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Decomposition of vague α-soft open sets in Vague Soft Topological Spaces

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Abstract

In this paper, we introduce some generalization of vague soft open sets in vague soft topological spaces and obtain a decomposition of vague α-soft open sets by using them.

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1. Introduction

Fuzzy set theory [16], intuitionistic fuzzy set theory [1], vague set theory [6] and other Mathematical tools are useful approaches to describing uncertainty. However, all of these theories have their own difficulties. To overcome these difficulties Molodtsov [10] introduced the concepts of soft sets as a new mathematical tool for dealing with vagueness and uncertainties. Soft set theory has a rich potential for application in solving practical problems in economics, engineering, environment, social science, medical science and business management. Later on, Maji et al. [7], [8], [9] studied the theory of fuzzy soft sets and intuitionistic fuzzy soft sets. W. Xu et al. [14] introduce the notion of vague soft set which is an extension to the soft set and its basic properties are discussed. Alkhazaleh et al. [2], [3] and B.P. Varol et al. [12], Y. Yin et al. [15], I. Osmanoglu et al. [11] extended their studies on fuzzy soft sets and intuitionistic fuzzy soft sets respectively. Alhazaymeh and Hassan [4], [5] worked on vague soft set and its applications. C. Wang and Y. Li [13] introduced the notion of vague soft topological spaces which are defined over an initial universe with a fixed set of parameters. They also studied some of basic concepts of vague soft topological spaces like vague soft connectedness and vague soft compactness.

In the present study, we introduce some new concepts in vague soft topological spaces such as vague semi-soft sets, vague pre-soft sets, vague α-soft sets and vague regular-soft sets. We also study the relationship between vague semi-soft interior, vague semi-soft closure and vague soft interior, vague soft closure. Also the decomposition of vague α-soft sets are presented.

2. Preliminaries

Definition 2.1. [6] A vague set A in the universe $X = \{x_1, x_2, \ldots, x_n\}$ can be expressed by the following notion: $A = \{(x_i, [t_A(x_i), 1 - f_A(x_i)])|x_i \in X\}$; that is, $A(x_i) = [t_A(x_i), 1 - f_A(x_i)]$, and the condition $0 \leq t_A(x_i) \leq 1 - f_A(x_i)$ should hold for any $x_i \in X$, where $t_A(x_i)$ is called the membership degree (true membership) of element x_i to the vague set A, while $f_A(x_i)$ is the degree of nonmembership (false membership) of the element x_i to the vague set A.

Definition 2.2. [6] Let A and B be vague sets of the form $A = \{(x, [t_A(x), 1 - f_A(x)]) \mid x \in X\}$ and $B = \{(x, [t_B(x), 1 - f_B(x)])|x \in X\}$. Then

i. $A \subseteq B$ if and only if $t_A(x) \leq t_B(x)$ and $1 - f_A(x) \leq 1 - f_B(x)$ for all $x \in X$.

ii. $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

iii. $A^c = \{(x, [f_A(x), 1 - t_A(x)]) \mid x \in X\}$.

iv. $A \cap B = \{(x, [(t_A(x) \wedge t_B(x)), (1 - f_A(x) \wedge 1 - f_B(x))]) \mid x \in X\}$.

v. $A \cup B = \{(x, [(t_A(x) \vee t_B(x)), (1 - f_A(x) \vee 1 - f_B(x))]) \mid x \in X\}$.

for simplicity we shall use the notion $A = \{(x, [t_A(x), 1 - f_A(x)])\}$ instead of $A = \{(x, [t_A(x), 1 - f_A(x)]) \mid x \in X\}$.

Definition 2.3. [14] Let X be an initial universe set, V(X) the set of all vague sets on X, E a set of parameters, and $A \subseteq E$. A pair (F,A) is called a vague soft set over X, where F is a mapping given by F: A→ V(X). The set of all vague soft sets over X is denoted by $\mathcal{V}\tilde{\mathcal{S}}(X, E)$.

Definition 2.4. [14] Let (F,A) and (G,B) be two vague soft sets over X. If $A \subseteq B$ and for all $e \in A$, F(e) is a vague subset of G(e), then (F,A) is called a vague soft subset of (G,B). This relation is denoted by $(F, A) \subseteq (G, B)$.

Definition 2.5. [14] Two vague soft sets (F,A) and (G,B) over X are said to be vague soft equal if (F,A) is a vague soft subset of (G,B) and (G,B) is a vague soft subset of (F,A). This relation is denoted by $(F, A) = (G, B)$.

Definition 2.6. [14] The complement of vague soft set (F,A) is denoted by $(F, A)^c$ and is defined by $(F, A)^c = (F^c, A)$ and is given by $t_{F^c(e)}(x) = f_{F(e)}(x)$, $1 - f_{F^c(e)}(x) = 1 - t_{F(e)}(x)$, for all $e \in A,\ x \in X$.

Definition 2.7. [14] A vague soft set (F,A) over X is said to be a null vague soft set denoted by $\hat{\emptyset}$, if $\forall e \in A$, $t_{F(e)}(x) = 0, 1 - f_{F(e)}(x) = 0, x \in X$.

Definition 2.8. [14] A vague soft set (F,A) over X is said to be an absolute vague soft set denoted by $\hat{X}$, if $\forall e \in A$, $t_{F(e)}(x) = 1, 1 - f_{F(e)}(x) = 1, x \in X$.

