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Best Proximity and Fixed Point Results in Complex Valued Rectangular Metric Spaces

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Abstract

The concept of best proximity points for non-self mappings between two subsets in the setting of complex valued rectangular metric spaces, is introduced. During the process the concept of P- property is defined and utilized in the same spaces. As a consequence, certain fixed point results in the complex valued rectangular metric spaces are also obtained. Our results are substantiated by some useful examples.

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1. Introduction and Preliminaries

If T is a non self mapping then it is not necessary that the fixed point equation $x = Tx$ has a solution, in this case it is of a certain interest to determine an approximate solution x that is optimal in the sence that the distance between x and Tx is minimum. In this context best proximity point theory is an useful tool in studying such kind of elements. The concept of best proximity point is introduced by Eldered and Veeramani [7], which reduces in fixed point when the underlying map is self map. The best proximity point theorems for many contraction are proved by different authors ([2],[5],[8]-[12],[15]-[18],[20],[21]). On the other hand Branciary [4] introduced the notion of rectangular metric space and proved an analogue of the Banach contraction principle in this space. The concept of complex valued metric spaces is introduced by Azam et al. [3] and proved fixed point result in this space. Choudhury et al. [6] introduced the concept of best proximity points for non-self maps between two subsets of complex valued metric spaces. Recently the complex valued rectangular (generalized) metric space is introduced by Abbas et al. [1] and obtained common fixed point result for mappings in such spaces.

In this article the best proximity point in complex valued rectangular metric space is introduced and proved best proximity point result by using the concept of P-property with the supportive example and proved fixed point results as it's consequences.

Consistent with Azam et al. [3], the following definitions and results will be needed in the sequel. Let C be the set of complex numbers and $z_1, z_2 \in C$. Define a partial order $\precsim$ on C as follows :

$$z_1 \precsim z_2 \ if \ and \ only \ if \ Re(z_1) \leq Re(z_2) \ and \ Im(z_1) \leq Im(z_2).$$

It follows that $z_1 \precsim z_2$ if one of the following conditions are satisfied :
(C1) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$;
(C2) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$;
(C3) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$;
(C4) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.
In particular, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of $(C2)$, $(C3)$ and $(C4)$ is satisfied and we write $z_1 \prec z_2$ if only $(C4)$ is satisfied. Note that

$$0 \precsim z_1 \precnsim z_2 \Rightarrow |z_1| < |z_2|,$$
$$z_1 \precsim z_2, z_2 \prec z_3 \Rightarrow z_1 \prec z_3.$$

Definition 1.1. [3] Let X be a nonempty set such that the mapping $d : X \times X \to C$ satisfies the following conditions:

(CM1) $0 \precsim d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0 \Leftrightarrow x = y$;

(CM2) $d(x, y) = d(y, x)$, for all $x, y \in X$;

(CM3) $d(x, y) \precsim d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a complex valued metric on X and (X, d) is called a complex valued metric space.

Abbas et al. [1] defined the notion of complex valued rectangular(generalized) metric spaces as follows:

Definitionn 1.2. [1] Let X be a non empty set. If a mapping $d : X \times X \to C$ satisfies:

(a) $0 \precsim d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0 \Leftrightarrow x = y$;

(b) $d(x, y) = d(y, x)$, for all $x, y \in X$;

(c) $d(x, y) \precsim d(x, u) + d(u, v) + d(v, y)$ for all $x, y \in X$ and all distinct $u, v \in X$, each one is different from x and y.

Then d is called a complex valued rectangular (generalized) metric on X and (X, d) is called a complex valued rectangular (generalized) metric space.

Example 1.3. **[13]** Let $X = \{i, -i, 1, -1\}$, and defined $d : X \times X \to C$ as follows: $d(1, -1) = d(-1, 1) = 3e^{i\theta}$,

$$d(-1, i) = d(i, -1) = d(1, i) = d(i, 1) = e^{i\theta}$$
$$d(1, -i) = d(-i, 1) = d(-1, -i) = d(-i, -1) = d(i, -i) = d(-i, i) = 4e^{i\theta}$$
$$d(1, 1) = d(-1, -1) = d(i, i) = d(-i, -i) = 0.$$

Then (X, d) is complex valued rectangular(generalized) metric space when $\theta \in [0, \frac{\pi}{2}]$. But (X, d) is not a complex valued metric space, since $d(-1, 1) = 3e^{i\theta} > d(-1, i) + d(i, 1) = 2e^{i\theta}$.

