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A New Approach to Infinite Matrices of Interval Numbers

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Abstract

In this paper, Cesàro interval null, Cesàro interval convergent and Cesàro interval bounded sequence spaces of interval numbers are introduced and proved some inclusion relations on them. Additionally, an isomorphism is constructed on these interval sequence spaces. Furthermore, $M(u), Y(u)$ and $O(u)$ symbols and some properties of these notions are investigated. Besides, definition of infinite dimensional Cesàro interval matrix and its left and right parts are introduced. Moreover, an useful comparison is given between classical Cesàro matrix transformation and Cesàro interval matrix transformation. Finally, completion of interval metric spaces is given.

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1. Introduction

Interval arithmetic was introduced by Dwyer [7]. In, [5], Chiao established sequence of interval numbers and gave the definition of usual convergence of sequence of interval numbers. Bounded and convergent sequence spaces of interval numbers are studied by Ṣengönül and Eryılmaz. In recent years, Esi [8] introduced lacunary sequence spaces of interval numbers. Hansen and Smith [9] make matrix calculations by means of interval arithmetic, firstly. After, many others such as Neumaier [11], Jaulin et al [10] and Rohn [13], etc. have worked on interval matrices. It is clear that matrices have an important place in various fields such as mathematics, engineering and statistics. Furthermore, intervals are generalization of real numbers. Therefore, arithmetic operations on intervals are a generalization of the arithmetic operations defined on the real number set. In addition, intervals will be seen as a bridge between fuzzy and classical sets.

To build a new sequence space by means of the matrix domain of a certain limitation method, many studies were employed by many authors, for example you can see: Altay, Başar and Mursaleen [2], Başar and Altay [3], [4], Ng and Lee [12] and Wang [15].

The rest of our paper is organized, as follows:

In Section 2, some basic definitions and theorems related with the interval numbers are given. Also, definitions of interval metric metric space, sequence of interval numbers, interval Cauchy sequence are given. In Section 3, we have introduced Cesàro interval null, Cesàro interval convergent and Cesàro interval bounded sequence spaces of interval numbers as the set of all sequences such that $\mathcal{C}$-transforms of them are in the spaces c^i, c_0^i and ℓ_∞^i, respectively, by means of Cesàro interval matrix and proved some inclusion relations on these sequence spaces. It is also established in Section 3 that the sequence spaces showed by $\mathbb{E}_0^{\mathcal{C}}, \mathbb{E}_c^{\mathcal{C}}$ and $\mathbb{E}_b^{\mathcal{C}}$ are linearly isomorphic to the spaces c_0^i, c^i and ℓ_∞^i, respectively. Additionally, it is proved that the spaces $\mathbb{E}_0^{\mathcal{C}}, \mathbb{E}_c^{\mathcal{C}}$ and $\mathbb{E}_b^{\mathcal{C}}$ are complete metric spaces. Furthermore, $M(u), Y(u)$ and $O(u)$ symbols and some properties are investigated. Finally, in Section 3, interval matrix notion and algebraic structures on matrices are given. Moreover, Cesàro interval and left, right Cesàro interval matrices are introduced and a comparison is given between classical Cesàro matrix transformation and Cesàro interval matrix transformation. In the final Section 4, completion of interval metric spaces is given.

2. Preliminaries, Background and Notation

In this section, we recall some of the basic definitions and notions in the theory of interval numbers and sequence spaces such as notions of interval metric space, algebraic operations on $\mathbb{E}$, $(\mathbb{E}, +)$ triple, interval convergence and interval Cauchy sequence.

Let suppose that $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{E}$ are the set of all non-negative integers, all real numbers and all bounded and closed intervals on $\mathbb{R}$, respectively. Any element of $\mathbb{E}$ is denoted by u and called as interval number. That is, $\mathbb{E} = \{u = [u^-, u^+] : u^-, u^+ \in \mathbb{R}, u^- \leq u^+\}$. For $u, v \in \mathbb{E}$, we have $u = v \Leftrightarrow u^- = v^-, u^+ = v^+$. Additionally, we give the algebraic operations addition, scalar multiplication and multiplication as follows, respectively:

$$+ : \mathbb{E} \times \mathbb{E} \to \mathbb{E}, +(u, v) = u + v = [u^- + v^-, u^+ + v^+],$$

$$\cdot : \mathbb{R} \times \mathbb{E} \to \mathbb{E}, \alpha u = \begin{cases} [\alpha u^-, \alpha u^+], & \alpha \geq 0 \\ [\alpha u^+, \alpha u^-], & \alpha < 0, \end{cases}$$

$$\cdot : \mathbb{E} \times \mathbb{E} \to \mathbb{E}, \cdot(u, v) = u \cdot v = [\min R, \max R], R = \{u^-v^-, u^-v^+, u^+v^-, u^+v^+\}.$$

We will use abbreviation $u_0 \mp \epsilon = [u_0 - \epsilon, u_0 + \epsilon] = u_0 + \epsilon[-1, 1]$ where $\epsilon > 0$, in appropriate places.

By w, we denote the set of all complex valued sequences. w is a linear space with the defined addition and scalar multiplication. Additionally, each linear subspace of w is called a sequence space. We show ℓ_∞, c and c_0 for the classical sequence spaces of all bounded, convergent and null sequences, respectively. We can give the most general linear operator between two sequence spaces by means of infinite matrices. For this reason, matrix transformations have an important place in sequence space studies. For brevity in notation, through all the text, we shall write $\sum_n$, $\sup_n$, and $\lim_n$ instead of $\sum_{n=0}^{\infty}$, $\sup_{n\in\mathbb{N}}$, and $\lim_{n\to\infty}$.

Let λ and μ be two sequence spaces and $\mathcal{A} = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, we can say that $\mathcal{A}$ defines a matrix mapping from λ to μ, and we denote it by writing $\mathcal{A} \in (\lambda : \mu)$, if for every sequence $x = (x_k)$ is in λ and the sequence $\mathcal{A}x = \{(\mathcal{A}x)_n\}$, the $\mathcal{A}$- transform of x, is in μ, where k runs from 0 to ∞. The domain $\lambda_{\mathcal{A}}$ of an infinite matrix $\mathcal{A}$ in a sequence space λ is defined by

$$\lambda^{\mathcal{A}} = \{x = (x_k) \in w : \mathcal{A}x \in \lambda\} \tag{1}$$

which is a sequence space. If we take $\lambda = c$, then $c_{\mathcal{A}}$ is called convergence domain of $\mathcal{A}$. We write the limit of $\mathcal{A}x$ as $\mathcal{A} - \lim_n x_n = \lim_n \sum_{k=0}^{\infty} a_{nk}x_k$, and the $\mathcal{A}$ is called regular if $\lim_{\mathcal{A}} x = \lim x$ for every $x \in c$. Also, a matrix $\mathcal{A} = (a_{nk})$ is called triangle if $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$ for all $n \in \mathbb{N}$.