Definition 2.9. [14] The union of two vague soft sets (F,A) and (G,B) over a universe X is a vague soft set (H,C), where $C = A \cup B$ and $\forall\ e \in C$,

$$t_{H(e)}(x) = \begin{cases} t_{F(e)}(x) & e \in A - B,\ x \in X, \\ t_{G(e)}(x) & e \in B - A,\ x \in X, \\ t_{F(e)}(x) \vee t_{G(e)}(x) & e \in A \cap B,\ x \in X, \end{cases}$$

$$1 - f_{H(e)}(x) = \begin{cases} 1 - f_{F(e)}(x) & e \in A - B,\ x \in X, \\ 1 - f_{G(e)}(x) & e \in B - A,\ x \in X, \\ (1 - f_{F(e)}(x) \vee 1 - f_{G(e)}(x)) & e \in A \cap B,\ x \in X. \end{cases}$$

We denote it by $(F, A) \cup (G, B) = (H, C)$.

Definition 2.10. [14] The intersection of two vague soft sets (F,A) and (G,B) over a universe X is a vague soft set (H,C), where $C = A \cup B$ and $\forall\ e \in C$,

$$t_{H(e)}(x) = \begin{cases} t_{F(e)}(x) & e \in A - B,\ x \in X, \\ t_{G(e)}(x) & e \in B - A,\ x \in X, \\ t_{F(e)}(x) \wedge t_{G(e)}(x) & e \in A \cap B,\ x \in X, \end{cases}$$

$$1 - f_{H(e)}(x) = \begin{cases} 1 - f_{F(e)}(x) & e \in A - B,\ x \in X, \\ 1 - f_{G(e)}(x) & e \in B - A,\ x \in X, \\ (1 - f_{F(e)}(x) \wedge 1 - f_{G(e)}(x)) & e \in A \cap B,\ x \in X. \end{cases}$$

We denote it by $(F, A) \cap (G, B) = (H, C)$.

Proposition 2.11. [14] If (F,A) and (G,B) are two vague soft sets over a universe X, then

1) $((F, A) \cup (G, B))^c = (F, A)^c \cap (G, B)^c$,

2) $((F, A) \cap (G, B))^c = (F, A)^c \cup (G, B)^c$.

2.1. Vague Soft Topological Spaces

Definition 2.12. [13] Let τ be the collection of vague soft sets over X; then τ is said to be a vague soft topology on X if

1. $\hat{\emptyset}, \hat{X}$ belongs to τ.
2. the union of any number of vague soft sets in τ belongs to τ.
3. the intersection of any two vague soft sets in τ belongs to τ.

The triple (X, τ, E) is called a vague soft topological space over X.

Definition 2.13. [13] Let (X, τ, E) be a vague soft topological space over X, then the members of τ are said to be vague soft open set in X.

Definition 2.14. [13] Let (X, τ, E) be a vague soft topological space over X. A vague soft set (F,A) over X is said to be a vague soft closed set in X, if its relative complement $(F, A)^c$ belongs to τ.

Proposition 2.15. [13] If (X, τ, E) is a vague soft topological space over X, then

1. $\hat{\emptyset}, \hat{X}$ belongs to τ.
2. the intersection of any number of vague soft closed sets is a vague soft closed set over X.
3. the union of any two vague soft closed sets is a vague soft closed set over X.

Proof. It follows from the definition of vague soft topological spaces and De-Morgan's laws for vague soft sets which are given in Proposition 2.11. ■

Definition 2.16. [13] Let (X, τ, E) be a vague soft topological space and let (F,A) be a vague soft set over X. Then vague soft interior of (F,A) denoted by vs̃int(F,A) and is defined by,
vs̃int(F,A)=$\bigcup \{(G, A) \,/\, (G, A) \in \tau \ and \ (G, A) \subseteq (F, A)\}$.
Clearly, vs̃int(F,A) is the largest vague soft open set contained in (F,A) and it is vague soft open set.

Theorem 2.17. [13] Let (X, τ, E) be a vague soft topological space over X, and let (F,A) and (G,B) be a vague soft sets over X. Then the following properties hold:

1. $v\tilde{s}int(\hat{\emptyset}) = \hat{\emptyset}$ and $v\tilde{s}int(\hat{X}) = \hat{X}$.
2. $v\tilde{s}int(F, A) \subseteq (F, A)$.
3. (F,A) is a vague soft open set if and only if $v\tilde{s}int(F, A) = (F, A)$.

4. $v\tilde{s}int(v\tilde{s}int(F,A)) = v\tilde{s}int(F,A)$.

5. $(F,A) \subseteq (G,B)$ implies $v\tilde{s}int(F,A) \subseteq v\tilde{s}int(G,B)$.

6. $v\tilde{s}int((F,A) \cap (G,B)) = v\tilde{s}int(F,A) \cap v\tilde{s}int(G,B)$.

7. $v\tilde{s}int((F,A) \cup (G,B)) \supseteq v\tilde{s}int(F,A) \cup v\tilde{s}int(G,B)$.

8. $(v\tilde{s}int(F,A))^c = v\tilde{s}cl((F,A)^c)$

Definition 2.18. [13] Let (X,τ,E) be a vague soft topological space and let (F,A) be a vague soft set over X. Then vague soft closure of (F,A) denoted by $v\tilde{s}cl(F,A)$ and is defined by,
$v\tilde{s}cl(F,A) = \bigcap \{(H,A) \,/\, (H,A) \in \tau \text{ and } (F,A) \subseteq (H,A)\}$.

Clearly, $v\tilde{s}cl(F,A)$ is the smallest vague soft closed set containing (F,A) and it is vague soft closed set.

