Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 14, Number 6 (2018), pp. 775–785

http://www.ripublication.com/gjpam.htm

Approximate Controllability of the Stokes system

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Abstract

In this paper we establish some approximate controllability results for system of parabolic equations of the Stokes kind. Motivated by the solution of controllability problems for the Navier-Stokes equations modeling incompressible viscous flow, we will discuss now controllability issues for a system of partial differential equations which is not of the Cauchy-Kowalewska type, namely, the classical Stokes system.

AMS subject classification: 93B05, 35K40, 76D07.
Keywords: Stokes system, Controllability, Control optimal.

1. Generalities. Synopsis

In the original text (that is, Acta Numerica [17]), the content of this work was considered as a preliminary step to a more ambitious goal, namely, the control of systems governed by the Navier-Stokes equations modeling incompressible viscous flow. Indeed, substantial progress concerning this objective took place in the late 1990s.

Back to the original text, let us say that the control problems and methods which have been discussed so far in this book have been mostly concerned with systems governed by linear diffusion equations of the parabolic type, associated with second order elliptic operators. Indeed, these methods have been applied in, for example, Berggren

[19]. Glowinski, and J.L. Lions [16], to the solution of approximate boundary controllability problems for systems governed by strongly advection dominated linear advection-diffusion equations. These methods can also be applied to systems of linear advection-diffusion equations and to higher-order parabolic equations (or systems of such equations). Motivated by the solution of controllability problems for the Navier-Stokes equations modeling incompressible viscous flow, we will discuss now controllability issues for a system of partial differential equations which is not of the Cauchy-Kowalewska type, namely, the classical Stokes system.

2. Formulation of the Stokes system. A fundamental controllability result

In the following, we equip the Euclidean space $\mathbb{R}^n$ $(n \geq 2)$ with its classical scalar product and with the corresponding norm, that is,

$$\left| \begin{array}{l} a.b = \sum_{i=1}^{n} a_i b_i, \;\; \forall a = \{a_i\}_{i=1}^{n}, b = \{b_i\}_{i=1}^{n} \in \mathbb{R}^n; \\ |a| = (a.a)^{\frac{1}{2}}, \;\; \forall a \in \mathbb{R}^n. \end{array} \right.$$

Let Ω be a bounded connected open set (that is, a bounded domain) in $\mathbb{R}^n$.

We shall also assume that $\Gamma = \partial\Omega$ is "sufficiently smooth," which is also not mandatory. Let O be an open subset of Ω. We emphasize here, at the very beginning, that O can be arbitrary "small". The control function u will be with support in O; it is a distributed control. The state equation is given by

$$\left\{ \begin{array}{rcll} \frac{\partial y}{\partial t} - \Delta y & = & u\chi_O - \nabla \pi & \text{in } \; Q, \\ \nabla . y & = & 0 & \text{in } \; Q, \end{array} \right. \tag{2.1}$$

subject to the following initial and boundary conditions:

$$y(0) = 0 \;\text{ and }\; y = 0 \;\text{ on }\; \Sigma (= \Gamma \times (0,T)). \tag{2.2}$$

where χ_O is the characteristic function of the set O. From now on, we shall denote by Q the space-time domain $\Omega \times (0,T)$.

In (2.1), we shall assume that

$$u \in \mathcal{U} = \;\text{ a closed subspace of }\; \left(L^2 (O \times (0,T))\right)^n \tag{2.3}$$

To fix ideas, we shall take $n = 3$, and consider the following cases for $\mathcal{U}$:

$$\mathcal{U} = \left(L^2 (O \times (0,T))\right)^3, \tag{2.4}$$

$$\mathcal{U} = \left\{ u, u = \{u_1, u_2, 0\}, \{u_1, u_2\} \in \left(L^2 (O \times (0,T))\right)^2 \right\}, \tag{2.5}$$

$$\mathcal{U} = \{u, u = \{u_1, 0, 0\}, u_1 \in L^2(O \times (0,T))\}. \tag{2.6}$$

Problem (2.1), (2.2) has a unique solution, such that (in particular)

$$\left| \begin{array}{l} y(u) \in L^2\left(0,T;\left(H_0^1(\Omega)\right)^3\right), \\ \nabla . y(u) = 0, \\ \dfrac{\partial y(u)}{\partial t} \in L^2(0,T;V'), \end{array} \right. \tag{2.7}$$

where V' is the dual space of

$$V = \left\{\varphi, \varphi \in \left(H_0^1(\Omega)\right)^3,\ \nabla.\varphi = 0\right\}. \tag{2.8}$$