Definition 1.4. **[1]** Let (X, d) be a complex valued rectangular (generalized) metric space and $\{x_n\}$ be a sequence in X.

1. If for every $c \in C$ with $0 \prec c$, there exist $n_0 \in N$ such that $d(x_n, x) \prec c$ for all $n > n_0$, then $\{x_n\}$ is said to be convergent to $x \in X$, and we denote this by $x_n \to x$ as $n \to \infty$ or $lim_{n\to\infty} x_n = x$.

2. If for every $c \in C$ with $0 \prec c$, there exist $n_0 \in N$ such that for all $n, m > n_0$, $d(x_n, x_m) \prec c$, then $\{x_n\}$ is called Cauchy sequence in X.

3. If every Cauchy sequence in X is convergent in X, then (X, d) is called a complete complex valued rectangular(generalized) metric space.

Verma et al. [22] define max function for complex numbers as follows.

Definition 1.5. The max function for complex numbers with partial order relation $\precsim$ is defined as

1. $\max\{z_1, z_2\} = z_2 \Rightarrow z_1 \precsim z_2$;

2. $z_1 \precsim \max\{z_2, z_3\} \Rightarrow z_1 \precsim z_2$ or $z_1 \precsim z_3$.

On the similar lines Singh et al.[19] defined min function as

1. $\min\{z_1, z_2\} = z_1 \Rightarrow z_1 \precsim z_2$;

2. $\min\{z_1, z_2\} \precsim z_3 \Rightarrow z_1 \precsim z_3$ or $z_2 \precsim z_3$.

Now we introduce the best proximity point and some related concept in complex valued rectangular metric space.

Definition 1.6. Let A and B be two non empty bounded subsets of a complex valued rectangular metric space (X, d). Then $\{d(x, y) : x \in A, y \in B\}$ is always bounded below by $z_0 = 0 + i0$ and hence $\inf\{d(x, y) : x \in A, y \in B\}$ exists. Here we define

$$d(A, B) = \inf\{d(x, y) : x \in A, y \in B\},$$

$$A_0 = \{x \in A : d(x, y) = d(A, B) \;\; for \;\; some \;\; y \in B\},$$

$$B_0 = \{y \in B : d(x, y) = d(A, B) \;\; for \;\; some \;\; x \in A\}.$$

From the above definition, it is clear that for every $x \in A_0$ there exists $y \in B_0$ such that $d(x, y) = d(A, B)$ and conversely, for every $y \in B_0$ there exists $x \in A_0$ such that $d(x, y) = d(A, B)$.

Definition 1.7. Let A and B be two nonempty bounded subsets of a complex valued rectangular metric space (X, d) and $T : A \to B$ be a non-self-mapping. A point $x \in A$ is called a best proximity point of T if $d(x, Tx) = d(A, B)$.

The definition of P-property was introduced in [14]. Now we define them in complex valued rectangular metric space.

Definition 1.8. Let A and B be two nonempty subsets of a complex valued rectangular metric space (X, d) with $A_0 \neq \phi$. Then the pair (A, B) is said to have the P-property if, for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$

$$d(x_1, y_1) = d(A, B) \text{ and } d(x_2, y_2) = d(A, B) \Rightarrow d(x_1, x_2) = d(y_1, y_2)$$

2. Main result

In context of Matkowski [10] the function $\phi : [0, \infty)^2 \to [0, \infty)^2$ such that $\phi(t) \prec t$ and $\phi(0) = 0$ [whrer $t = (t_1, t_2) \in [0, \infty)^2$] is considered in whole article. Family of such type of function is denoted by Φ.

Example 2.1. The simple function $\phi(t_1, t_2) = \left(\frac{t_1}{2}, \frac{t_2}{2}\right)$ is well defined and satisfies the conditions of functions in Φ.