Definition 2.1. Let $(\mathbb{E}, +)$ be Abel monoid and the following conditions are satisfied for all $u, v \in \mathbb{E}$ and for all $\alpha, \beta \in \mathbb{R}$:

1. $\alpha(u + v) = \alpha u + \alpha v$
2. $\alpha(\beta u) = (\alpha\beta)u$
3. $[1, 1]u = u$
4. $(\alpha + \beta)u = \alpha u + \beta u, \alpha, \beta \geq 0.$

Here, $\theta = [0, 0]$ and $[1, 1]$ are identity elements of $\mathbb{E}$ according to addition and multiplication operations, respectively. In this case, we can say that $(\mathbb{E}, +)$ triple is called almost linear space on $\mathbb{R}$.

Definition 2.2. The sequence which terms consists of compact subsets of $\mathbb{R}$ is called the sequence of interval numbers. Let us show the set of all sequence spaces of interval numbers by $w(\mathbb{E})$, for $k \in \mathbb{N}$ as written below:

$$w(\mathbb{E}) = \{u^k = ([u_k^-, u_k^+]) : f, g : \mathbb{N} \to \mathbb{R}, f(k) = u_k^-, g(k) = u_k^+, u_k^- \leq u_k^+\}.$$

Let $\lambda \subset w(\mathbb{E})$ and $d : \lambda \times \lambda \to \mathbb{R}$ be a metric. For $u^k \in w(\mathbb{E})$ we can give following definitions by means of [5].

1. A sequence $u^k = ([u_k^-, u_k^+]) \in w(\mathbb{E})$ is convergent to $[u_0^-, u_0^+]$ according to metric d if and only if for every $\epsilon > 0$ there exists a $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$, $d([u_k^-, u_k^+], [u_0^-, u_0^+]) < \epsilon$. It is shown by $\lim_k [u_k^-, u_k^+] = [u_0^-, u_0^+]$ or $[u_k^-, u_k^+] \to [u_0^-, u_0^+], k \to \infty$.

2. A sequence $u^k = ([u_k^-, u_k^+])$ of interval numbers is said to be Cauchy sequence if for every $\epsilon > 0$ there exist a $k_0 \in \mathbb{N}$ such that $d([u_k^-, u_k^+], [u_n^-, u_n^+]) < \epsilon$ for all $k, n \geq k_0$. It is easy to see that every convergent sequence of interval numbers is a Cauchy sequence.

Definition 2.3. Let X be a non-empty interval set and $d : X \times X \to \mathbb{E}^+$ be interval metric function. The couple (X, d) is called interval metric space, if

1. $d(u, v) = 0$ if and only if $u = v$,

2. $d(u, v) = d(v, u) = 0$ for all $u, v \in X$,

3. $d(u, z) \leq d(u, v) + d(v, z)$.

In the following, some special sub-sets of $\mathbb{E}$ will be given [14]:

$$c^i = \{u^k = ([u_k^-, u_k^+]) : \lim_k [u_k^-, u_k^+] = [u_0^-, u_0^+]\} \tag{2}$$

$$c_0^i = \{u^k = ([u_k^-, u_k^+]) : \lim_k [u_k^-, u_k^+] = [0, 0]\} \tag{3}$$

$$\ell_\infty^i = \{u^k = ([u_k^-, u_k^+]) : \sup_k M([u_k^-, u_k^+]) < \infty\}. \tag{4}$$

We called these subsets as interval convergent, null interval convergent and interval bounded sequence spaces, respectively. If we take $u_k^- = u_k^+$ for all $k \in \mathbb{N}$ then (2), (3) and (4) are correspond to convergent, null and bounded sequence spaces of real numbers, respectively.

3. Cesàro Interval Matrices and sequence spaces

In this section, matrices which consist of intervals are introduced and some properties of these matrices are examined.

Definition 3.1. Let us suppose that $\mathbb{N}_n = \{0, 1, 2, \ldots, n\}, \mathbb{N}_m = \{0, 1, 2, \ldots, m\}$ and

$$f : \mathbb{N}_n \times \mathbb{N}_m \to \mathbb{E}$$

$$(i, j) \to f(i, j) = a_{ij} = a \in \mathbb{E}, (1 \leq i \leq n, 1 \leq j \leq m)$$

are given. From here, we obtain interval matrix $\mathcal{A} = [[a_{ij}^-, a_{ij}^+]]_{n\times m}$ that consists of elements $a_{ij} = [a_{ij}^-, a_{ij}^+]$. It is easy to see that if we take $a_{ij}^- = a_{ij}^+$ for $1 \leq i \leq n$ and $1 \leq j \leq m$ then $[[a_{ij}^-, a_{ij}^+]]_{n\times m}$ is reduced to real matrices which has $n \times m$ dimension. We show the set of all interval matrices with $n \times m$ dimension as below:

$$\mathbb{E}^{n\times m} = \{\mathcal{A} : \mathcal{A} \text{ is a interval matrix with } n \times m \text{ dimension}\}.$$

The matrix in the form showed as in the following is called infinite dimensional interval matrix.

$$\mathcal{A} = \begin{bmatrix} [a_{11}^-, a_{11}^+] & \dots & [a_{1m}^-, a_{1m}^+] & \dots \\ [a_{21}^-, a_{21}^+] & \dots & [a_{2m}^-, a_{2m}^+] & \dots \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ [a_{n1}^-, a_{n1}^+] & \dots & [a_{nm}^-, a_{nm}^+] & \dots \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \end{bmatrix}_{\infty\times\infty}$$

Let $\mathcal{A}, \mathcal{B} \in \mathbb{E}^{n\times m}, \lambda \in \mathbb{R}$. We give the algebraic structures of interval matrices addition and scalar multiplication as in the following:

$$\mathcal{A} + \mathcal{B} = [[a_{ij}^-, a_{ij}^+]]_{n\times m} + [[b_{ij}^-, b_{ij}^+]]_{n\times m},$$

$$\lambda u = \begin{cases} [[\lambda a_{ij}^-, \lambda a_{ij}^+]]_{n\times m}, & \lambda \geq 0, \\ [[\lambda a_{ij}^+, \lambda a_{ij}^-]]_{n\times m}, & \lambda < 0. \end{cases}$$