Theorem 2.19. [13] Let (X,τ,E) be a vague soft topological space over X, and let (F,A) and (G,B) be a vague soft sets over X. Then the following properties hold:

1. $v\tilde{s}cl(\hat{\emptyset}) = \hat{\emptyset}$ and $v\tilde{s}cl(\hat{X}) = \hat{X}$.

2. $(F,A) \subseteq v\tilde{s}cl(F,A)$.

3. (F,A) is a vague soft closed set if and only if $v\tilde{s}cl(F,A) = (F,A)$.

4. $v\tilde{s}cl(v\tilde{s}cl(F,A)) = v\tilde{s}cl(F,A)$.

5. $(F,A) \subseteq (G,B)$ implies $v\tilde{s}cl(F,A) \subseteq v\tilde{s}cl(G,B)$.

6. $v\tilde{s}cl((F,A) \cup (G,B)) = v\tilde{s}cl(F,A) \cup v\tilde{s}cl(G,B)$.

7. $v\tilde{s}cl((F,A) \cap (G,B)) \subseteq v\tilde{s}cl(F,A) \cap v\tilde{s}cl(G,B)$.

8. $(v\tilde{s}cl(F,A))^c = v\tilde{s}int((F,A)^c)$

3. Vague Semi-Soft Open (Closed) Sets

Definition 3.1. Let (X,τ,E) be a vague soft topological space and let (F,A) be a vague soft set over X. Then (F,A) is said to be a vague semi-soft open set of (X,τ,E), if there exist a vague soft open set (O,A) such that $(O,A) \subseteq (F,A) \subseteq v\tilde{s}cl(O,A)$. The set of all vague semi-soft open sets in $\mathcal{V}\tilde{\mathcal{S}}$(X,E) is denoted by $\mathcal{VS}\tilde{\mathcal{S}}\mathcal{O}(X,\tau,E)$ or $\mathcal{VS}\tilde{\mathcal{S}}\mathcal{O}(X)$.

Theorem 3.2. Let $\{(F_\alpha,E)\}_{\alpha\in\Delta}$ be a collection of vague semi-soft open sets in (X,τ,E). Then $\bigcup_{\alpha\in\Delta}(F_\alpha,E)$ is vague semi-soft open.

Proof. Since $\{(F_\alpha, E)\}_{\alpha\in\Delta}$ is a collection of vague semi-soft open sets, we have for each $\alpha \in \Delta$ there is a vague soft open set (O_α,E), such that (O_α,E)$\subseteq$ (F_α,E) $\subseteq$ v$\tilde{s}$cl(O_α,E). Then $\bigcup\limits_{\alpha\in\Delta}$($O_\alpha$,E)$\subseteq$ $\bigcup\limits_{\alpha\in\Delta}$ (F_α,E)$\subseteq$ $\bigcup\limits_{\alpha\in\Delta} v\tilde{s}cl(O_\alpha, E) = v\tilde{s}cl \bigcup\limits_{\alpha\in\Delta}$ (O_α, E). Because $\bigcup\limits_{\alpha\in\Delta}$ (O_α,E) is vague soft open set, then $\bigcup\limits_{\alpha\in\Delta} (F_\alpha, E)$ is vague semi-soft open. ■

Theorem 3.3. Let (F,A) be a vague semi-soft open set in a vague soft topological space (X, τ, E) and suppose (F,A)$\subseteq$(G,A) $\subseteq$ v$\tilde{s}$cl(F,A). Then (G,A) is vague semi-soft open.

Proof. Since (F,A) is a vague semi-soft open set, there exists a vague soft open set (O,A) such that $(O, A) \subseteq (F, A) \subseteq v\tilde{s}cl(O, A)$. Then $(O, A) \subseteq (G, A)$. But $v\tilde{s}cl(F, A) \subseteq v\tilde{s}cl(O, A)$ and thus $(G, A) \subseteq v\tilde{s}cl(O, A)$. Hence $(O, A) \subseteq (G, A) \subseteq v\tilde{s}cl(O, A)$ and (G,A) is vague semi-soft open. ■

Theorem 3.4. A vague soft set (F,A) in a vague soft topological space (X, τ, E) is vague semi-soft open iff $(F, A) \subseteq v\tilde{s}cl(v\tilde{s}int(F, A))$.

Proof. *Sufficiency*: Let $(F, A) \subseteq v\tilde{s}cl(v\tilde{s}int(F, A))$. Then for $(O, A) = v\tilde{s}int(F, A)$, we have $(O, A) \subseteq (F, A) \subseteq v\tilde{s}cl(O, A)$. Therefore (F,A) is vague semi-soft open.
Necessity: Let (F,A) be a vague semi-soft open set. Then $(O, A) \subseteq (F, A) \subseteq v\tilde{s}cl(O, A)$ for some vague soft open set (O,A). But $(O, A) = v\tilde{s}int(O, A) \subseteq v\tilde{s}int(F, A)$ and $v\tilde{s}cl(O, A) \subseteq v\tilde{s}cl(v\tilde{s}int(F, A))$. Hence $(F, A) \subseteq v\tilde{s}cl(O, A) \subseteq v\tilde{s}cl(v\tilde{s}int(F, A))$. ■

Theorem 3.5. Every vague soft open set in a vague soft topological space is vague semi-soft open.

Proof. Let (F,A) be vague soft open set. Since (F,A) is vague soft open set, we have $(F, A) = v\tilde{s}int(F, A)$. Then, $(F, A) \subseteq v\tilde{s}cl(F, A) = v\tilde{s}cl(v\tilde{s}int(F, A))$. Hence (F,A) is vague semi-soft open. ■

Remark 3.6. The converse of the above Theorem 3.5 is not true in general as shown in the following example.