Theorem 2.2. Let A and B be two non-empty closed and bounded subsets of a complete complex valued rectangular metric space $(X, d)$with pair (A, B) satisfies the P-property. Let a continuous mapping $T : A \to B$ with $T(A_0) \subset B_0$, where A_0 is non-empty, if there exist $L \geq 0$ and a continuous $\phi \in \Phi$, such that

$$d(Tx, Ty) \precsim k\phi\left(\max\left\{\frac{\{d(x, Ty) - d(A, B)\}\{d(y, Tx) - d(A, B)\}\{d(x, Tx) + d(y, Ty) - 2d(A, B)\}}{1 + d(x, y)}, d(x, y)\right\}\right)$$
$$+ L\min\big(\{d(x, Tx) - d(A, B)\}, \{d(y, Ty) - d(A, B)\}, \{d(x, Ty) - d(A, B)\}, \{d(y, Tx) - d(A, B)\}\big)$$

where k is any real number with $0 < k < 1$.

Then T has a unique best proximity point in A.

Proof. Assume that $x_0 \in A_0$, so $Tx_0 \in B_0$ then there exist $x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A, B)$. Again $Tx_1 \in B_0$, then there exists $x_2 \in A_0$ such that

$$d(x_2, Tx_1) = d(A, B).$$

By continuing this manner we can form a sequence $\{x_n\}$ in A_0, with

$$d(x_{n+1}, Tx_n) = d(A, B), \quad \forall n \in N.$$

By using the P-property, we have

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n).$$

If there exists $n_0 \in N$ such that $x_{n_0-1} = x_{n_0}$, then $d(x_{n_0}, Tx_{n_0-1}) = d(A, B) = d(x_{n_0-1}, Tx_{n_0-1})$. Hence proof is complete. Now assume that $x_{n-1} \neq x_n, \ \forall n \in N$. Consider

$$\begin{aligned}
&d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)\\
&\precsim k\phi\left(\max\left\{\frac{\{d(x_{n-1}, Tx_n) - d(A,B)\}\{d(x_n, Tx_{n-1}) - d(A,B)\}\{d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n) - 2d(A,B)\}}{1 + d(x_{n-1}, y)}, d(x_{n-1}, x_n)\right\}\right)\\
&\quad + L\min\big(\{d(x_{n-1}, Tx_{n-1}) - d(A,B)\}, \{d(x_n, Tx_n) - d(A,B)\}, \{d(x_{n-1}, Tx_n) - d(A,B)\},\\
&\{d(x_n, Tx_{n-1}) - d(A,B)\}\big)
\end{aligned}$$

$$d(x_n, x_{n+1}) \precsim k\phi\big(d(x_{n-1}, x_n)\big)$$

$$i.e \ \ d(x_n, x_{n+1}) \precsim kd(x_{n-1}, x_n)$$

$$d(x_n, x_{n+1}) \precsim kd(x_{n-1}, x_n) \precsim k^2 d(x_{n-2}, x_{n-1}) \precsim \cdots \precsim k^n d(x_0, x_1)$$

for any $m > n$

$$\begin{aligned}
d(x_m, x_n) &\precsim d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)\\
&\precsim d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + \cdots + d(x_{m-1}, x_m)\\
&\precsim [k^n + k^{n+1} + k^{n+2} + \cdots + k^{m-1}]d(x_0, x_1)\\
&\precsim \frac{k^n}{1-k} d(x_0, x_1) \to 0 \ \ as \ \ m, n \to \infty.
\end{aligned}$$

Thus $\{x_n\}$ is a Cauchy sequence in A. Since X is complete, so there exists $u \in X$ such that $x_n \to u \in X$. Again, since A is closed subset of X, we have $u \in A$. Using the rectangular inequality Definition 1.2, we get

$$\begin{aligned}
d(Tu, u) &\precsim d(u, x_{n+1}) + d(x_{n+1}, Tx_n) + d(Tx_n, Tu)\\
&\precsim d(u, x_{n+1}) + d(x_{n+1}, Tx_n)\\
&\quad + k\phi\left(\max\left\{\frac{\{d(x_n, Tu) - d(A,B)\}\{d(u, Tx_n) - d(A,B)\}\{d(x_n, Tx_n) + d(u, Tu) - 2d(A,B)\}}{1 + d(x_n, u)}, d(x_n, u)\right\}\right)\\
&\quad + L\min\big(\{d(x_n, Tx_n) - d(A,B)\}, \{d(u, Tu) - d(A,B)\}, \{d(x_n, Tu) - d(A,B)\},\\
&\quad \{d(u, Tx_n) - d(A,B)\}\big).
\end{aligned}$$

