Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 14, Number 6 (2018), pp. 809-817

http://www.ripublication.com

Existence and Uniqueness Results for Random Impulsive Integro-Differential Equation

K. Malar
Department of Mathematics, Erode Arts and Science College,
Erode , Tamil Nadu. India

Abstract

In this article we concerned with the existence and uniqueness of mild solutions for random impulsive integro-differential equation through fixed point technique. Moreover, Lipschitz condition as to be relaxed on the impulsive terms in the deriving results. It is investigated under sufficient conditions. The results are obtained by using the Banach fixed point theorem.

Keywords: random impulsive integro- differential equation; mild solution; fixed point.

1.. INTRODUCTION

Impulsive differential systems have been highly recognized and applied in the fields as diverse as physics, population dynamics, aeronautics, economics, telecommunications and engineering are characterized by the fact that they undergo abrupt change of state at certain moments of time between intervals of continuous evolution. The duration of these changes are often negligible compared to the total duration of process act instantaneously in the form of impulses see [6,7,8]. The impulses may be deterministic or random. There are lot of papers which investigate the qualitative properties of deterministic impulses see [1,3,5] and the references therein.

When the impulses are exist at random points, the solutions of the differential systems are stochastic processes. It is very different from deterministic impulsive differential systems and also it is different from stochastic differential equations. Thus the random impulsive systems give more realistic than deterministic impulsive systems. The study of random impulsive differential equations is a new research area. Actual impulses do not always happen at fixed points but usually at random points, i.e., impulsive moments t_k are random variable, see [2,5,9-16]

Recently in [5], A.Vinodkumar etl. Studied the existence and uniqueness and stability of random impulsive fractional differential equation. In [4], the existence and uniqueness of random impulsive differential system by relaxing the linear growth conditions, sufficient conditions for stability through continuous dependence on initial conditions and exponential stability via fixed point theory. Moreover, Lipschitz condition has to be relaxed on the impulsive terms in the deriving results.

2. PRELIMINARIES

Let X be a real separable Hilbert space and Ω a nonempty set. Assume that τ_k is a random variable defined from $\Omega \; to \, D_k \underline{\underline{def}}.(0,d_k)$ for all $k=1,2,...,$ where $0<d_k<+\infty$. Furthermore, assume that $\tau_i \; and \; \tau_j$ are independent with each other as $i \neq j \; for \, i,j=1,2,...$ Let $\tau \in \Re$ be a constant. For the sake of simplicity, we denote $\Re_\tau = [\tau, T]$. We consider the integro-differential equations with random impulses of the form

$$x'(t) = Ax(t) + f(t,x(t)) + \int_0^T f_1(\theta, x(t+\theta))\,d\theta, \qquad t \neq \xi_k, t \geq \tau,$$
$$x(\xi_k) = b_k(\tau_k)x(\xi_k^-), \qquad k=1,2,.... \qquad (2.1)$$
$$x_{t_0} = x_0,$$

where A is the infinitesimal generator of a strongly continuous semi group of bounded linear operators T(t) domain D(A) in X. $f:\Re_T \times X \to X, b_k : D_k \to \Re$ for each $k=1,2,...;\ \xi_0 = t_0 \in [\tau,T] \, and \ \xi_k = \xi_{k-1} + \tau_k \ for \, k=1,2,...,$ here $t_0 \in \Re_\tau$ is arbitrary real number. Obviously, $t_0 = \xi_0 < \xi_1 < \xi_2 < .... < \xi_k < ...;\ x(\xi_k^-) = \lim_{t \uparrow \xi_k} x(t)$ according to their paths with the norm $\|x\|_t = \sup_{\tau \le s \le t} |x(s)|$ for each t satisfying $\tau \le t \le T$. Let us denote $\{B_t, t \ge 0\}$ be the simple counting process generated by $\{\xi_n\}$ that is, $\{B_t \ge n\} = \{\xi_n \le t\}$, and denote $\mathcal{F}_t \; the \, \sigma - a\lg ebra$ generated by $\{B_t, t\ge 0\}$. Then $(\Omega, P, \{\mathcal{F}_t\})$ is a probability space. Let $L_2 = L_2(\Omega, \{\mathcal{F}_t\}, X)$ denote the Hilbert Space of all $\{\mathcal{F}_t\}$- measurable square integrable random variables with values in X.