Example 3.7. Let $X = \{a, b\}, E = \{e_1, e_2, e_3\}$ and $\tau = \{\hat{\emptyset}, \hat{X}, (F, E)\}$ be a vague soft topology defined on X, where (F,E) is a vague soft open set over X, defined as follows,

$$F(e_1) = \left\{\frac{[0.1, 0.6]}{a}, \frac{[0, 0.5]}{b}\right\}$$

$$F(e_2) = \left\{\frac{[0.4, 0.5]}{a}, \frac{[0.3, 0.6]}{b}\right\}$$

$$F(e_3) = \left\{\frac{[0.2, 0.4]}{a}, \frac{[0.3, 0.4]}{b}\right\}$$

Let (G,E) be a vague soft set in $\mathcal{V}\tilde{\mathcal{S}}$(X,E) defined as follows:

$$G(e_1) = \left\{ \frac{[0.2, 0.7]}{a}, \frac{[0.4, 0.5]}{b} \right\}$$

$$G(e_2) = \left\{ \frac{[0.4, 0.6]}{a}, \frac{[0.3, 0.7]}{b} \right\}$$

$$G(e_3) = \left\{ \frac{[0.4, 0.5]}{a}, \frac{[0.5, 0.6]}{b} \right\}$$

Then (G,E) is vague semi-soft open set of (X, τ, E) but not vague soft open.

Definition 3.8. Let (X, τ, E) be a vague soft topological space and let (G,B) be a vague soft set over X. Then (G,B) is said to be a vague semi-soft closed set of (X, τ, E), if there exist a vague soft closed set (H,B) such that $v\tilde{s}int(H, B) \subseteq (G, B) \subseteq (H, B)$. The set of all vague semi-soft closed sets in $\mathcal{V}\tilde{\mathcal{S}}$(X,E) is denoted by $\mathcal{VS}\tilde{\mathcal{SC}}(X, \tau, E)$ or $\mathcal{VS}\tilde{\mathcal{SC}}(X)$.

Theorem 3.9. Let $\{(G_\alpha, E)\}_{\alpha\in\Delta}$ be a collection of vague semi-soft closed sets in (X, τ, E). Then $\bigcap_{\alpha\in\Delta} (G_\alpha, E)$ is vague semi-soft closed.

Proof. Since $\{(G_\alpha, E)\}_{\alpha\in\Delta}$ is a collection of vague semi-soft closed sets, we have for each $\alpha \in \Delta$ there is a vague soft closed set (H_α,E), such that $v\tilde{s}int(H_\alpha, E) \subseteq (G_\alpha, E) \subseteq (H_\alpha, E)$. Then $v\tilde{s}int(\bigcap_{\alpha\in\Delta} (H_\alpha, E)) = \bigcap_{\alpha\in\Delta} v\tilde{s}int(H_\alpha, E) \subseteq \bigcap_{\alpha\in\Delta} (G_\alpha, E) \subseteq \bigcap_{\alpha\in\Delta} (H_\alpha, E)$. Because $\bigcap_{\alpha\in\Delta} (H_\alpha, E) = (H, E)$ is vague soft closed set by Proposition 2.15, then $\bigcap_{\alpha\in\Delta} (G_\alpha, E)$ is vague semi-soft closed. ■

Theorem 3.10. Let (G,B) be vague semi-soft closed set in a vague soft topological space (X, τ, E) and suppose $v\tilde{s}int(G, B) \subseteq (F, B) \subseteq (G, B)$. Then (F,B) is vague semi-soft closed.

Proof. Since (G,B) is vague semi-soft closed set, there exists a vague soft closed set (H,B) such that $v\tilde{s}int(H, B) \subseteq (G, B) \subseteq (H, B)$. Then $(F, B) \subseteq (H, B)$. But $v\tilde{s}int(v\tilde{s}int(H, B)) = v\tilde{s}int(H, B) \subseteq v\tilde{s}int(G, B)$ and thus $v\tilde{s}int(H, B) \subseteq (F, B)$. Hence $v\tilde{s}int(H, B) \subseteq (F, B) \subseteq (H, B)$ and (F,B) is vague semi-soft closed. ■

Theorem 3.11. A vague soft set (G,B) in a vague soft topological space (X, τ, E) is vague semi-soft closed iff $v\tilde{s}int(v\tilde{s}cl(G, B)) \subseteq (G, B)$.

Proof. Sufficiency: Let $v\tilde{s}int(v\tilde{s}cl(G, B)) \subseteq (G, B)$. Then for $(H, B) = v\tilde{s}cl(G, B)$, we have $v\tilde{s}int(H, B) \subseteq (G, B) \subseteq (H, B)$. And hence (G,B) is vague semi-soft closed.

Necessity: Let (G,B) be a vague semi-soft closed set. Then $v\tilde{s}int(H, E) \subseteq (G, B) \subseteq (H, E)$ for some vague soft closed set (H,B). But $v\tilde{s}cl(G, B) \subseteq v\tilde{s}cl(H, B) = (H, B)$ and $v\tilde{s}int(v\tilde{s}cl(G, B)) \subseteq v\tilde{s}int(H, B)$. Hence $v\tilde{s}int(v\tilde{s}cl(G, B)) \subseteq v\tilde{s}int(H, B) \subseteq (G, B)$. ■

Corollary 3.12. A vague soft set (G,B) in a vague soft topological space (X, τ, E) is vague semi-soft closed iff $(G, B) = (G, B) \cup v\tilde{s}int(v\tilde{s}cl(G, B))$.

Theorem 3.13. Every vague soft closed set in a vague soft topological space is vague semi-soft closed.