Above, $H^1(\Omega)$ and $H_0^1(\Omega)$ are the functional spaces defined as follows:

$$H^1(\Omega) = \left\{\phi, \phi \in L^2(\Omega), \frac{\partial \phi}{\partial x_i} \in L^2(\Omega),\ \forall i = 1, \ldots, n\right\},$$

and

$$H_0^1(\Omega) = \{\phi, \phi \in H^1(\Omega),\ \phi = 0 \text{ on } \Gamma\}.$$

It follows from (2.7) that

$$t \longrightarrow y(t;u) \text{ belongs to } \mathcal{C}^0([0,T];H), \tag{2.9}$$

where

$$\begin{aligned} H &= \text{closure of } V \text{ in } \left(L^2(\Omega)\right)^3 \\ &= \left\{\varphi, \varphi \in \left(L^2(\Omega)\right)^3,\ \nabla.\varphi = 0,\ \varphi.\upsilon = 0 \text{ on } \Gamma\right\} \end{aligned} \tag{2.10}$$

(υ denotes the unit outward normal vector at Γ).

We are now going to prove the following:

Proposition 2.1. If V is defined by either (2.4) or (2.5), then the space spanned by $y(T;u)$ is dense in H.

Proof. It suffices to prove the above results for the case where V is defined by (2.5).

Let us therefore consider $f \in H$ such that,

$$\int_\Omega y(T;u).f dx = 0,\ \forall u \in \mathcal{U}. \tag{2.11}$$

With f we associate the solution q of the following backward Stokes problem:

$$\left\{ \begin{array}{rcll} -\dfrac{\partial q}{\partial t} - \Delta q &=& -\nabla\sigma & \text{in } Q, \\ \nabla.q &=& 0 & \text{in } Q, \end{array} \right. \tag{2.12}$$

$$q(T) = f, \;\; q = 0 \;\text{ on }\; \Sigma. \tag{2.13}$$

Multiplying the first equation in (2.12) by $y = y(u)$ and integrating by parts we obtain

$$\int_Q q.udxdt = 0, \;\; \forall u \in \mathcal{U}. \tag{2.14}$$

Suppose that $q = \{q_1, q_2, q_3\}$; it follows then from (2.14) that

$$q_1 = q_2 = 0 \;\text{ in }\; O \times (0,T). \tag{2.15}$$

Since q is (among other things) continuous in t and real analytic in x in $Q(= \Omega \times (0,T))$, it follows from (2.15) that

$$q_1 = q_2 = 0 \;\text{ in }\; Q. \tag{2.16}$$

Since (from (2.12)) $\nabla.q = 0$, it follows from (2.16) that $\frac{\partial q_3}{\partial x_3} = 0$ in Q, which combined with the boundary condition $q_3 = 0$ on Σ implies that $q_3 = 0$ on Q; the t-continuity of q implies that $q(T) = 0$, that is, $f = 0$ (from (2.13)), which completes the proof. ■

Remark 2.2. The above density result does not always hold if $\mathcal{U}$ is defined by (2.6), as shown in Diaz and Fursikov [15].

Remark 2.3. Proposition 2.1 was proved in the lectures of the second author (J.L. Lions) at College de France in 1990-91. Other results along these lines are due to Fursikov [20].

The density result in Proposition 2.1 implies (at least) approximate controllability. Thus, we shall formulate and discuss, in the following sections, two approximate controllability problems.

3. Two approximate controllability problems

The first problem is defined by

$$\min_{u \in \mathcal{V}_f} \frac{1}{2} \int_Q |u|^2 \, dxdt, \tag{3.1}$$

where

$$\mathcal{V}_f = \{u, u \in \mathcal{U}, (u,y) \text{ verifies (2.1), (2.2), and } y(T) \in y_T + \beta B_H\}; \tag{3.2}$$

in (3.2), y_T is given in H, β is a positive number arbitrarily small, B_H is the unit ball of H and “to fix ideas” the control space $\mathcal{U}$ is defined by (2.5).