Let $\mathcal{B}$ denote Banach space $\mathcal{B}([\tau,T], L_2)$, the family of all $\{\mathcal{F}_t\}$- measurable random variables ψ with the norm

$$\|\psi\|^2 = \sup_{t \in [\tau,T]} E\|\psi(t)\|^2$$

DEFINITION 2.1. Consider the inhomogeneous problem where $f:[0,T] \to X$.

$$x'(t) = Ax(t) + f(t)$$
$$x(0) = x_0$$

Let A be the infinitesimal generator of a C0 semi group T(t). Let $x_0 \in X$ and $f \in L^1(0,T;X)$. Then the function $x \in C([0,T];X)$ is given by

$$x(t) = T(t)x_0 + \int_0^t T(t-s) f(s)ds, \quad 0 \le t \le T$$

is the mild solution of the above initial value problem for $t \in [0,T]$.

DEFINITION 2.2. A semigroup $\{T(t), t \ge 0\}$ is said to be uniformly bounded if there exists a constant $M \ge 1$ such that

$\| T(t) \| \le M, \ for \, t \ge 0$

DEFINITION 2.3. For a given $T \in (\tau, +\infty)$, a stochastic process $\{x(t) \in \mathcal{B}, \tau \le t \le T\}$ is called a mild solution to equation (2.1) in $(\Omega, P, \{\mathcal{F}_t\})$, if

(i) $x(t) \in X \ is \ \mathcal{F}_t - adapted;$

$$x(t) = \sum_{k=0}^{+\infty} \Bigg[\prod_{i=1}^{k} b_i(\tau_i) T(t-t_0)x_0 + \sum_{i=1}^{k} \prod_{j=1}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} T(t-s) f(s, x(s))ds + \int_{\xi_k}^{t} T(t-s) f(s, x(s))ds$$
$$+ \sum_{i=1}^{k} \prod_{j=1}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} T(t-s) \int_0^T f_1(\theta, x(s+\theta)) d\theta ds$$
$$+ \int_{\xi_k}^{t} T(t-s) \int_0^T f_1(\theta, x(s+\theta)) d\theta ds \Bigg] I_{[\xi_k, \xi_{k+1})}(t) \qquad t \in [\tau, T],$$

where,

$\prod_{j=m}^{n}(.) = 1 \, as \, m > n, \prod_{j=i}^{k} b_j(\tau_j) = b_k(\tau_k) b_{k-1}(\tau_{k-1}) ... b_i(\tau_i), \ and \ I_A(.) \, is \, the \, index \ function, i.e.,$

$$I_A(t) = \begin{cases} 1, if \ t \in A, \\ 0, if \ t \notin A. \end{cases}$$

3. MAIN RESULTS

In the section, we discuss the existence and uniqueness of the mild solution for the system (2.1). Before proving the main results, we introduce the following hypothesis which is used in our results.

(H1) The function $f : [t_0, T] \times C \to X$ satisfies the Lipschitz condition. That is, for $\zeta, \varsigma \in X \ and \ \tau \le t \le T$ there exists a constants $L_1 = L_1(T) \ and \ L_2 = L_2(T) > 0$ such that

$E\|f(t,\zeta)-f(t,\varsigma)\|^2\le L_1 E\|\zeta-\varsigma\|^2,$

$E\|f(t,\zeta)\|^2\le L_2(1+E\|\zeta\|^2),$

$E\|f(t,0)\|^2\le\kappa, where\,\kappa\ge 0\, is\, a$ constant.