Additionally, if we take $\mathcal{D} \in \mathbb{E}^{m\times r}$ then multiplication of $\mathcal{A} \cdot \mathcal{D}$ is defined as follows for $1 \leq i \leq n, 1 \leq j \leq r$:

$$\mathcal{A} \cdot \mathcal{D} = [[a_{ij}^-, a_{ij}^+]]_{n\times m} \cdot [[d_{ij}^-, d_{ij}^+]]_{m\times r} = [[c_{ij}^-, c_{ij}^+]]_{n\times r}$$

where

$$c_{ij}^- = \sum_{k=1}^{m} \min\{a_{ik}^- d_{kj}^-, a_{ik}^- d_{kj}^+, a_{ik}^+ d_{kj}^-, a_{ik}^+ d_{kj}^+\}$$

$$c_{ij}^+ = \sum_{k=1}^{m} \max\{a_{ik}^- d_{kj}^-, a_{ik}^- d_{kj}^+, a_{ik}^+ d_{kj}^-, a_{ik}^+ d_{kj}^+\}.$$

Theorem 3.2. Let us suppose that $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{E}^{n\times m}$. Then, "+" operation defined $+ : \mathbb{E}^{n\times m} \times \mathbb{E}^{n\times m} \to \mathbb{E}^{n\times m}$ satisfies the following properties:

1. $\mathcal{A} + \mathcal{B} = \mathcal{B} + \mathcal{A}$

2. $(\mathcal{A}+\mathcal{B})+\mathcal{C}=\mathcal{A}+(\mathcal{B}+\mathcal{C})$

3. $\mathcal{A}+\theta=\mathcal{A}$.

Here $\theta=[[\theta_{ij}^{-},\theta_{ij}^{+}]]=[[0,0]]\in\mathbb{E}^{n\times m}, 1\le i\le n, 1\le j\le m$.

Definition 3.3. $\mathcal{A}=[[a_{ij}^{-},a_{ij}^{+}]]_{n\times m}$ be an interval matrix of interval numbers. If there exists $\mathcal{B}=[[b_{ij}^{-},b_{ij}^{+}]]_{n\times m}$ such that $[b_{ij}^{-},b_{ij}^{+}]\subseteq[a_{ij}^{-},a_{ij}^{+}]$ for all $i,j\in\mathbb{N}$, then we called $\mathcal{B}$ by sub-matrix of $\mathcal{A}$ and showed by $\mathcal{B}\subseteq\mathcal{A}$.

Because of the fact that every real number is an interval which has same first and end terms as mentioned in Definition 3.3, it is clear that each finite dimensional matrix of real numbers is sub-matrix of an interval matrix.

Theorem 3.4. $\mathcal{A},\mathcal{B},\mathcal{C}$ and $\mathcal{D}$ be interval matrices with type $n\times n$ and $A=[a_{ij}]$ be real matrix with $n\times m$ type. Then, following features are obtained easily:

1. If $\mathcal{C}\subseteq\mathcal{B}$ and $\mathcal{D}\subseteq\mathcal{A}$ then $\mathcal{C}\mathcal{D}\subseteq\mathcal{B}\mathcal{A}$,

2. $\mathcal{A}(\mathcal{B}+\mathcal{C})\subseteq\mathcal{A}\mathcal{B}+\mathcal{A}\mathcal{C}$,

3. $A(\mathcal{A}+\mathcal{B}=A\mathcal{A}+A\mathcal{B}$.

Interval matrices of type $1\times n$ are called vectors of interval numbers. The interval matrices of type $n\times m$ can be expanded to infinite dimensional intervals of type $\infty\times\infty$ as mentioned in the following definition.

Definition 3.5. Now, let us take $u^k=([u_k^{-},u_k^{+}])=([u_1^{-},u_1^{+}],[u_2^{-},u_2^{+}],\dots,[u_k^{-},u_k^{+}],\dots)$ be a sequence of interval numbers. If series in the form $\sum_k [a_{nk}^{-},a_{nk}^{+}][u_k^{-},u_k^{+}]$ are convergent for all $n\in\mathbb{N}$ then $\mathcal{A}u^k=v^k$ is called as $\mathcal{A}-$ transformation of interval sequence u^k. In addition this, if we choose $a_{nk}^{-}=a_{nk}^{+}$ and $u_k^{-}=u_k^{+}$ for all $n,k\in\mathbb{N}$, in this case the equation

$$\begin{bmatrix} [a_{11}^{-},a_{11}^{+}] & \dots & [a_{1m}^{-},a_{1m}^{+}] & \dots \\ [a_{21}^{-},a_{21}^{+}] & \dots & [a_{2m}^{-},a_{2m}^{+}] & \dots \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ [a_{n1}^{-},a_{n1}^{+}] & \dots & [a_{nm}^{-},a_{nm}^{+}] & \dots \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}_{\infty\times\infty} \begin{bmatrix} [u_{11}^{-},u_{11}^{+}] \\ [u_{21}^{-},u_{21}^{+}] \\ \cdot \\ \cdot \\ \cdot \\ [u_k^{-},u_k^{+}] \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}_{\infty\times 1} = \begin{bmatrix} \sum_k [a_{1k}^{-},a_{1k}^{+}][u_k^{-},u_k^{+}] \\ \sum_k [a_{2k}^{-},a_{2k}^{+}][u_k^{-},u_k^{+}] \\ \cdot \\ \cdot \\ \cdot \\ \sum_k [a_{nk}^{-},a_{nk}^{+}][u_k^{-},u_k^{+}] \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}_{\infty\times 1}$$

is reduced to matrix transformations of sequences in the classical sense.

Definition 3.6. $\mathcal{A}$ be an infinite dimensional interval matrix in type $\infty \times \infty$. Then, norm of the matrix $\mathcal{A}$ is given as below:

$$||\mathcal{A}|| = \sup_i \sum_j M([a_{ij}^-, a_{ij}^+]).$$

where $M([a_{ij}^-, a_{ij}^+]) = \max\{|a_{ij}^-|, |a_{ij}^+|\}$.