The second problem is obtained by penalization of the final condition $y(T) = y_T$; we have then

$$\min_{u \in \mathcal{U}} \left\{ \frac{1}{2} \int_Q |u|^2 \, dxdt + \frac{1}{2} k \int_\Omega |y(T) - y_T|^2 \, dx \right\}, \tag{3.3}$$

where, in (3.3), k is an arbitrarily large positive number, y is obtained from u via (2.1), (2.2), and $\mathcal{U}$ is as above.

It follows from Proposition 2.1 that both control problems (3.1) and (3.3) have a unique solution.

4. Optimality conditions and dual problems

We start with problem (3.3), since it is (by far) simpler than problem (3.1). Ifwe denote by J_k the cost functional in (3.3), we have

$$\begin{aligned} \left(J_k'(u), w\right) &= \lim_{\substack{\theta\to 0\\ \theta\neq 0}} \frac{J_k(u+\theta w) - J_k(u)}{\theta} \\ &= \int_Q (u-\psi).wdxdt, \ \forall u, w \in \mathcal{U}, \end{aligned} \tag{4.1}$$

where, in (4.1), the adjoint velocity field ψ is solution of the following backward Stokes problem:

$$\begin{cases} -\dfrac{\partial\psi}{\partial t} - \Delta\psi + \nabla\sigma &= 0 \quad \text{in} \quad Q, \\ \nabla.\psi &= 0 \quad \text{in} \quad Q, \end{cases} \tag{4.2}$$

$$\psi = 0 \ \text{ on } \ \Sigma, \ \ \psi(T) = k\left(y(T) - y_T\right). \tag{4.3}$$

Suppose now that v is the unique solution of problem (3.3); it is characterized by

$$\left| \begin{array}{l} v \in \mathcal{U}, \\ \left(J_k'(v), w\right) = 0, \ \ \forall w \in \mathcal{U}, \end{array} \right. \tag{4.4}$$

which implies in turn that the optimal triple $\{v, y, \psi\}$ is characterized by

$$\left| \begin{array}{l} v_1 = \psi_1|_O, \\ v_2 = \psi_2|_O, \\ v_3 = 0, \end{array} \right. \tag{4.5}$$

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y + \nabla\pi &= v\chi_O \quad \text{in} \quad Q, \\ \nabla.y &= 0 \quad\ \ \text{in} \quad Q, \end{cases} \tag{4.6}$$

$$y(0) = 0, \ \ y = 0 \ \text{ on } \ \Sigma, \tag{4.7}$$

to be completed by (4.2), (4.3).

To obtain the dual problem of (3.3) from the above optimality conditions, we proceed by introducing an operator $\Lambda \in \mathcal{L}(H, H)$ defined as follows:

$$\Lambda g = \Phi_g(T), \ \ \forall g \in H, \tag{4.8}$$

where to obtain $\Phi_g(T)$ we solve first

$$\begin{cases} -\dfrac{\partial \digamma_g}{\partial t} - \Delta \digamma_g + \nabla \sigma_g &= 0 \text{ in } Q, \\ \nabla . \digamma_g &= 0 \text{ in } Q, \end{cases} \tag{4.9}$$

$$\digamma_g(T) = g, \;\; \digamma_g = 0 \;\text{ on }\; \Sigma, \tag{4.10}$$

and then (with obvious notation)

$$\begin{cases} -\dfrac{\partial \phi_g}{\partial t} - \Delta \phi_g + \nabla \pi_g &= \{\digamma_{1g}, \digamma_{2g}, 0\} \chi_O \text{ in } Q, \\ \nabla . \Phi_g &= 0 \quad\quad\quad\quad\quad\;\; \text{ in } Q, \end{cases} \tag{4.11}$$

$$\Phi_g(0) = 0, \;\; \Phi_g = 0 \;\text{ on }\; \Sigma, \tag{4.12}$$

(the two above Stokes problems are well posed).

Integrating by parts in time and using Green's formula, we can show (again with obvious notation) that

$$\int_\Omega (\Lambda g) . g' dx = \int_Q \left(\digamma_1 \digamma_1' + \digamma_2 \digamma_2' \right) dxdt, \;\; \forall g, g' \in H. \tag{4.13}$$

It follows from relation (4.13) that the operator Λ is symmetric and positive semi definite over H; indeed, using the approach taken in Section 2 to prove Proposition 2.1, we can show that the operator Λ is positive definite over H. Back to the optimality conditions, let us denote by f the function $\psi(T)$; it follows then from (4.3) and from the definition of Λ that f satisfies

$$k^{-1} f + \Lambda f = y_T \tag{4.14}$$

which is precisely the dual problem of (3.3). From the symmetry and positivity of Λ, the dual problem (4.14) can be solved by a conjugate gradient algorithm operating in the space H.