(H2) The condition $\max_{i,k}\left\{\prod_{j=1}^{k}\|b_j(\tau_j)\|\right\}$ is uniformly bounded if, there is a constant $C>0$ such that

$$E\left\{\max_{i,k}\left\{\|\prod_{j=i}^{k}b_j(\tau_j)\|\right\}\right\}\le C \qquad for\ all\ \tau_j\in D_j,\ j=1,2,3,....$$

Theorem 3.1

Let the hypothesis **(H1) – (H2)** be hold. If the following inequality

$$\Lambda = M^2\max\{1,C^2\}(T-\tau)^2 L_1<1, \qquad (2.2)$$

is satisfied, then the system (2.1) has a unique mild solution in $\mathcal{B}$.

Proof.

Let T be an arbitrary number $\tau<T<+\infty$. First we define the nonlinear operator $\mathcal{S}:\mathcal{B}\to\mathcal{B}$ as follows

$$(\mathcal{S}x)(t)=\sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k}b_i(\tau_i)T(t-t_0)x_0+\sum_{i=1}^{k}\prod_{j=1}^{k}b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}T(t-s)f(s,x)ds+\int_{\xi_k}^{t}T(t-s)f(s,x)ds$$
$$+\sum_{i=1}^{k}\prod_{j=1}^{k}b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}T(t-s)\int_0^T f_1(\theta,x(s+\theta))d\theta ds$$

$$+\int_{\xi_k}^{t}T(t-s)\int_0^T f_1(\theta,x(s+\theta))d\theta ds\Bigg]I_{[\xi_k,\xi_{k+1})}(t) \qquad t\in[\tau,T],$$

It is easy to prove the continuity of S. Now, we have to show that S maps B into itself.

$$\|(\mathcal{S}x)(t)\|^2\le\Bigg[\sum_{k=0}^{+\infty}\Bigg[\|\prod_{i=1}^{k}b_i(\tau_i)\|\|T(t-t_0)\|\|x_0\|\sum_{k=0}^{k}\|\prod_{j=1}^{k}b_j(\tau_j)\|$$

$$\times\left\{\int_{\xi_{i-1}}^{\xi_i}\|T(t-s)f(s,x(s))\|ds\right\}+\int_{\xi_i}^{t}\|T(t-s)f(s,x(s))\|ds$$

$$+\sum_{k=0}^{k}\|\prod_{j=1}^{k}b_j(\tau_j)\|\left\{\int_{\xi_{i-1}}^{\xi_i}\|T(t-s)\int_0^T f_1(\theta,x(s+\theta))\|\,d\theta ds\right\}$$

$$\left.\left.+\int_{\xi_k}^{t}\|T(t-s)\int_0^T f_1(\theta,x(s+\theta))\|\,d\theta ds\right]I_{[\xi_k,\xi_{k+1})}(t)\right]^2$$

$$\leq 2\left[\sum_{k=0}^{+\infty}\left[\|\prod_{i=1}^{k}b_i(\tau_i)\|^2\|T(t-t_0)\|^2\|x_0\|^2\,I_{[\xi_k,\xi_{k+1})}(t)\right]\right.$$

$$+\left[\sum_{k=0}^{+\infty}\left[\sum_{k=0}^{k}\|\prod_{j=1}^{k}b_j(\tau_j)\|\left\{\int_{\xi_{i-1}}^{\xi_i}\|T(t-s)f(s,x(s))\|\,ds\right\}\right.\right.$$

$$\left.\left.+\int_{\xi_k}^{t}\|T(t-s)f(s,x(s))\|\,ds\right]I_{[\xi_k,\xi_{k+1})}(t)\right]^2$$

$$+\left[\sum_{k=0}^{+\infty}\left[\sum_{k=0}^{k}\|\prod_{j=1}^{k}b_j(\tau_j)\|\left\{\int_{\xi_{i-1}}^{\xi_i}\|T(t-s)\int_0^T f_1(\theta,x(s+\theta))\|\,d\theta ds\right\}\right.\right.$$

$$\left.\left.\left.+\int_{\xi_k}^{t}\|T(t-s)\int_0^T f_1(\theta,x(s+\theta))\|\,d\theta ds\right]\right]I_{[\xi_k,\xi_{k+1})}(t)\right]^2$$