Definition 3.7. We give generalized definition of Cesàro matrix by means of interval numbers for all $n, k \in \mathbb{N}$. Here, first of all we introduce Cesàro interval matrix as follows:

$$\mathcal{C} = (\mathcal{C}_{nk}) = \begin{cases} [\frac{-1}{n+1}, \frac{1}{n+1}], & n \geq k, \\ [0, 0], & \text{otherwise.} \end{cases}$$

After, we divide Cesàro interval matrix into two parts named left Cesàro interval matrix and right Cesàro interval matrix. We showed left and right Cesàro interval matrices by $\mathcal{C}^-$ and $\mathcal{C}^+$, respectively, as given in the following:

$$\mathcal{C}^- = (\mathcal{C}_{nk}^-) = \begin{cases} [\frac{-1}{n+1}, 0], & n \geq k, \\ [0, 0], & \text{otherwise.} \end{cases}$$

$$\mathcal{C}^+ = (\mathcal{C}_{nk}^+) = \begin{cases} [0, \frac{1}{n+1}], & n \geq k, \\ [0, 0], & \text{otherwise.} \end{cases}$$

Norm of the matrices $\mathcal{C}^-, \mathcal{C}^+$ and $\mathcal{C}$ is equal to 1. It means that $||\mathcal{C}^-|| = ||\mathcal{C}^+|| = ||\mathcal{C}|| = 1$.

Definition 3.8. The matrix

$$\mathcal{A} = \begin{cases} [a_{nk}^-, a_{nk}^+], & n \leq k, \\ [0, 0], & \text{otherwise} \end{cases}$$

is defined as the lower triangular interval matrix of type $\infty \times \infty$. Additionally, if there is no element on the principal diagonal, then $\mathcal{A}$ is called normal interval matrix.

A number of rules and properties can be extended to intervals in Summability Theory.

Let $r = (r_k)$ be a sequence of real numbers. We say that $[L^-, L^+]$ interval number is $\mathcal{A}$ interval limit of the sequence (r_k) if and only if the sequence $(\mathcal{A}r)_n = \sum_k [a_{nk}^-, a_{nk}^+] r_k$ is convergent to $[L^-, L^+]$.

Interval matrices can be used to determine the range of the limit, when there is difficulty in the determination phase of the limit of a reel valued, bounded but non-convergent sequence.

Cesàro limit of the real valued but non-convergent sequence (-1^k) is equal to 0. $\mathcal{C}^-, \mathcal{C}^+$ Cesàro limits of the sequence (-1^k) is given as in the following:

$$\lim_n (\mathcal{C}^- r)_n = \lim_n = \begin{cases} [\dfrac{-1}{2}, \dfrac{1}{2}], & n \geq k, \\ [\dfrac{-(n+1)}{2n}, \dfrac{n-1}{2n}], & \text{otherwise.} \end{cases}$$

$$\lim_n (\mathcal{C}^+ r)_n = \lim_n \begin{cases} [\dfrac{-1}{2}, \dfrac{1}{2}], & \text{if } n \text{ is even,} \\ [\dfrac{-(n+1)}{2n}, \dfrac{n+1}{2n}], & \text{if } n \text{ is odd.} \end{cases}$$

The result of both limits is equal to $\left[\dfrac{-1}{2}, \dfrac{1}{2}\right]$. Limits are coincident. If we calculate the $\mathcal{C}$- limit of the sequence $(-1)^k$ with a straightforward calculation it can be written as a result that $\lim_n (\mathcal{C}r)_n = [-1, 1]$. In spite of the fact that there are three intervals which include 0, Cesàro limit of the sequence $(-1)^k$, the interval $\left[\dfrac{-1}{2}, \dfrac{1}{2}\right]$ is narrower than $[-1, 1]$. Taking account this example we give the following theorem for explain our idea.

Theorem 3.9. If there exists both left and right Cesàro limits of a sequence (r_k), it means that $\lim_n (\mathcal{C}^- r)_n = [L_1^-, L_1^+]$ and $\lim_n (\mathcal{C}^+ r)_n = [L_2^-, L_2^+]$ are present then $\lim_n (\mathcal{C}^- r)_n + \lim_n (\mathcal{C}^+ r)_n = \lim_n (\mathcal{C}r)_n = [L_1^- + L_2^-, L_1^+, L_2^+]$.

Proof. When matrix addition is considered proof is obtained easily. ■

A generalization of Theorem 3.9 will be given as below.

Theorem 3.10. Let $\mathcal{A}$ and $\mathcal{B}$ be infinite matrices of intervals. In this case,

1. If $r = (r_k)$ is a sequence of real numbers then $\lim_n ((\mathcal{A} + \mathcal{B})r)_n = \lim_n (\mathcal{A}r)_n + \lim_n (\mathcal{B}r)_n$.
2. $\lim_n ((\mathcal{A} + \mathcal{B})u)_n = \lim_n (\mathcal{A}u)_n + \lim_n (\mathcal{B}u)_n$ where u^k is a sequence of interval numbers.

Proof. Proof is clear from ([11], Page 79 Proposition 3.1.2). ■

Now, we give the theorem about regularity of interval infinite dimensional matrices which is expressed and proved by [6].

Theorem 3.11. The matrix $\mathcal{A} = \left[\sum_{k=1}^{\infty} a_{nk}^-, \sum_{k=1}^{\infty} a_{nk}^+\right]$ of interval numbers is regular for every $n = 1, 2, \ldots$ if and only if it satisfies the following statements:

1. There exists $M > 0$ such that $\sum_{k=1}^{\infty} |a_{nk}^-| \leq M$ and $\sum_{k=1}^{\infty} |a_{nk}^+| \leq M$.

2. $\lim_n \left[\sum_{k=1}^{\infty} a_{nk}^-, \sum_{k=1}^{\infty} a_{nk}^+\right] = [0,0]$ for every $k = 1, 2, \ldots$

3. $\lim_n \left[\sum_{k=1}^{\infty} a_{nk}^-, \sum_{k=1}^{\infty} a_{nk}^+\right] = [1,1]$.

From here, it is easy to see that interval Cesàro matrix is regular.