We consider now the control problem (3.1); applying, as done previously, the Fenchel-Rockafellar duality theory it can be shown that the unique solution v of problem (3.1) can be obtained via

$$\left| \begin{array}{l} v_1 = \psi_1 \chi_O, \\ v_2 = \psi_2 \chi_O, \\ v_3 = 0, \end{array} \right. \tag{4.15}$$

where, in (4.15), ψ is the solution of the following backward Stokes problem:

$$\begin{cases} -\dfrac{\partial \psi}{\partial t} - \Delta \psi + \nabla \sigma &= 0 \text{ in } Q, \\ \nabla . \psi &= 0 \text{ in } Q, \end{cases} \tag{4.16}$$

$$\psi(T) = f, \;\; \psi = 0 \;\text{ on }\; \Sigma, \tag{4.17}$$

where, in (4.17), f is the solution of the following variational inequality:

$$\left| \begin{array}{l} f \in H, \ \forall g \in H, \\ \displaystyle\int_\Omega (\Lambda f) \cdot (g-f)\, dx + \beta \|g\|_H - \beta \|f\|_H \geq \int_\Omega y_T \cdot (g-f)\, dx, \end{array} \right. \tag{4.18}$$

with $\|g\|_H = \left(\int_\Omega |g|^2 \, dx \right)^{\frac{1}{2}}$. Problem (4.18) can be viewed as the dual of problem (3.1).

5. Application to sentinels

5.1. Definition

The notion of sentinel was introduced by J.L.Lions to study systems of incomplete data [18]. The notion permits to distinguish and to analyze two types of incomplete data: the so called pollution terms on which we look for information's, independently of the other type of incomplete data which is the missing terms, and that we do not want to identify.

Typically, the Lions' sentinel is a functional defined from an open set O on which we consider three functions: the "observation" y_{obs} corresponding to measurements, a given "mean" function h_0, and a control function u to be determined.

In this article, we propose a notion of sentinel which revisits the one of Lions.

Let us remind that Lions' sentinel theory [18] relies on the following three features: the state equation y which is governed by a system of PDE, the observation system and some particular evaluation function: the sentinel itself.

More precisely, we consider in the first step the Navier-Stokes system

$$\left\{ \begin{array}{rcll} \dfrac{\partial y}{\partial t} - \Delta y + \nabla \pi & = & \xi + \lambda \widehat{\xi} & \text{in } Q \\ \operatorname{div} y & = & 0 & \text{in } Q \\ y(0) & = & y_0 + \tau \widehat{y_0} & \text{in } \Omega \\ y & = & 0 & \text{on } \Sigma \end{array} \right. \tag{5.1}$$

We are interested in systems with data that are not completely known. The functions ξ and y_0 are known with ξ in $L^2(Q)$ and y_0 in $L^2(\Omega)$. However, the terms $\lambda\widehat{\xi}$ and $\tau\widehat{y_0}$ are unknown, but are such that

$$\left| \begin{array}{l} \left\| \widehat{\xi} \right\|_{L^2(Q)} \leq 1, \ \ \|\widehat{y_0}\|_{L^2(\Omega)} \leq 1, \\ \text{and that the reals } \lambda \text{ and } \tau \text{ are small enough.} \end{array} \right. \tag{5.2}$$

This growth condition is classical (see [14]). Under this growth condition, it is proved in **[16]**, p. 63 that there exists $\alpha > 0$ such that when

$$\left\| \xi + \lambda \widehat{\xi} \right\|_{L^2(Q)} + \|y_0 + \tau \widehat{y_0}\|_{L^2(\Omega)} \leq \alpha$$

the problem (5.1) admits a unique solution. For the sake of simplicity, we denote

$$y(x,t;\lambda,\tau) = y(\lambda,\tau) \tag{5.3}$$

The general question we want to address is

$$\left| \begin{array}{l} \text{given some observation of the state of system, can one obtain} \\ \lambda\widehat{\xi} \text{ without any attempt at computing } \tau\widehat{y_0}? \end{array} \right. \tag{5.4}$$

In this context, we refer to $\lambda\widehat{\xi}$ as pollution term, the one we are trying to identify, and the term $\tau\widehat{y_0}$ as the missing one that on which we do not want to identify.