$$\leq 2M^2\max_k\left\{\prod_{i=1}^{k}\|b_i(\tau_i)\|^2\right\}\|x_0\|^2$$

$$+2M^2\left[\max_{i,k}\left\{1,\prod_{j=1}^{k}\|b_j(\tau_j)\|\right\}\right]^2\left(\int_{t_0}^{t}\|f(s,x(s))\|\,ds\,I_{[\xi_k,\xi_{k+1})}(t)\right)^2$$

$$+2M^2\left[\max_{i,k}\left\{1,\prod_{j=1}^{k}\|b_j(\tau_j)\|\right\}\right]^2\left(\int_{t_0}^{t}\int_0^T\|f_2(\theta,x(s+\theta))\|\,d\theta ds\,I_{[\xi_k,\xi_{k+1}]}(t)\right)^2$$

$$\leq 2M^2C^2\|\varphi(0)\|^2+2M^2\max\{1,C^2\}\Big(\int_{t_0}^{t}\|f(s,x_s)\|\,ds\Big)^2$$

$$+2M^2\max\{1,C^2\}\left(\int_{t_0}^{t}\int_{0}^{T}\| f_1(\theta,x_{(s+\theta)})\| d\theta ds\right)^2$$

$$\leq 2M^2C^2\|x_0\|^2+2M^2\max\{1,C^2\}(t-t_0)\int_{t_0}^{t}\| f(s,x(s))\|^2 ds$$

$$+2M^2\max\{1,C^2\}(t-t_0)\int_{t_0}^{t}\left(\int_{0}^{T}\| f_1(\theta,x(s+\theta))\| d\theta ds\right)^2$$

$$\|(\mathcal{S}x)(t)\|^2\leq 2M^2C^2\|x_0\|^2+2M^2\max\{1,C^2\}(t-t_0)\int_{t_0}^{t}\| f(s,x(s))\|^2 ds$$

$$+2M^2\max\{1,C^2\}(t-t_0)T\int_{t_0}^{t}\int_{0}^{T}k(\theta)\|x(s+\theta)\|^2 d\theta ds$$

$$E\|(\mathcal{S}x)(t)\|^2\leq 2M^2C^2\|x_0\|^2+2M^2\max\{1,C^2\}(T-\tau)\int_{t_0}^{t}E\| f(s,x(s))\|^2 ds$$

$$+2M^2\max\{1,C^2\}(T-\tau)(T)\int_{t_0}^{t}\int_{0}^{T}k(\theta)E\|x(s+\theta)\|^2 d\theta ds$$

$$\leq 2M^2C^2\|x_0\|^2+4M^2\max\{1,C^2\}(T-\tau)^2L_2$$

$$+4M^2\max\{1,C^2\}(T-\tau)L_2\int_{t_0}^{t}E\|x(s)\|^2 ds+4M^2\max\{1,C^2\}(T-\tau)^2T^2L_2$$

$$+4M^2\max\{1,C^2\}(T-\tau)TL_2\int_{t_0}^{t}\int_{0}^{T}E\|x(s+\theta)\|^2 d\theta ds$$

Thus,

$$\sup_{t\in[\tau,T]}E\|(\mathcal{S}x)(t)\|^2\leq 2M^2C^2\|x_0\|^2+4M^2\max\{1,C^2\}(T-\tau)^2L_2$$

$$+4M^2\max\{1,C^2\}(T-\tau)L_2\int_{t_0}^{t}\sup_{s\in[\tau,t]}E\|x(s)\|^2 ds+4M^2\max\{1,C^2\}(T-\tau)^2T^2L_2$$

$$+4M^2\max\{1,C^2\}(T-\tau)TL_2\int_{t_0}^{t}\int_{0}^{T}\sup_{s\in[\tau,T]}E\|x(s+\theta)\|^2\,d\theta ds$$

$$\leq 2M^2C^2\|x_0\|^2+4M^2\max\{1,C^2\}(T-\tau)^2L_2$$

$$+4M^2\max\{1,C^2\}(T-\tau)^2L_2\sup_{t\in[\tau,t]}E\|x\|_t^2+4M^2\max\{1,C^2\}(T-\tau)^2T^2L_2$$