3.1. The Cesàro Interval Sequence Spaces $\mathbb{E}_0^{\mathfrak{C}}$, $\mathbb{E}_c^{\mathfrak{C}}$ and $\mathbb{E}_b^{\mathfrak{C}}$

In this section, we wish to introduce the Cesàro interval null, Cesàro interval convergent and Cesàro interval bounded sequence spaces of interval numbers represented by $\mathbb{E}_0^{\mathfrak{C}}, \mathbb{E}_c^{\mathfrak{C}}$ and $\mathbb{E}_b^{\mathfrak{C}}$, as the set of all sequences such that $\mathfrak{C}$-transforms of them are in the spaces c_0^i, c^i and ℓ_∞^i, respectively, that is

$$\mathbb{E}_0^{\mathfrak{C}} = \{u^k = ([u_k^-, u_k^+]) \in w(\mathbb{E}) : \mathfrak{C}u^k \in c_0^i\} \tag{5}$$

$$\mathbb{E}_c^{\mathfrak{C}} = \{u^k = ([u_k^-, u_k^+]) \in w(\mathbb{E}) : \mathfrak{C}u^k \in c^i\} \tag{6}$$

and

$$\mathbb{E}_b^{\mathfrak{C}} = \{u^k = ([u_k^-, u_k^+]) \in w(\mathbb{E}) : \mathfrak{C}u^k \in \ell_\infty^i\}. \tag{7}$$

It is easy to see that Cesàro interval matrix is lower triangular and regular interval matrix of type $\infty \times \infty$.

Now, we define the sequence of interval numbers $v^k = ([v_k^-, v_k^+])$ which will be frequently used, as the $\mathfrak{C}$- transform of a sequence of interval numbers $u^k = ([u_k^-, u_k^+])$ for $n, k \in \mathbb{N}$, i.e.

$$[v_k^-, v_k^+] = \left[\frac{-1}{k+1}, \frac{1}{k+1}\right] \sum_{j=0}^{k} [u_j^-, u_j^+]. \tag{8}$$

Theorem 3.12. The sequence spaces $\mathbb{E}_0^{\mathfrak{C}}, \mathbb{E}_c^{\mathfrak{C}}$ and $\mathbb{E}_b^{\mathfrak{C}}$ are linearly isomorphic to the spaces c_0^i, c^i and ℓ_∞^i, respectively, i.e. $\mathbb{E}_0^{\mathfrak{C}} \cong c_0^i$, $\mathbb{E}_c^{\mathfrak{C}} \cong c^i$ and $\mathbb{E}_b^{\mathfrak{C}} \cong \ell_\infty^i$.

Proof. We consider only the case $\mathbb{E}_b^{\mathfrak{C}} \cong \ell_\infty^i$ since others can be done in the same way. First of all, we should show the existence of a linear bijection between the spaces $\mathbb{E}_b^{\mathfrak{C}}$ and ℓ_∞^i. Consider the transformation defined T, with the notation of (8) by $T : \mathbb{E}_b^{\mathfrak{C}} \to \ell_\infty^i, u^k \to v^k = Tu^k = \left[\frac{-1}{k+1}, \frac{1}{k+1}\right] \sum_{j=0}^{k} [u_j^-, u_j^+]$. The following equations hold for T:

1. $T(u^k + v^k) = Tu^k + Tv^k$
2. $$T(\alpha u^k) = \begin{cases} \left[\frac{-1}{k+1}, \frac{1}{k+1}\right] \sum_{j=0}^{k} [\alpha u_j^-, \alpha u_j^+] = \alpha \left[\frac{-1}{k+1}, \frac{1}{k+1}\right] \sum_{j=0}^{k} [u_j^-, u_j^+] = \alpha T u^k, & \text{for } \alpha \geq 0 \\ \left[\frac{-1}{k+1}, \frac{1}{k+1}\right] \sum_{j=0}^{k} [\alpha u_j^+, \alpha u_j^-] = \alpha \left[\frac{-1}{k+1}, \frac{1}{k+1}\right] \sum_{j=0}^{k} [u_j^-, u_j^+] = \alpha T u^k, & \text{for } \alpha < 0 \end{cases}$$

   where $u^k, v^k \in \mathbb{E}_b^{\mathcal{C}}$. Thus, T is linear.

Let $v^k \in \ell_\infty^i$ and define the sequence $u^k = ([u_k^-, u_k^+])$ by

$$[u_k^-, u_k^+] = \frac{[v_k^-, v_k^+]}{\left[\frac{-1}{k+1}, \frac{1}{k+1}\right]} - \frac{[v_{k-1}^-, v_{k-1}^+]}{\left[\frac{-1}{k}, \frac{1}{k}\right]}, (k \in \mathbb{N}).$$

From here, we can write

$$\begin{aligned} \sup_{n\in\mathbb{N}} d(\mathcal{C}u^k, \theta) &= \sup_n d\left(\left[\frac{-1}{n+1}, \frac{1}{n+1}\right] \sum_{k=0}^{n} [u_k^-, u_k^+], \theta\right) \\ &= \sup_n d\left(\left[\frac{-1}{n+1}, \frac{1}{n+1}\right] \sum_{k=0}^{n} \frac{[v_k^-, v_k^+]}{\left[\frac{-1}{k+1}, \frac{1}{k+1}\right]} - \frac{[v_{k-1}^-, v_{k-1}^+]}{\left[\frac{-1}{k}, \frac{1}{k}\right]}, \theta\right) \\ &= \sup_n d([v_n^-, v_n^+], \theta) < \infty \end{aligned}$$

which obtain from the above equation is $u^k \in \mathbb{E}_b^{\mathcal{C}}$. Furthermore,

$$\begin{aligned} ||u^k||_{\mathbb{E}_b^{\mathcal{C}}} &= \sup_n d\left(\left[\frac{-1}{n+1}, \frac{1}{n+1}\right] \sum_{k=0}^{n} \frac{[v_k^-, v_k^+]}{\left[\frac{-1}{k+1}, \frac{1}{k+1}\right]} - \frac{[v_{k-1}^-, v_{k-1}^+]}{\left[\frac{-1}{k}, \frac{1}{k}\right]}, \theta\right) \\ &= \sup_n d([v_n^-, v_n^+], \theta) \\ &= ||v^k||_{\ell_\infty^i} < \infty. \end{aligned}$$

Namely, T is norm preserving. As a result, the spaces $\mathbb{E}_b^{\mathcal{C}}$ and ℓ_∞^i are linearly isomorphic. It is clear that if the spaces $\mathbb{E}_b^{\mathcal{C}}$ and ℓ_∞^i are replaced by the spaces $\mathbb{E}_0^{\mathcal{C}}$ and c_0^i, $\mathbb{E}_c^{\mathcal{C}}$ and c^i, we obtain the fact that $\mathbb{E}_0^{\mathcal{C}} \cong c_0^i$ and $\mathbb{E}_c^{\mathcal{C}} \cong c^i$. This completes the proof. ■

Let us define the sequence $c^k = \{c_k^{(n)}\}_{n\in\mathbb{N}}$ of the space $\mathbb{E}_c^{\mathcal{C}}$ as below:

$$c_k^{(n)} = \begin{cases} [(-1)^{n-k}(k+1), (-1)^{n-k}(k+1)], & n-1 \leq k \leq n \\ [0, 0], & 0 \leq k \leq n-1 \text{ or } k > n \end{cases}$$

for every fixed $k \in \mathbb{N}$. Then we can say that the sequence c^k is a basis for the space $\mathbb{E}_c^{\mathcal{C}}$ and every $u^k \in \mathbb{E}_c^{\mathcal{C}}$ has a unique representation of the form

$$u^k = \sum_k \mu_k c^k$$

where $\mu_k = (\mathcal{C}u)_k$ for all $k \in \mathbb{N}$.