Proof. Let (G,B) be a vague soft closed set. Since (G,B) is vague soft closed, we have $v\tilde{s}cl(G, B) = (G, B)$. Then, $v\tilde{s}int(v\tilde{s}cl(G, B)) = v\tilde{s}int(G, B) \subseteq (G, B)$. Hence (G,B) is vague semi-soft closed. ■

Remark 3.14. The converse of the above Theorem 3.13 is not true in general as shown in the following example.

Example 3.15. Consider the vague soft topological space (X, τ, E) is same as in Example 3.7.

Let (P,E) be a vague soft set in (X, τ, E) which is defined as follows:

$$P(e_1) = \left\{ \frac{[0.3, 0.7]}{a}, \frac{[0.4, 0.8]}{b} \right\}$$

$$P(e_2) = \left\{ \frac{[0.4, 0.6]}{a}, \frac{[0.3, 0.6]}{b} \right\}$$

$$P(e_3) = \left\{ \frac{[0.2, 0.5]}{a}, \frac{[0.6, 0.7]}{b} \right\}$$

Then (P,E) is vague semi-soft closed set of (X, τ, E) but not vague soft closed set.

Definition 3.16. Let (X, τ, E) be a vague soft topological space and let (F,A) be a vague soft set over X. Then vague semi-soft interior of (F,A) is the vague soft set $vs\tilde{s}int(F, A) = \bigcup\{(G, A)/(G, A)$ is vague semi-soft open and $(G, A) \subseteq (F, A)\}$.

Clearly, $vs\tilde{s}int(F, A)$ is the largest vague semi-soft open set contained in (F,A) and it is vague semi-soft open set.

Definition 3.17. Let (X, τ, E) be a vague soft topological space and let (F,A) be a vague soft set over X. Then vague semi-soft closure of (F,A) is the vague soft set $vs\tilde{s}cl(F, A) = \bigcap\{(S, A)/(S, A)$ is vague semi-soft closed and $(F, A) \subseteq (S, A)\}$.

Clearly, $vs\tilde{s}cl(F, A)$ is the smallest vague semi-soft closed set containing (F,A) and it is vague semi-soft closed set.

Remark 3.18. Let (F,A) be a vague soft set in $\mathcal{V}\tilde{\mathcal{S}}$(X,E) and let (X, τ, E) be a vague soft topological space, then

(i) $vs\tilde{s}int(F, A) = (F, A) \cap v\tilde{s}cl(v\tilde{s}int(F, A))$.

(ii) $vs\tilde{s}cl(F, A) = (F, A) \cup v\tilde{s}int(v\tilde{s}cl(F, A))$.

Example 3.19. Let $X = \{a, b, c\}, E = \{e_1, e_2\}$ and $\tau = \{\hat{\emptyset}, \hat{X}, (F, E), (G, E)\}$ be a vague soft topology defined on X, where (F,E), (G,E) are vague soft sets over X, defined as follows,

$$F(e_1) = \left\{\frac{[0.2, 0.7]}{a}, \frac{[0.1, 0.9]}{b}, \frac{[0.3, 0.6]}{c}\right\}$$

$$F(e_2) = \left\{\frac{[0.2, 0.8]}{a}, \frac{[0.4, 0.5]}{b}, \frac{[0, 0.7]}{c}\right\}$$

and

$$G(e_1) = \left\{\frac{[0.5, 0.9]}{a}, \frac{[0.6, 0.9]}{b}, \frac{[0.7, 0.8]}{c}\right\}$$

$$G(e_2) = \left\{\frac{[0.8, 0.8]}{a}, \frac{[0.5, 0.8]}{b}, \frac{[0.9, 1]}{c}\right\}$$

Let (M,E) be a vague soft set over X defined as follows:

$$M(e_1) = \left\{\frac{[0.3, 0.9]}{a}, \frac{[0.2, 0.7]}{b}, \frac{[0.4, 0.8]}{c}\right\}$$

$$M(e_2) = \left\{\frac{[0.4, 0.8]}{a}, \frac{[0.5, 0.6]}{b}, \frac{[0.5, 0.9]}{c}\right\}$$

Then we have $vs\tilde{s}int(M, E) = (F, E)^c$ and $vs\tilde{s}cl(M, E) = \hat{X}$.

Theorem 3.20. Let (X, τ, E) be a vague soft topological space and (F,A) be a vague soft set over X. Then $v\tilde{s}int(F, A) \subseteq vs\tilde{s}int(F, A) \subseteq (F, A) \subseteq vs\tilde{s}cl(F, A) \subseteq v\tilde{s}cl(F, A)$.

Proof. By Theorem 3.5, Theorem 3.13, Definitions 3.16 and 3.17. ■

Theorem 3.21. Let (X, τ, E) be a vague soft topological space and (F,A) be a vague soft set over X. Then

(1) $(vs\tilde{s}cl(F, A))^c = vs\tilde{s}int((F, A)^c)$

(2) $(vs\tilde{s}int(F,A))^c = vs\tilde{s}cl((F,A)^c)$

(3) $vs\tilde{s}int(F,A) = (vs\tilde{s}cl((F,A)^c))^c$.