To make things more specific, we consider in the second step the observation process. The observation is the knowledge, along some time period, of some function y_{obs} which is defined on the strip $O \times (0,T)$ over some nonempty open subset $O \subset \Omega$, called observatory. The function y_{obs} is assumed to be of the form

$$y_{obs} = m_0 + \sum_{i=1}^{M} \eta_i m_i. \tag{5.5}$$

where the functions $m_0, m_1, \ldots, m_M$ are given measurements of y in $L^2(O \times (0,T))$, but where the real coefficients η_i are unknown. We assume that η_i are small. We refer to the terms $\eta_i m_i$ as the interference terms. We can assume without loss of generality that

$$\text{the functions } m_i \text{ are linearly independent on } O \times (0,T). \tag{5.6}$$

Finally, we introduce now the notion of sentinel. Let h_0 be a given function on $O \times (0,T)$ such that

$$h_0 \geq 0, \quad \int_0^T \int_O h_0 dxdt = 1. \tag{5.7}$$

Moreover let ω be an open and non-empty subset of Ω. For any control function $u \in L^2(\omega \times (0,T))$, set

$$\mathcal{S}(\lambda,\tau) = \int_0^T \int_\omega u y(\lambda,\tau) dxdt + \int_0^T \int_O h_0 y(\lambda,\tau) dxdt. \tag{5.8}$$

The role of the function u appears in the following definition. We shall say that $\mathcal{S}$ defines a discriminating sentinel (for the system (5.1), (5.5) and (5.7)) if there exists u such that the functional $\mathcal{S}$ satisfies the following conditions:

(i) $\mathcal{S}$ is stationary at first order with respect to the missing terms $\tau\widehat{y_0}$, for all $\varepsilon \geq 0$ we have

$$\left| \frac{\partial \mathcal{S}(0,0)}{\partial \tau} \right| \leq \varepsilon, \quad \forall \widehat{y_0} \tag{5.9}$$

(ii) $\mathcal{S}$ is stationary with respect to the interference terms $\eta_i m_i$, that is

$$\int_0^T \int_\omega u m_i dxdt + \int_0^T \int_O h_0 m_i dxdt = 0, \quad 1 \le i \le M. \tag{5.10}$$

(iii) u is of minimal norm in $L^2(\omega \times (0,T))$ among control functions in $L^2(\omega \times (0,T))$ which satisfy the above conditions. That is

$$\|u\|_{L^2(\omega\times(0,T))} = \text{ minimum.} \tag{5.11}$$

Remark 5.1. At this point, some comments must be made

1. According to (5.11), the function $\mathcal{S}$ if it exists, is unique. We refer to $\mathcal{S}$ as the sentinel.

2. If the functions m_i, $1 \le i \le M$, are null functions, the sentinel $\mathcal{S}$ is defined only by (5.9) and (5.11). If $m_i \neq 0$, the sentinel $\mathcal{S}$ is defined only by (5.9), (5.10) and (5.11) and it is called a discriminating sentinel.

3. The original Lions' sentinel $\mathcal{S}$ corresponds to the case $\omega = O$. In this case; if we choose $u = -h_0$, then (5.9) and (5.10) hold true, so that problem (5.9)-(5.10) admits a unique solution. Of course this solution may have an interest only if $u \neq -h_0$. Now, $\omega \neq O$ and if the support supp(h) of h does not in ω, we cannot have $u = -h_0$ except when $u = -h_0 = 0$. Therefore, the previous definition introduces a generalization of Lions's discriminating sentinel to the case where the observation and the control have their supports in two different open subsets.

4. The support supp(m_i) of functions m_i is assumed to be included in O. Suppose $\omega \cap O = \emptyset$ then $\int_0^T \int_\omega u m_i dxdt = 0$. Therefore, it suffices to choose h_0 such that h_0 is orthogonal to each m_i and then (5.10) would be readily verified. Therefore, for all ω can neglect the part of ω which is out of O. So, without loss of generality, it may be assumed that

$$\omega \subset O \tag{5.12}$$

5.2. Orientation

A detailed study of the notion of sentinel is given in [13], in particular sufficient conditions on h_0 to ensure the existence of a sentinel. We can also find in [13] other examples of sentinels in the case $\omega \subset O$. They lead to new controllability problems with constraints on the control.

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