Theorem 3.13. The sets $(\mathbb{E}_c^{\mathcal{C}}, d), (\mathbb{E}_0^{\mathcal{C}}, d)$ and $(\mathbb{E}_b^{\mathcal{C}}, d)$ are complete metric spaces with the metric defined by

$$\sup_n \max \left\{ \left| \sum_{k=1}^{n} c_{nk}^- u_k^- - \sum_{k=1}^{n} c_{nk}^- v_k^- \right|, \left| \sum_{k=1}^{n} c_{nk}^+ u_k^+ - \sum_{k=1}^{n} c_{nk}^+ v_k^+ \right| \right\}. \tag{9}$$

Proof. It was seen that in Theorem 3.12, the sequence spaces of interval numbers $(\mathbb{E}_c^{\mathcal{C}}, d), (\mathbb{E}_0^{\mathcal{C}}, d)$ and $(\mathbb{E}_b^{\mathcal{C}}, d)$ are linearly isomorphic to the spaces c^i, c_0^i and ℓ_∞^i, respectively. Additionally, since the Cesàro interval matrix is normal (see, [16]) and c^i, c_0^i and ℓ_∞^i are complete normed sequence spaces, it is clear that the sequence spaces $(\mathbb{E}_c^{\mathcal{C}}, d), (\mathbb{E}_0^{\mathcal{C}}, d)$ and $(\mathbb{E}_b^{\mathcal{C}}, d)$ are complete metric spaces with the metric defined in (9). ■

Theorem 3.14. The inclusions $\mathbb{E}_0^{\mathcal{C}} \subset \mathbb{E}_c^{\mathcal{C}} \subset \mathbb{E}_b^{\mathcal{C}}$ strictly hold.

Proof. It is clear that $\mathbb{E}_0^{\mathcal{C}} \subset \mathbb{E}_c^{\mathcal{C}}$. Additionally, if we take the sequence $((-1)^k) = ([(-1)^k, (-1)^k])$ then $\mathcal{C}((-1)^k) \in \mathbb{E}_c^{\mathcal{C}}$ but $\mathcal{C}((-1)^k) \notin \mathbb{E}_0^{\mathcal{C}}$. Now, we show $\mathbb{E}_c^{\mathcal{C}} \subset \mathbb{E}_b^{\mathcal{C}}$. Let $u^k \in \mathbb{E}_c^{\mathcal{C}}$ then $\mathcal{C}u^k \in c^i \subset \ell_\infty^i$. Namely, $u^k \in \mathbb{E}_b^{\mathcal{C}}$. It means that $\mathbb{E}_c^{\mathcal{C}} \subset \mathbb{E}_b^{\mathcal{C}}$. Furthermore, if we take into account the sequence $\left((-1)^n \frac{n}{n+1}\right)$ it is easy to see that $\mathcal{C}\left((-1)^n \frac{n}{n+1}\right) \in \mathbb{E}_b^{\mathcal{C}}$ but $\mathcal{C}\left((-1)^n \frac{n}{n+1}\right) \notin \mathbb{E}_c^{\mathcal{C}}$. This step completes the proof of the theorem. ■

Theorem 3.15. The inclusions $c_0^i \subset \mathbb{E}_0^{\mathcal{C}}, c^i \subset \mathbb{E}_c^{\mathcal{C}}$ and $\ell_\infty^i \subset \mathbb{E}_b^{\mathcal{C}}$ strictly hold.

Proof. Because of the fact that $\mathcal{C}$ is regular interval matrix, inclusions $c_0^i \subset \mathbb{E}_0^{\mathcal{C}}$ and $c^i \subset \mathbb{E}_c^{\mathcal{C}}$ are clear. Besides, let us take the sequence $u^k = ([u_k^-, u_k^+]) = ([(-1)^k, (-1)^k])$, then for every $n \in \mathbb{N}$ we can write that $u^k \notin c^i$ but $u^k \in \mathbb{E}_c^{\mathcal{C}}$. Hence, inclusion $c^i \subset \mathbb{E}_c^{\mathcal{C}}$ strictly holds.

Let us show that $u^k = ([u_k^-, u_k^+]) \in \ell_\infty^i$. From here, we can say that there is an element of $\mathbb{R}$ in the form $M > 0$ such that $\sup_k\{|u_k^-|, |u_k^+|\} < M$ i.e. $|u_k^-| < M$ and

$|u_k^+| < M$ for all $k \in \mathbb{N}$. That is, we can write the followings for each $n \in \mathbb{N}$:

$$\begin{aligned} |\mathcal{C}(u^k)| &= \left\| \left[\sum_{k=1}^{n} c_{nk}^- u_k^-, \sum_{k=1}^{n} c_{nk}^+ u_k^+ \right] \right\| \\ &= \max \left\{ \sum_{k=1}^{n} c_{nk}^- |u_k^-|, \sum_{k=1}^{n} c_{nk}^+ |u_k^+| \right\} \\ &\leq \max \left\{ \sum_{k=1}^{n} c_{nk}^- M, \sum_{k=1}^{n} c_{nk}^+ M \right\} \\ &= M. \end{aligned}$$

Consequently, we conclude the inclusion $\ell_\infty^i \subset \mathbb{E}_b^{\mathcal{C}}$. ∎

Some properties of the sets $\mathbb{E}_c^{\mathcal{C}}, \mathbb{E}_0^{\mathcal{C}}$ and $\mathbb{E}_b^{\mathcal{C}}$, defined above, are listed below. Their proof can be structured as if it were in real-valued sequences.