$$+4M^2\max\{1,C^2\}(T-\tau)^2TL\int_{0}^{T}\sup_{t\in[\tau,T]}E\|x(t+\theta)\|^2\,d\theta$$

for all $t\in[\tau,T]$,

Therefore $\mathcal{S}$ *maps* $\mathcal{B}$ into itself

Now, we have to show $\mathcal{S}$ is a contraction mapping

$$\|(\mathcal{S}x)(t)-(\mathcal{S}y)(t)\|^2\leq\left[\sum_{k=0}^{+\infty}\left[\sum_{i=1}^{k}\prod_{j=i}^{k}\|b_j(\tau_j)\|\int_{\xi_{i-1}}^{\xi_i}\|T(t-s)\|\,\|f(s,x(s))-f(s,y(s))\|\,ds\right.\right.$$

$$\left.\left.+\int_{\xi_k}^{t}\|T(t-s)\|\,\|f(s,x(s))-f(s,y(s))\|\,ds\right]I_{[\xi_k,\xi_{k+1})}(t)\right]^2$$

$$+\left[\sum_{k=0}^{+\infty}\left[\sum_{i=1}^{k}\prod_{j=i}^{k}\|b_j(\tau_j)\|\int_{\xi_{i-1}}^{\xi_i}\|T(t-s)\|\int_{0}^{T}\|f_1(\theta,x(s+\theta))-f(\theta,y(s+\theta))\|\,d\theta ds\right.\right.$$

$$\left.\left.+\int_{\xi_k}^{t}\|T(t-s)\|\int_{0}^{T}\|f_1(\theta,x(s+\theta))-f(\theta,y(s+\theta))\|\,d\theta ds\right]I_{[\xi_k,\xi_{k+1})}(t)\right]^2$$

$$\leq M^2\left[\max_{i,k}\left\{1,\prod_{j=i}^{k}\|b_j(\tau_j)\|\right\}\right]^2\left(\int_{t_0}^{t}\|f(s,x(s))-f(s,y(s))\|\,ds\,I_{[\xi_k,\xi_{k+1})}(t)\right)^2$$

$$+M^2\left[\max_{i,k}\left\{1,\prod_{j=i}^{k}\|b_j(\tau_j)\|\right\}\right]^2\left(\int_{t_0}^{t}\int_{0}^{T}\|f_1(\theta,x(s+\theta))-f(\theta,y(s+\theta))\|\,d\theta ds\,I_{[\xi_k,\xi_{k+1})}(t)\right)^2$$

$$\leq M^2\max\{1,C^2\}(t-t_0)\int_{t_0}^{t}\|f(s,x(s))-f(s,y(s))\|^2\,ds$$

$$+M^2 \max\{1, C^2\}(t-t_0)T \int_{t_0}^{t}\int_{0}^{T} \| f_1(\theta, x(s+\theta)) - f_1(\theta, y(s+\theta)) \|^2 d\theta\, ds$$

$$E\| (\mathcal{S}x)(t) - (\mathcal{S}y)(t) \|^2 \leq M^2 \max\{1, C^2\}(t-t_0) \int_{t_0}^{t} E\| f(s, x(s)) - f(s, y(s)) \|^2 ds$$

$$+M^2 \max\{1, C^2\}(t-t_0)(T) \int_{t_0}^{t}\int_{0}^{T} E\| f_1(\theta, x(s+\theta) - f_1(\theta, y(s+\theta)) \|^2 d\theta\, ds$$

$$\leq M^2 \max\{1, C^2\}(T-\tau) L_1 \int_{t_0}^{t} E\| x(s) - y(s) \|^2 ds$$

$$+M^2 \max\{1, C^2\}(T-\tau)(T) L_1 \int_{t_0}^{t}\int_{0}^{T} E\| x(s+\theta) - y(s+\theta) \|^2 d\theta\, ds$$