Proof. (1). $(vs\tilde{s}cl(F,A))^c = (\bigcap\{(H,A)/(H,A)$ is vague semi-soft closed and $(F,A) \subseteq (H,A)\})^c$.
$= \bigcup \{(H,A)^c$ / (H,A) is vague semi-soft closed and $(F,A) \subseteq (H,A)\}$.
$= \bigcup \{(H,A)^c$ / $(H,A)^c$ is vague semi-soft open and $(H,A)^c \subseteq (F,A)^c\}$.
$= vs\tilde{s}int((F,A)^c)$.
(2). $(vs\tilde{s}int(F,A))^c = (\bigcup \{(G,A)$ / (G,A) is vague semi-soft open and $(G,A) \subseteq (F,A)\})^c$.
$= \bigcap \{(G,A)^c$ / (G,A) is vague semi-soft open and $(G,A) \subseteq (F,A)\}$.
$= \bigcap \{(G,A)^c$ / $(G,A)^c$ is vague semi-soft closed and $(F,A)^c \subseteq (G,A)^c\}$.
$= vs\tilde{s}cl((F,A)^c)$.
(3). By part (2) $(vs\tilde{s}int(F,A))^c = vs\tilde{s}cl((F,A)^c)$
$((vs\tilde{s}int(F,A))^c)^c = (vs\tilde{s}cl((F,A)^c))^c$
Thus $vs\tilde{s}int(F,A) = (vs\tilde{s}cl((F,A)^c))^c$. ■

Theorem 3.22. Let (X,τ,E) be a vague soft topological space over X, and let (F,A) and (G,B) be a vague soft sets over X. Then the following properties hold:

(1) $vs\tilde{s}int(\hat{\emptyset}) = \hat{\emptyset}$ and $vs\tilde{s}int(\hat{X}) = \hat{X}$.

(2) $vs\tilde{s}int(F,A) \subseteq (F,A)$.

(3) $(F,A) \subseteq (G,B)$ implies $vs\tilde{s}int(F,A) \subseteq vs\tilde{s}int(G,B)$.

(4) (F,A) is a vague semi-soft open set if and only if $vs\tilde{s}int(F,A) = (F,A)$.

(5) $vs\tilde{s}int(vs\tilde{s}int(F,A)) = vs\tilde{s}int(F,A)$.

(6) $vs\tilde{s}int((F,A)\cup(G,B)) \supseteq vs\tilde{s}int(F,A) \cup vs\tilde{s}int(G,B)$.

Proof. (1), (2) and (3) are obvious.
(4) If (F,A) is a vague semi-soft open set, then (F,A) itself is a vague semi-soft open set over X which is contained in (F,A). So $vs\tilde{s}int(F,A)$ is the largest vague semi-soft open set contained in (F,A) and $vs\tilde{s}int(F,A) = (F,A)$.

Conversely, suppose that $vs\tilde{s}int(F,A) = (F,A)$. Since $vs\tilde{s}int(F,A)$ is a vague semi-soft open set, (F,A) is a vague semi-soft open set over X.
(5) Since $vs\tilde{s}int(F,A)$ is a vague semi-soft open set, by part (4) we have
$vs\tilde{s}int(vs\tilde{s}int(F,A)) = vs\tilde{s}int(F,A)$.
(6) Since $(F,A) \subseteq ((F,A)\cup(G,B))$ and $(G,B) \subseteq ((F,A)\cup(G,B))$, so $vs\tilde{s}int(F,A) \subseteq vs\tilde{s}int((F,A)\cup(G,B))$, $vs\tilde{s}int(G,B) \subseteq vs\tilde{s}int((F,A)\cup(G,B))$ by part (3). Thus $vs\tilde{s}int(F,A) \cup vs\tilde{s}int(G,B) \subseteq vs\tilde{s}int((F,A)\cup(G,B))$. ■

Theorem 3.23. Let (X, τ, E) be a vague soft topological space over X, and let (F,A) and (G,B) be a vague soft sets over X. Then the following properties hold:

(1) $vs\tilde{s}cl(\hat{\emptyset}) = \hat{\emptyset}$ and $vs\tilde{s}cl(\hat{X}) = \hat{X}$.

(2) $(F, A) \subseteq vs\tilde{s}cl(F, A)$.

(3) $(F, A) \subseteq (G, B)$ implies $vs\tilde{s}cl(F, A) \subseteq vs\tilde{s}cl(G, B)$.

(4) (F, A) is a vague semi-soft closed set if and only if $vs\tilde{s}cl(F, A) = (F, A)$.

(5) $vs\tilde{s}cl(vs\tilde{s}cl(F, A)) = vs\tilde{s}cl(F, A)$.

(6) $vs\tilde{s}cl((F, A) \cap (G, B)) \subseteq vs\tilde{s}int(F, A) \cap vs\tilde{s}int(G, B)$.

Proof. Immediate. ■

Theorem 3.24. Let (X, τ, E) be a vague soft topological space and (F,A) be a vague soft set over X. Then

i. $(vs\tilde{s}int(v\tilde{s}int(F, A))) = (v\tilde{s}int(vs\tilde{s}int(F, A))) = v\tilde{s}int(F, A)$.

ii. $(vs\tilde{s}cl(v\tilde{s}cl(F, A))) = (v\tilde{s}cl(vs\tilde{s}cl(F, A))) = v\tilde{s}cl(F, A)$.

Proof.

i. Since $v\tilde{s}int(F, A)$ is vague soft open, we have $v\tilde{s}int(F, A)$ is vague semi-soft open by Theorem 3.5. So we can get $vs\tilde{s}int(v\tilde{s}int(F, A)) = v\tilde{s}int(F, A)$ by Theorem 3.22 (4).