As $n \to \infty$, we get

$$d(Tu, u) \precsim d(A, B).$$

Thus

$$d(Tu,u)=d(A,B).$$

That is $u\in A$ is the best proximity point of T.
Uniqueness: Let $u^*\in A$ is another best proximity point of T. Then $d(u^*,Tu^*)=d(A,B)$. Also we have $d(Tu,u)=d(A,B)$. By using P-property, we have

$$\begin{aligned}
d(u,u^*)&=d(Tu,Tu^*)\\
&\precsim k\phi\left(\max\left\{\frac{\{d(u,Tu^*)-d(A,B)\}\{d(u^*,Tu)-d(A,B)\}\{d(u,Tu)+d(u^*,Tu^*)-2d(A,B)\}}{1+d(u,u^*)},d(u,u^*)\right\}\right)\\
&\quad+L\min\big(\{d(u,Tu)-d(A,B)\},\{d(u^*,Tu^*)-d(A,B)\},\{d(u,Tu^*)-d(A,B)\},\\
&\quad\{d(u^*,Tu)-d(A,B)\}\big)\\
d(u,u^*)&\precsim k\phi\big(d(u,u^*)\big)\\
&\precsim kd(u,u^*),
\end{aligned}$$

which is a contradiction. Hence $d(u,u^*)=0$ or $u=u^*$ is a unique best proximity point of T. ■

The following fixed point result is a consequence of the above best proximity point Theorem 2.2.

Theorem 2.3. Let (X,d) be a complete complex valued rectangular metric space.
Let $T:X\to X$ be a continuous mapping and there exist $L\geq 0$, a continuous $\phi\in\Phi$, such that

$$\begin{aligned}
d(Tx,Ty)\precsim k\phi\big(\max\big\{\frac{d(x,Ty)d(y,Tx)\{d(x,Tx)+d(y,Ty)\}}{1+d(x,y)},d(x,y)\big\}\big)\\
+L\min\big(d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\big),
\end{aligned}$$

for all $x,y\in X$ and k is any real number with $0<k<1$. Then T has a unique fixed point in X.

Proof. Assume that $x_0\in X$,then there exists $x_1\in X$ such that $x_1=Tx_0$ and for $x_1\in X$,then there exists $x_2\in X$ such that $x_2=Tx_1$.
Thus we can construct a sequence $x_{n+1}=Tx_n$.

$$\begin{aligned}
d(x_{n+1},x_{n+2})&=d(Tx_n,Tx_{n+1})\\
&\precsim k\phi\left(\max\left\{\frac{d(x_n,Tx_{n+1})d(x_{n+1},Tx_n)\{d(x_n,Tx_n)+d(x_{n+1},Tx_{n+1})\}}{1+d(x_n,x_{n+1})},d(x_n,x_{n+1})\right\}\right)\\
&\quad+L\min\big(d(x_n,Tx_n),d(x_{n+1},Tx_{n+1}),d(x_n,Tx_{n+1}),d(x_{n+1},Tx_n)\big)\\
d(x_{n+1},x_{n+2})&\precsim k\phi\big(d(x_n,x_{n+1})\big)\\
d(x_{n+1},x_{n+2})&\precsim kd(x_n,x_{n+1})
\end{aligned}$$

for any $m > n$

$$\begin{aligned} d(x_m, x_n) &\precsim d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m) \\ &\precsim d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + \cdots + d(x_{m-1}, x_m) \\ &\precsim [k^n + k^{n+1} + k^{n+2} + \cdots + k^{m-1}] d(x_0, x_1) \\ &\precsim \frac{k^n}{1-k} d(x_0, x_1) \to 0 \ \ as \ \ m, n \to \infty. \end{aligned}$$