Proposition 3.16. Let us suppose that $\alpha \in \mathbb{R}$, $\lim_k [u_k^-, u_k^+] = [u_0^-, u_0^+]$ and $\lim_k [v_k^-, v_k^+] = [v_0^-, v_0^+]$. Then we can write the followings:

1. If equation $\lim_k [u_k^-, u_k^+] = [u_0^-, u_0^+]$ hold, then $[u_0^-, u_0^+]$ is one and only.
2. $\lim_k ([u_k^-, u_k^+] + [v_k^-, v_k^+]) = \lim_k [u_k^-, u_k^+] + \lim_k [v_k^-, v_k^+] = [u_0^-, u_0^+] + [v_0^-, v_0^+]$.
3. $\lim_k \alpha[u_k^-, u_k^+] = \alpha \lim_k [u_k^-, u_k^+] = \alpha[u_0^-, u_0^+]$.
4. $\lim_k \dfrac{[u_k^-, u_k^+]}{[v_k^-, v_k^+]} = \lim [u_k^-, u_k^+] \lim [\dfrac{1}{v_k^+}, \dfrac{1}{v_k^-}] = [u_0^-, u_0^+][\dfrac{1}{v_0^+}, \dfrac{1}{v_0^-}]$, (For every $k \in \mathbb{N}, 0 \notin v^k$).
5. Let us suppose that $a = \min\{\lim_k u_k^- v_k^-, \lim_k u_k^- v_k^+, \lim_k u_k^+ v_k^-, \lim_k u_k^+ v_k^+\}$ and $b = \max\{\lim_k u_k^- v_k^-, \lim_k u_k^- v_k^+, \lim_k u_k^+ v_k^-, \lim_k u_k^+ v_k^+\}$. Then it is clear that $\lim_k ([u_k^-, u_k^+][v_k^-, v_k^+]) = [a, b]$.
6. $\mathbb{E}_0^{\mathcal{C}} \subseteq \mathbb{E}_c^{\mathcal{C}} \subseteq \mathbb{E}_B^{\mathcal{C}}$ hold.

Definition 3.17. A sequence space W of interval numbers is called solid if $y^k = ([y_k^-, y_k^+]) \in W$ whenever $||y^k|| \leq ||x^k||$ for all $k \in \mathbb{N}$ and $x^k = ([x_k^-, x_k^+]) \in W$.

Theorem 3.18. The spaces $\mathbb{E}_0^{\mathcal{C}}, \mathbb{E}_c^{\mathcal{C}}$ and $\mathbb{E}_b^{\mathcal{C}}$ are solid.

Proof. Let $||y^k|| \leq ||x^k||$ for all $k \in \mathbb{N}$ and for some $x^k \in \mathbb{E}_0^{\mathfrak{C}}$. From here, it is easy to see that $d(y^k, \theta) \leq d(x^k, \theta)$. Namely, we have $y_k^- \leq x_k^-$ and$y_k^+ \leq x_k^+$. So, $y^k \in \mathbb{E}_0^{\mathfrak{C}}$. Consequently, $\mathbb{E}_0^{\mathfrak{C}}$ is solid. The others can be proved in similar way. ∎

3.2. M(u), Y(u), O(u) Symbols and Some Properties

Definition 3.19. Let us take $u = [u^-, u^+] \in \mathbb{E}$. Now, we give the following definitions with [1]. The real numbers $B(u) = u^+ - u^-$, $M(u) = \max\{|u^-|, |u^+|\}$ and $O(u) = \frac{1}{2}(u^+ - u^-)$ are called size of u, norm of u and middle point of u, respectively. Additionally, by considering the set $B(u)$ it is clear that the set of real numbers will be written by $\mathbb{R} = \{u = [u^-, u^+] : B(u) = 0\}$.

Definition 3.20. Let $u, v \in \mathbb{E}$. If $u^- \leq v^-$ and $u^+ \leq v^+$, then u is said to be less than v. This can be showed by $u \preceq v$.

It is obvious that this sort of order can not compare any element of $\mathbb{E}$. For instance, we can not compare the interval numbers $u = [0, 8]$ and $v = [-1, 9]$. Shortly, if $u \subset v$ or $v \subset u$, it is not possible to determine which of u and v interval numbers is smaller, by using Definition 3.20.

A second way to compare intervals is given as follows. Let $S = \{u_i = [u_i^-, u_i^+] : i \in \mathbb{N}\} \subset E$ be given. For each $i \in \mathbb{N}$, length of u_i is equal to $B(u_i) = u_i^+ - u_i^-$. In this case, if $B(v_i) \leq B(u_i)$ for $v_i \in \mathbb{E}$ for all $i \in \mathbb{N}$ then S is said to be bounded below. In the same way, if $B(u_i) \leq B(w_i)$ for $w_i \in \mathbb{E}$ for all $i \in \mathbb{N}$ then S is said to be bounded above.

Definition 3.21. Let $[u_i^-, u_i^+] \in \mathbb{E}, 1 \leq i \leq n, (n \in \mathbb{N})$ and $B(u_i) = u_i^+ - u_i^-$. In this case, if $\max_i B(u_i) = M$ for $i = k$, then $\max_i u_i = u_k$.

In Definition 3.21, length of interval is considered in order to choose the largest size of intervals instead of end points of intervals. With this sort of order, the problems that occur in the order given by using the end points of intervals, will be removed. Additionally, we have the possibility to compare each element of $\mathbb{E}$.

4. Completion of Interval Metric Spaces

Definition 4.1. Let (X, d_1) and (Y, d_2) be interval metric spaces. A mapping T of X to Y is called interval isometry if for all $u, v \in X, d_2(Tx, Ty) = d_1(u, v)$. In addition this, X is considered as interval-isometric to the space Y, if there exists a bijective interval isometry between the spaces X and Y. Then, X and Y are said to be interval-isometric spaces.

Theorem 4.2. Let $(\tilde{X}, \tilde{d})$ be a complete interval metric space. Every interval metric space (X, d) has a completion $\tilde{X}$ which has a subspace Z dense in $\tilde{X}$ and there exists an isometry between X and $\tilde{X}$.

Proof. Let (X,d) be an interval metric space and $u^n = ([u_n^-, u_n^+])$ and $v^n = ([v_n^-, v_n^+])$ are Cauchy sequences in X. Define a relation between u^n and v^n as in the following:

$$u^n \sim v^n \Leftrightarrow \lim_n d(u^n, v^n) = \theta.$$

This clearly defines an equivalence relation. In fact:

1. $\lim_n d(u^n, u^n) = \max\{|u_n^- - u_n^-|, |u_n^+ - u_n^+|\} = 0 \Rightarrow u^n \sim u^n$,
2. $u^n \sim v^n \Leftrightarrow \lim_n d(u^n, v^n) = 0 = \lim_n d(v^n, u^n) \Leftrightarrow v^n \sim u^n$,
3. Since $\lim_n d(u^n, v^n) \leq \lim_n d(u^n, z^n) + \lim_n d(z^n, v^n), u^n \sim z^n$ and $z^n \sim v^n$ means that $u^n \sim v^n$.