Taking supremum over t, we get,

$$\| (\mathcal{S}x) - (\mathcal{S}y) \|^2 \leq M^2 \max\{1, C^2\}(T-\tau)^2 L_1 \| x - y \|^2$$

$$+M^2 \max\{1, C^2\}(T-\tau)^2 (T) L_1 \int_{0}^{T} E(\sup_{t\in[\tau,T]} \| x(t+\theta) - y(t+\theta) \|^2) d\theta$$

$$\leq \Lambda \| x - y \|^2 + \Lambda \int_{0}^{T} E(\sup_{t\in[\tau,T]} \| x(t+\theta) - y(t+\theta) \|^2) d\theta$$

By applying the Gronwall inequality in the above equation, it follows that

$$\| (\mathcal{S}x) - (\mathcal{S}y) \|^2 \leq \Lambda \| x - y \|^2$$

Since $0 < \Lambda < 1$. This shows that the operator $\mathcal{S}$ satisfies the contraction mapping principle and therefore, $\mathcal{S}$ has a unique fixed point which is the mild solution of the system (2.1). This completes the proof.

REFERENCES

[1] A.Anguraj and A.Vinodkumar, Existence, uniqueness and stability results of random impulsive semilinear differential systems, Nonlinear Analysis Hybrid systems, 3(2010), 475 – 483.

[2] A.Anguraj, S.Wu and A.Vinodkumar, Existence and exponential stabilty of semilinear functional differential equations with random impulses under non – uniqueness, Nonlinear Analysis: Theory, methods & Applications, 74(2011), 331-342.

[3] A.Anguraj and A.Vinodkumar, Existence and uniqueness of neutral functional differential equations with random impulses, International Journal of nonlinear science, 8(4)(2009) 1135-1158.

[4] A.Vinodkumar, Existence, uniqueness of solutions for random impulsive differential equation, Malaya Journal of Matematik 1(1)(2012)8-13.

[5] K.Malar and A.Vinodkumar, Existence, Uniqueness and stability of random impulsive fractional differential equations, Acta Mathematica Scientia 2016, 36B(2):428–442.

[6] K. Karthikeyan, A. Anguraj, K. Malar and Juan J. Trujilo, Existence of Mild and Classical Solutions for Nonlocal Impulsive Integrodifferential Equations in Banach Spaces with Measure of Noncompactness, Hindawi Publishing Corporation International Journal of Differential Equations, Volume 2014, Article ID 319250, 10 pages.

[7] V.Lakshmikanthan, D.D.Bainov and P.S.Simeonov, Theory of Impulsive Differential Equations, world scientific, Singapore, 1989.

[8] A.M.Samoilenko and N.A.Perestyuk, Impulsive Differential Equations, World Scientific, Singapore, 1995.

[9] J.M.Sanz-Serna and A.M.Stuart, Ergodicity of dissipative differential equations subject to random impulses, J.Differential Equations, 155(1999),262-284.

[10] A.Vinodkumar, Existence results on random impulsive semilinear functional differential inclusions with delays, Ann. Funct. Anal., 3(2012), 89-106.

[11] A.Vinodkumar and A.Anguraj, Existence of random impulsive abstract neutral non-autonomous differential inclusions with delays, Nonlinear Anal. Hybird Systems, 5(2011), 413426.

[12] A.Vinodkumar, Stability results of Random impulsive semilinear differential equations, Science direct, Acta Mathematica Scientica 2014, 34(B):1055-1071.

[13] S.J.Wu and X.Z.Meng, Boundedness of nonlinear differential systems with impulsive effect on random moments, Acta Math. Appl., Sin., 20(1)(2004),147-154.

[14] S.J.Wu and Y.R.Duan, Oscillation, stability, and boundedness of second-order differential systems with random impulses, Comput. Math. Appl., 49(9-10)(2005),1375-1386.

[15] S.J.Wu,X.L.Guo and S.Q.Lin, Existence and uniqueness of solutions to random impulsive differential systems, Acta Math.Appl. Sin., 22(4)(2006),595-600.

[16] S.J.Wu, X.L. Guo and Y.Zhou, p-moment stability of functional differential equations with random impulses, Comput. Math. Appl., 52(2006), 1683-1694.