Thus $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, so there exists $z \in X$ such that $x_n \to z \in X$.
Using the rectangular inequality Definition 1.2, we get

$$\begin{aligned} d(z, Tz) &\precsim d(z, x_{n+1}) + d(x_{n+1}, Tx_n) + d(Tx_n, Tz) \\ &\precsim d(z, x_{n+1}) + d(x_{n+1}, Tx_n) + \\ &\quad k\phi\left(\max\left\{\frac{d(x_n, Tz) d(z, Tx_n)\{d(x_n, Tx_n) + d(z, Tz)\}}{1 + d(x_n, z)}, d(x_n, z)\right\}\right) \\ &\quad + L\min\big(d(x_n, Tx_n), d(z, Tz), d(x_n, Tz), d(z, Tx_n)\big) \\ &\precsim k\phi\big(d(x_n, z)\big) \\ &\precsim kd(x_n, z) \quad \to 0 \ \text{ as } \ n \to \infty. \end{aligned}$$

Implies that $z = Tz$. Hence z is a fixed point of T.
Uniqueness: Let $z^* \in X(\neq z)$ is another fixed point of T. Consider

$$\begin{aligned} d(z, z^*) &= d(Tz, Tz^*) \\ &\precsim k\phi\left(\max\left\{\frac{d(z, Tz^*) d(z^*, Tz)\{d(z, Tz) + d(z^*, Tz^*)\}}{1 + d(z, z^*)}, d(z, z^*)\right\}\right) \\ &\quad + L\min\big(d(z, Tz), d(z^*, Tz^*), d(z, Tz^*), d(z^*, Tz)\big) \\ &\precsim k\phi\big(d(z, z^*)\big) \\ &\precsim kd(z, z^*), \end{aligned}$$

which is a contradiction. Hence $d(z, z^*) = 0$ or $z = z^*$ is a unique fixed point of T. ■

If we set $L = 0$ in the Theorems 2.2 and2.3, then we get the following corollaries respectively.

Corollary 2.4. Let A and B be two non-empty closed and bounded subsets of a complete complex valued rectangular metric space $(X, d)$with pair (A, B) satisfies the P-property. Let a continuous mapping $T : A \to B$ with $T(A_0) \subset B_0$, where A_0 is non-empty, if there exist a continuous $\phi \in \Phi$, such that

$$d(Tx, Ty) \precsim k\phi\left(\max\left\{\frac{\{d(x, Ty) - d(A, B)\}\{d(y, Tx) - d(A, B)\}\{d(x, Tx) + d(y, Ty) - 2d(A, B)\}}{1 + d(x, y)}, d(x, y)\right\}\right)$$

where k is any real number with $0 < k < 1$.

Then T has a unique best proximity point in A.

Corollary 2.5. Let (X, d) be a complete complk2ex valued rectangular metric space. Let $T : X \to X$ be a continuous mapping and there exist a continuous $\phi \in \Phi$, such that

$$d(Tx, Ty) \precsim k\phi\left(\max\left\{\frac{d(x, Ty)d(y, Tx)\{d(x, Tx) + d(y, Ty)\}}{1 + d(x, y)}, d(x, y)\right\}\right),$$

for all $x, y \in X$ and k is any real number with $0 < k < 1$.
Then T has a unique fixed point in X.

Example 2.6. Let $X = A \cup B$ where

$$A = \{2 + iy : 0 \le y \le \frac{1}{2}\}$$

$$B = \{3 + iy : 0 \le y \le \frac{1}{2}\}.$$

Define a metric d on X as follows
$d(z_1, z_2) = |(x_1 - x_2| + i|y_1 - y_2|)$, where $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$
then (X, d) is a complex valued rectangular metric space. Also (A, B) is a pair of non-empty closed and bounded subsets of X such that

$$d(A, B) = 1, \quad A_0 = A, \quad B_0 = B.$$

It is verified that pair (A, B) satisfies the $P-$property.
Let $T : A \to B$ be defined as follows

$$Tz = 3 + \left(\frac{1}{2} - y\right)i, \quad for \quad z = x + iy \in A.$$

Then T satisfies the conditions mentioned in Theorem 2.2.
Hence T has a unique best proximity point $z = 2 + \frac{1}{4}i$.