Let $\tilde{X}$ be the set of all equivalence classes of interval Cauchy sequences. It means that

$$\tilde{X} = \{\tilde{u} : u^n \text{ is a Cauchy sequence in } X\}.$$

Now, we define $\tilde{d} : \tilde{X} \times \tilde{X} \to \mathbb{E}^+$ by

$$\tilde{d}(\tilde{u}, \tilde{v}) = \lim_n d(u^n, v^n) \tag{10}$$

for $\tilde{u}, \tilde{v} \in \tilde{X}$. Let u^{n_1} and v^{n_1} be two Cauchy sequence of X such that $u^n \sim u^{n_1}$ and $v^n \sim v^{n_1}$. Then

$$\lim_n d(u^n, u^{n_1}) = \lim_n d(v^n, v^{n_1}) = \theta.$$

By taking into consideration the triangle inequality, we conclude

$$d(u^n, v^n) \leq d(u^n, u^{n_1}) + d(u^{n_1}, v^{n_1}) + d(v^{n_1}, v^n)$$

and

$$d(u^{n_1}, v^{n_1}) \leq d(u^{n_1}, u^n) + d(u^n, v^n) + d(v^n, v^{n_1}).$$

From here, we can write,

$$|d(u^n, v^n) - d(u^{n_1}, v^{n_1})| \leq d(u^n, u^{n_1}) + d(v^n, v^{n_1}) \to 0.$$

Because of the fact that $(d(u^n, v^n))$ and $(d(u^{n_1}, v^{n_1}))$ are convergent $\tilde{d}$ is well-defined. Now, we prove that $\tilde{d}$ in (10) is a interval metric on $\tilde{X}$. In fact,

1. $\tilde{d}(\tilde{u}, \tilde{u}) = \lim_n d(u^n, u^n) = 0$,
2. $\tilde{d}(\tilde{u}, \tilde{v}) = \lim_n d(u^n, v^n) = \lim_n d(v^n, u^n) = \tilde{d}(\tilde{v}, \tilde{u})$,
3. $\tilde{d}(\tilde{u}, \tilde{v}) = \lim_n d(u^n, v^n) \leq \lim_n d(u^n, z^n) + \lim_n d(z^n, v^n) = \tilde{d}(\tilde{u}, \tilde{z}) + \tilde{d}(\tilde{z}, \tilde{v})$.

Consequently, $\tilde{d}$ is a metric on $\tilde{X}$.

With each $u \in X$, we construct the class $\tilde{u} = ([u^-, u^+], [u^-, u^+], \dots) \in \tilde{X}$, the equivalence classes of the constant sequence $([u^-, u^+], [u^-, u^+], \dots)$. This deduce a mapping $T : X \to \tilde{X}$ with $T(u) = \tilde{u}$. Then, for any $u, v \in X$

$$\tilde{d}(T(u), T(v)) = \tilde{d}(\tilde{u}, \tilde{v}) = \lim_n d(u, v) = d(u, v).$$

It means that T is an isometry from X into $\tilde{X}$. Now, let us show the denseness of $Z \in \tilde{X}$. Supposing that $\tilde{u} \in \tilde{X}, \varepsilon > 0$ and u^n is a member of $\tilde{u}^n$. Because of the fact that u^n is interval Cauchy sequence there exists a $n_0 \in \mathbb{N}$ such that for any $m, n \geq \mathbb{N}, d(u^m, u^n) < \frac{\varepsilon}{2}$. Consider the constant Cauchy sequence $u^N = (u_N, u_N, \dots)$ and $\tilde{u}^N$ be its equivalence class. Since,

$$\tilde{d}(\tilde{u}, \tilde{u}^N) = \lim_n d(u^n, u^N) = \lim_n d(u^n, u^N) \leq \frac{\varepsilon}{2} < \varepsilon.$$

Hence, $\tilde{u}^N \in Z$. Thus, Z is dense in $\tilde{X}$.

The completeness of $\tilde{X}$ is shown as below:

Let $\tilde{u}^n$ be any Cauchy sequence in $\tilde{X}$. Since Z is dense $\tilde{X}$, there exists $\tilde{z}^n \in Z$ for every $\tilde{u}^n$ as follows:

$$\tilde{d}(\tilde{u}^n, \tilde{z}^n) < \frac{1}{n}. \tag{11}$$

From here, we have by the triangle inequality $\tilde{d}(\tilde{z}^m, \tilde{z}^n) \leq \tilde{d}(\tilde{z}^m, \tilde{u}^m) + \tilde{d}(\tilde{u}^m, \tilde{u}^n) + \tilde{d}(\tilde{u}^n, \tilde{z}^n) < \frac{1}{m} + \tilde{d}(\tilde{u}^m, \tilde{u}^n) + \frac{1}{n} < \varepsilon$. Since $\tilde{u}^m$ is Cauchy, $\tilde{z}^m$ is also Cauchy and $T : X \to Z$ is isometric, $\tilde{z}^m \in Z$, then the sequence z^m, where $z^m = T^{-1}\tilde{z}^m$, is Cauchy in X. Now, let us show that $\tilde{u}$ is the limit of $\tilde{u}^n$. By using (11) we obtain

$$\tilde{d}(\tilde{u}^n, \tilde{u}) \leq \tilde{d}(\tilde{u}^n, \tilde{z}^n) + \tilde{d}(\tilde{z}^n, \tilde{u}) < \frac{1}{n} + \tilde{d}(\tilde{z}^n, \tilde{u}). \tag{12}$$

Because of the fact that $z^m \in \tilde{u}$ and $(z^n, z^n, ...) \in \tilde{z}^n \in Z$, (12) turn into

$$\tilde{d}(\tilde{u}^n, \tilde{u}) < \frac{1}{n} + \lim_m d(z^n, z^m) < \varepsilon.$$

Hence, arbitrary Cauchy sequence $\tilde{u}^n \in \tilde{X}$ has the limit $\tilde{u} \in \tilde{X}$. From here, $\tilde{X}$ is complete. ■

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