerighteq a \wedge b$. Therefore $X \leq A$. Similarly $X \leq B$. Suppose $Y \in CS(L)$ is such that

$Y \leq A$ and $Y \leq B$. Then $y \in Y$ implies $y \sqsubseteq_L a$ for some $a \in A$ and $y \sqsubseteq_L b$ for some $b \in B$. Hence, by the weak $\wedge$-perfectness of L, $y \sqsubseteq_L a \wedge b \in X$. On the other hand, if $x \in X$, then $x \sqsupseteq_L a \wedge b$ for some $a \in A, b \in B$. But then $a \sqsubseteq_L y_1$ and $b \sqsupseteq_L y_2$ for some $y_1 \wedge y_2 \in Y$. Hence by the weak $\wedge$-perfectness of L, $a \wedge b \sqsupseteq_L y_1 \wedge y_2$. Therefore, for each $x \in X, x \sqsupseteq_L a \wedge b \sqsupseteq_L y_1 \wedge y_2 \in Y$ so that $Y \leq X$. Hence $X = A \wedge B$. By the dual argument it follows that $A \vee B = \{x \in L | a \vee b \sqsubseteq_L x \sqsubseteq_L a_1 \vee b_1$ for some $a, a_1 \in A$ and $b, b_1 \in B\}$. ■

Remark 3.12. In Theorem 3.11, $A \wedge B = X = \langle a \wedge b | a \in A, b \in B \rangle$.

For, let $C = \langle a \wedge b | a \in A, b \in B \rangle$. Clearly, $X \supseteq \langle a \wedge b | a \in A, b \in B \rangle$. Therefore $X \supseteq \{a \wedge b | a \in A, b \in B\} = C$. Let $x \in X$. Then $a \wedge b \sqsubseteq_L x \sqsubseteq_L a_1 \wedge b_1$ for some $a, a_1 \in A$ and $b, b_1 \in B$. Now $a \wedge b, a_1 \wedge b_1 \in C$ and C is convex imply that $x \in C$. Thus $X \leq C$ and therefore $X = C$.

Remark 3.13. The condition in Theorem 3.11 is only a sufficient condition and not a necessary one for $\langle CS(L); \leqslant \rangle$ to be a lattice. For, consider the trellis L of Figure 1. It is not weakly perfect as $a \sqsubseteq_L c, c \sqsubseteq_L c$ do not imply $a \vee c \sqsubseteq_L c$. However, $\langle CS(L); \leqslant \rangle$ given in Figure 7 is a lattice. In fact,
$CS(L) = \{\{0\}, \{a\}, \{b\}, \{c\}, \{1\}, \{0, a\}, \{a, b\}, \{b, c\}, \{c, 1\},$
$\{0, a, b\}, \{b, c, 1\}, \{L\}\}$.

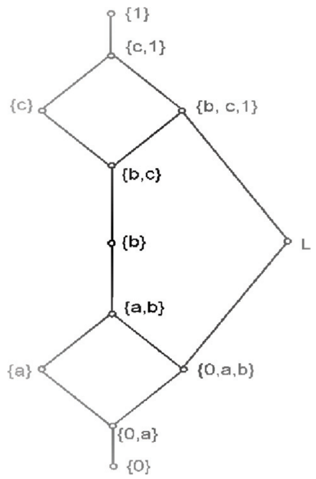

Figure 7:

Following two theorems are the generalizations of results in lattices [3] to weakly perfect trellises and they can be proved easily.

Theorem 3.14. The set Int $(L) = \{[a, b] | a, b \in L, a \sqsubseteq_L b\}$ of all intervals of a weakly perfect trellis L forms a sublattice of the lattice $\langle CS(L); \leqslant \rangle$.

Theorem 3.15. For any two weakly perfect trellises L and K, $CS(L \times K) \backsimeq CS(L) \times CS(K)$.

Let L be a weakly perfect trellis and $CS(L)$ be the set of all convex subtrellises of L. From the definition of $\leq$, it follows clearly that for $A, B \in CS(L)$, $A \leq B$ in $CS(L)$ if and only if $(A] \subseteq (B]$ and $[A) \supseteq [B)$. In particular, if $I, J \in I(L)$, the lattice of all ideals of L, then $I \leq J$ in $CS(L)$ if and only if $I \subseteq J$. Similarly if $D, K \in D(L)$, the lattice of all dual ideals of L, then $D \leq K$ in CS(L) if and only if $D \supseteq K$.

The following theorem shows that $I(L)$ and $D(L)$ are convex sublattices of $CS(L)$ and it generalizes a Theorem in [3].

Theorem 3.16. In a weakly perfect trellis $L, I(L)$ is the principal ideal $(L]$ generated by L and $D(L)$ is the principal dual ideal $[L)$ generated by L in $\langle CS(L); \leqslant\rangle$.

Proof. Let $I \in I(L)$. Then since $L \in I(L)$ and $I \subseteq L$, by the above note we get $I \leq L$ in CS(L). Hence $I(L) \subseteq (L]$ in CS(L). On the other hand, if $X \in (L]$, then $X \leq L$. Let $x \in X$ and $a \in L$ with $a \trianglelefteq x$. Since $X \leq L$, clearly $a \sqsupseteq_L x_1$, for some$x_1 \in X$. Now $x_1 \sqsubseteq_L a \trianglelefteq x$ so that $a \in X$ by the convexity of X. Already X is a subtrellis of L. Therefore $X \in I(L)$. Hence $I(L) = (L]$. Second part of the theorem follows by duality. ■

The following theorem is a generalization of a corresponding result in lattices [3] and can be proved similarly.

Theorem 3.17. For any weakly perfect trellis L, the mapping $f : CS(L) \to I(L) \times D(L)$ defined by, for $X \in CS(L)$, $f(X) = ((X], [X))$ is an embedding. In fact, $CS(L)$ is isomorphic to the sublattice $\{(I, D)I \in I(L), D \in D(L), I \cap D \neq \phi\}$ of $I(L) \times D(L)$.

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