References

[1] M. Abbas. V. C. Rajic, T. Nazir, S. Radenovic, Common fixed point of mappings satisfying rational inequalities in ordered complex valued generalized metric spaces, Afr. Mat. DOI 10.1007/s 13370-013-0185-z(2013).

[2] H. Aydi, E. Karapinar, I. Erhan, P. Salimi, Best proximity points of generalized almost ψ-Geraghty contractive non-self-mappings, Fixed Point Theory and Applications, 2014 (2014), 32.

[3] A. Azam, B. Fisher, M. Khan, Common fixed point theorems in complex valued metric spaces, Num. Functional Analysis and optimization, 32, No.3 (2011), 243–253.

[4] A. Branciari, A fixed point theorem of Banach-Caccippoli type on a class of generalized metric spaces, Publ. Math.Debrecen, 57 (2000), 31–37.

[5] J. Caballero, J. Harjani, K. Sadarangani, A best proximity point theorem for Geraghty-contractions, Fixed Point Theory and Applications, 2012 (2012), 231.

[6] B. S. Choudhury, N. Metiya, P. Maity, Best proximity points results in complex valued metric spaces, Int. Jour. of Analysis, Vol. 2014, Article ID 827862, 6 pages.

[7] A. A. Eldered, P. Veeramani, Existence and convergence of best proximity points, J.Math. Anal. Appl.,323(2006), 1001–1006.

[8] S. Karpagam, S. Agrawal, Best proximity point theorems for p-cyclic Meir-Keeler contractions, Fixed Point Theory and Applications, 2009 (2009), Article ID 197308, 1–10.

[9] E. Karapinar, On best proximity point of ψ-Geraghty contractions, Fixed Point Theory and Applications, 2013 (2013), 200.

[10] J. Matkowski, Fixed point theorems for mappings with a contractive iterate at a point, Proceedings of the American Mathematical Society, Vol.62, No. 2 (1977), 344–348.

[11] C. Mongkolkeha, Y.J. Cho, P. Kumam, Best proximity points for generalized proximal C-contraction mappings in metric spaces with partial orders, Journal of Inequalities and Applications, 2013 (2013), 94.

[12] H.K. Nashine, P. Kumam, C. Vetro, Best proximity point theorems for rational proximal contractions, Fixed Point Theory and Applications, 2013 (2013), 95.

[13] S. R. Patil ,J.N. Salunke, Common fixed point theorems in complex valued rectangular metric spaces, South Asian Journal of Mathematics, Vol.6(2016), 10–23.

[14] V. S. Raj, Abest proximity point theorem for weakly contractive non-self-mappings, Nonlinear Analysis: Theory, Methods and Applications, vol. 74, no. 14, pp. 4804–4808, 2011.

[15] S. Reich, Approximate selections, best approximations, fixed points and invariant sets, J. Math. Anal. Appl., 62 (1978), 104–113.

[16] S. Sadiq Basha, P. Veeramani, Best proximity pair theorems for multi-functions with open fibres, J. Approx. Theory, 103 (2000), 119–129.

[17] S. Sadiq Basha, Best proximity point theorems on partially ordered sets, Optim. Lett., 7 (2012), 1035–1043.

[18] S. Sadiq Basha, N. Shahzad, Best proximity point theorems for generalized proximal contractions, Fixed Point Theory and Applications, 2012 (2012), 42.

[19] D. Singh, V. Joshi,M. Imdad, P. Kumam, Unified implicit common fixed point theorems under non-negative complex valued functions satisfying the identity of indiscernible, Journal of Nonlinear Sciences and Applications (JNSA). 2016, Vol. 9 Issue 5, 2049–2069. 21p.

[20] W. Sintunavarat, P. Kumam, Coupled best proximity point theorem in metric spaces, Fixed Point Theory Appl., 2012 (2012), 93.

[21] M. A. Al-Thagafi, N. Shahzad, Convergence and existence results for best proximity points, Nonlinear Anal., 70 (2009), 3665–3671.

[22] R. K. Verma and H. K. Pathak, Common fixed point theorems using property (E.A) in complex-vauled metric spaces, Thai J. of Mathematics. Vol. 11 (1013), No.2: 347–355.