   By Theorem 3.20, we have $v\tilde{s}int(F, A) \subseteq vs\tilde{s}int(F, A) \subseteq (F, A)$. Then we can get, $v\tilde{s}int(F, A) \subseteq v\tilde{s}int(vs\tilde{s}int(F, A)) \subseteq v\tilde{s}int(F, A)$ and so $v\tilde{s}int(vs\tilde{s}int(F, A)) = v\tilde{s}int(F, A)$.

ii. Since $v\tilde{s}cl(F, A)$ is vague soft closed, we have $v\tilde{s}cl(F, A)$ is vague semi-soft closed by Theorem 3.13. So we can get $vs\tilde{s}cl(v\tilde{s}cl(F, A)) = v\tilde{s}cl(F, A)$ by Theorem 3.23 (4).
   By Theorem 3.20, we have $(F, A) \subseteq vs\tilde{s}cl(F, A) \subseteq v\tilde{s}cl(F, A)$. Then we can get, $v\tilde{s}cl(F, A) \subseteq v\tilde{s}cl(vs\tilde{s}cl(F, A)) \subseteq v\tilde{s}cl(F, A)$ and so $v\tilde{s}cl(vs\tilde{s}cl(F, A)) = v\tilde{s}cl(F, A)$. ■

4. Decomposition of vague α-soft open sets

Definition 4.1. A vague soft set (F,A) of a vague soft topological space (X, τ, E) is said to be

1. vague pre-soft open if $(F, A) \subseteq v\tilde{s}int(v\tilde{s}cl(F, A))$.

2. vague α-soft open if $(F, A) \subseteq v\tilde{s}int(v\tilde{s}cl(v\tilde{s}int(F, A)))$.

3. vague regular-soft open if $(F, A) = v\tilde{s}int(v\tilde{s}cl(F, A))$.

The relative complement of a vague pre-soft open (resp. vague α-soft open, vague regular-soft open) set is called vague pre-soft closed (resp. vague α-soft closed, vague regular-soft closed) set.

We will denote the family of all vague pre-soft open sets (resp. vague α-soft open sets, vague regular-soft open sets) of a vague soft topological space (X, τ, E) by $\mathcal{VP}\tilde{\mathcal{S}}\mathcal{O}(X)$ (resp. $\mathcal{V}\alpha\tilde{\mathcal{S}}\mathcal{O}(X), \mathcal{VR}\tilde{\mathcal{S}}\mathcal{O}(X)$).

Theorem 4.2. For a vague soft topological space (X, τ, E) the following statements hold:

1. Every vague soft open (resp. vague soft closed) set is a vague pre-soft open (resp. vague pre-soft closed) set.

2. Every vague soft open (resp. vague soft closed) set is a vague α-soft open (resp. vague α-soft closed) set.

3. Every vague regular-soft open (resp. vague regular-soft closed) set is a vague soft open (resp. vague soft closed) set.

Proof.

1. Let (F,A) be a vague soft open set. Since (F,A) is vague soft open and $(F, A) \subseteq v\tilde{s}cl(F, A)$, we have $(F, A) = v\tilde{s}int(F, A) \subseteq v\tilde{s}int(v\tilde{s}cl(F, A))$. That is $(F, A) \subseteq v\tilde{s}int(v\tilde{s}cl(F, A))$. Hence (F,A) is vague pre-soft open.

2. Let (F,A) be a vague soft open set. And now $(F, A) \subseteq v\tilde{s}cl(F, A) = v\tilde{s}cl(v\tilde{s}int(F, A))$.
   That is, $(F, A) \subseteq v\tilde{s}cl(v\tilde{s}int(F, A))$.
   $\Rightarrow$ $(F, A) = v\tilde{s}int(F, A) \subseteq v\tilde{s}int(v\tilde{s}cl(v\tilde{s}int(F, A)))$.
   Hence (F,A) is vague α-soft open.

3. Let (F,A) be a vague regular-soft open set.
   Then $(F, A) = v\tilde{s}int(v\tilde{s}cl(F, A))$.
   $\Rightarrow$ $v\tilde{s}int(F, A) = v\tilde{s}int(v\tilde{s}cl(F, A)) = (F, A)$.
   Hence, (F,A) is vague soft open. ∎

Theorem 4.3. For a vague soft topological space (X, τ, E) we have the followings:

i. Every vague α-soft open (resp. vague α-soft closed) set is a vague semi-soft open (resp. vague semi-soft closed) set.

ii. Every vague α-soft open (resp. vague α-soft closed) set is a vague pre-soft open (resp. vague pre-soft closed) set.

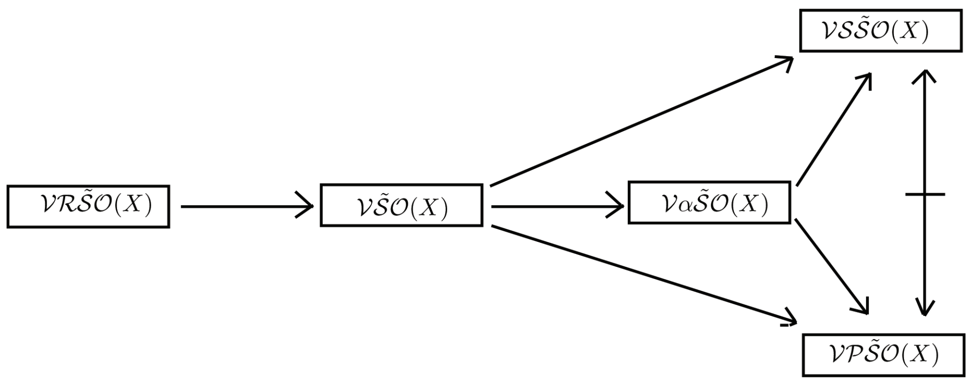
Proof.

i. Let (G,B) be a vague α-soft open set. Then $(G,B) \subseteq v\tilde{s}int(v\tilde{s}cl(v\tilde{s}int(G,B))) \subseteq v\tilde{s}cl(v\tilde{s}int(G,B))$. Hence (G,B) is vague semi-soft open.

ii. Let (G,B) be a vague α-soft open set. Then $(G,B) \subseteq v\tilde{s}int(v\tilde{s}cl(v\tilde{s}int(G,B)))$. Also since, $v\tilde{s}int(G,B) \subseteq (G,B)$, we obtain $v\tilde{s}int(v\tilde{s}cl(v\tilde{s}int(G,B))) \subseteq v\tilde{s}int(v\tilde{s}cl(G,B))$. Thus, $(G,B) \subseteq v\tilde{s}int(v\tilde{s}cl(G,B))$. Hence (G,B) is vague pre-soft open. ■

Remark 4.4. The notions of Vague semi-soft open (resp. vague semi-soft closed) sets and vague pre-soft open (resp. vague pre-soft closed) sets are independent of each other.

The following diagram shows that the relationship between the stronger and some weaker forms of vague soft sets discussed above.



Remark 4.5. The converse implications of the above diagram need not be true as shown in the following Examples.

Example 4.6. Let $X = \{x_1, x_2, x_3, x_4\}$, $E = \{e_1, e_2\}$ and let (X, τ, E) be a vague soft topological space with vague soft topology $\tau = \{\hat{\emptyset}, (G,E), \hat{X}\}$,

$$where,\ (G,E) = \left\{\left\langle e_1, \frac{[0.4,0.5]}{x_1}, \frac{[0.1,0.6]}{x_2}, \frac{[0.4,0.6]}{x_3}, \frac{[0.3,0.5]}{x_4}\right\rangle,\right.$$

$$\left.\left\langle e_2, \frac{[0.2,0.3]}{x_1}, \frac{[0.2,0.4]}{x_2}, \frac{[0,0.1]}{x_3}, \frac{[0.3,0.7]}{x_4}\right\rangle\right\}.$$

And let

$$(F_1,E) = \left\{\left\langle e_1, \frac{[0.7,0.8]}{x_1}, \frac{[0.8,0.9]}{x_2}, \frac{[0.5,0.8]}{x_3}, \frac{[0.5,0.9]}{x_4}\right\rangle,\right.$$

$$\left.\left\langle e_2, \frac{[0.8,0.9]}{x_1}, \frac{[0.8,0.9]}{x_2}, \frac{[1,1]}{x_3}, \frac{[0.6,0.8]}{x_4}\right\rangle\right\}.$$

Then the vague soft set (F_1,E) is vague pre-soft open but it is neither vague semi-soft open nor vague α-soft open.

Example 4.7. In Example 4.6, the vague soft set

$$(F_2, E) = \left\{ \left\langle e_1, \frac{[0.4, 0.5]}{x_1}, \frac{[0.3, 0.7]}{x_2}, \frac{[0.4, 0.6]}{x_3}, \frac{[0.4, 0.6]}{x_4} \right\rangle, \right.$$

$$\left. \left\langle e_2, \frac{[0.4, 0.6]}{x_1}, \frac{[0.3, 0.5]}{x_2}, \frac{[0.2, 0.3]}{x_3}, \frac{[0.3, 0.7]}{x_4} \right\rangle \right\}.$$

is vague semi-soft open but it is neither vague pre-soft open nor vague α-soft open.

Example 4.8. Let $X = \{x_1, x_2, x_3\}$, $E = \{e_1, e_2\}$ and let (X, τ, E) be a vague soft topological space with $\tau = \{\hat{\emptyset}, (G, E), \hat{X}\}$,where (G,E) is vague soft open set defined as,

$$(G, E) = \left\{ \left\langle e_1, \frac{[0.2, 0.8]}{x_1}, \frac{[0.4, 0.7]}{x_2}, \frac{[0.4, 0.6]}{x_3} \right\rangle, \right.$$

$$\left. \left\langle e_2, \frac{[0.6, 0.9]}{x_1}, \frac{[0.5, 0.7]}{x_2}, \frac{[0.4, 0.7]}{x_3} \right\rangle \right\}.$$

Then (G,E) is not a vague regular-soft open set.

Example 4.9. In Example 4.8, the vague soft set

$$(F_3, E) = \left\{ \left\langle e_1, \frac{[0.4, 0.8]}{x_1}, \frac{[0.7, 0.8]}{x_2}, \frac{[0.6, 0.7]}{x_3} \right\rangle, \right.$$

$$\left. \left\langle e_2, \frac{[0.9, 1]}{x_1}, \frac{[0.8, 0.9]}{x_2}, \frac{[0.6, 0.8]}{x_3} \right\rangle \right\}.$$

is a vague α-soft open set but not vague soft open.

Theorem 4.10. A vague soft set (F,A) of a vague soft topological space (X, τ, E) is vague α-soft open set iff (F,A) is both vague semi-soft open and vague pre-soft open set.

Proof. Necessity: It follows from the fact that, every vague α-soft open set is vague semi-soft open and vague pre-soft open.
Sufficiency: Let (G,B) be vague semi-soft open and vague pre-soft open. Then, we have
$(G, B) \subseteq v\tilde{s}int(v\tilde{s}cl(G, B)) \subseteq v\tilde{s}int(v\tilde{s}cl(v\tilde{s}cl(v\tilde{s}int(G, B)))) = v\tilde{s}int(v\tilde{s}cl(v\tilde{s}int(G, B)))$.
This shows that (G,B) is vague α-soft open. ■

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