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Generalization of the primal-dual composite method of partial inverses

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Abstract

The main purpose of this article is to propose an algorithm that generalize the primal-dual composite method of partial inverses and to study the resolution of the primal inclusion problem:

$$0 \in \mathcal{L}_{A,B,L,L^*}x$$

on the cartesian product of two Hilbert spaces $\mathcal{H}_1 \times \mathcal{H}_2$
and the dual inclusion problem:

$$0 \in \mathcal{L}^*{}_{A^{-1},B^{-1},L,L^*}v$$

on the cartesian product of two Hilbert spaces $\mathcal{G}_1 \times \mathcal{G}_2$
where the operators $\mathcal{L}_{A,B,L,L^*}$ and $\mathcal{L}^*{}_{A^{-1},B^{-1},L,L^*}$ are defined by

$$\mathcal{L}_{A,B,L,L^*} = A + L^*BL$$

and

$$\mathcal{L}^*{}_{A^{-1},B^{-1},L,L^*} = -LA^{-1}(-L^*) + B^{-1}$$

with A and B are two set-valued operators satisfying some conditions that we will define thereafter, and L is a bounded linear operator, and L^* its adjoint. And apply this algorithm to the resolution of the optimization problem.

AMS subject classification:
Keywords: Convex optimization, monotone inclusion, monotone operator, primal-dual composite method of partial inverses.

1. Introduction

The fundamental problem that arises in several domains of the applied mathematics is

$$\text{Find } \bar{x} \in \mathcal{H} \text{ such that } 0 \in A\bar{x} \tag{1.1}$$

where $A : \mathcal{H} \to 2^{\mathcal{H}}$ is maximally monotone operator and $2^{\mathcal{H}}$ the set of parts of $\mathcal{H}$.

This problem has been studied extensively in the literature (see [1, 4, 6, 12, 16, 21, 22, 28]).

In the case where the operator A of the problem **(1.1)** is replaced by A_V the partial inverse of A with respect to V (A_V was introduced in [25]) defined by:

$$GraA_V = \{(P_V x + P_{V^\perp} u, P_V u + P_{V^\perp} x) \mid (x, u) \in GraA\}$$

where $GraA = \{(x, u) \in \mathcal{H}^2 \mid u \in Ax\}$ denote the graph of A, V be closed vector subspace of $\mathcal{H}$, P_V and $P_{V^\perp}$ also denote respectively the projection onto V and onto its orthogonal complement, we arrive at the problem of the forme:

$$\text{Find } \bar{x} \in V \text{ and } \bar{u} \in V^\perp \text{ such that } \bar{u} \in A\bar{x} \tag{1.2}$$

the method of parial inverses has been proposed in [25] can be solve this problem.

This method results from the application of the proximal point algorithm to the parial inverse A_V (see [5, 7, 11, 13, 15, 17, 18, 20, 24, 25, 26]).

In many synthetic formulations **(1.1)**, the operator A can be expressed as the sum of two monotone operators, one of which is the composition of a monotone operator with a linear bounded operator and its adjoint:

$$\text{Find } \bar{x} \in \mathcal{H} \text{ such that } 0 \in A\bar{x} + L^* B L\bar{x} \tag{1.3}$$

together with the dual inclusion

$$\text{Find } \bar{v} \in \mathcal{G} \text{ such that } 0 \in -LA^{-1}(-L^*\bar{v}) + B^{-1}\bar{v} \tag{1.4}$$

Problems **(1.3)** and **(1.4)** were studied in (see [2, 3, 9, 14, 23]).

The purpose of this work is to generalize the primal-dual composite method of parial inverses (which solve the problems **(1.3)** and **(1.4)**) and to solve a more general problem with a more complex space structure (cartesian product) for problems **(1.3)** and **(1.4)**.

The generic problem that we consider is the following:

Problem 1.1. Let $\mathcal{H}_1, \mathcal{H}_2, \mathcal{G}_1, \mathcal{G}_2$ be real Hilbert spaces. Let $A_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ and $A_2 : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ be two maximally monotone operators, and let

$$B_1 : \mathcal{G}_1 \to 2^{\mathcal{G}_1} \text{ and } B_2 : \mathcal{G}_2 \to 2^{\mathcal{G}_2}$$

are also two maximally monotone operators.Let

$$L_1 : \mathcal{H}_1 \to \mathcal{G}_1 \text{ and } L_2 : \mathcal{H}_2 \to \mathcal{G}_2$$

be two linear bounded operators such that:

$$A : \mathcal{H}_1 \times \mathcal{H}_2 \longrightarrow 2^{\mathcal{H}_1 \times \mathcal{H}_2} : (x_1, x_2) \longmapsto A_1 x_1 \times A_2 x_2$$
$$B : \mathcal{G}_1 \times \mathcal{G}_2 \longrightarrow 2^{\mathcal{G}_1 \times \mathcal{G}_2} : (y_1, y_2) \longmapsto B_1 y_1 \times B_2 y_2$$
$$L : \mathcal{H}_1 \times \mathcal{H}_2 \longrightarrow \mathcal{G}_1 \times \mathcal{G}_2 : (x_1, x_2) \longmapsto (L_1 x_1, L_2 x_2)$$

The problem is to solve the primal inclusion

$$\text{Find } \bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2) \in \mathcal{H}_1 \times \mathcal{H}_2 \text{ such that } 0 \in A\bar{\mathbf{x}} + L^* B L \bar{\mathbf{x}} \tag{1.5}$$

and dual inclusion

$$\text{Find } \bar{\mathbf{v}} = (\bar{v}_1, \bar{v}_2) \in \mathcal{G}_1 \times \mathcal{G}_2 \text{ such that } 0 \in -LA^{-1}(-L^*\bar{\mathbf{v}}) + B^{-1}\bar{\mathbf{v}} \tag{1.6}$$

The notes of this work are organized as follows: after some preliminaires in section **2** concerning the monotone operator, we recall the primal-dual composiste method of parial inverses in section **3**, and then we propose an algorithm that solves the problem **(1.5)** and **(1.6)**. Finally, in section **4** we present an application of the problem **1.1**.

2. Notation and preliminary results

We recall a few notions and some basic definition in the theory of monotone operators (see [4] for more detail).

2.1. General notions

◇ $\mathcal{H}_1, \mathcal{H}_2, \mathcal{G}_1, \mathcal{G}_2$: Real Hilbert spaces.

◇ $\langle ., . \rangle_{\mathcal{H}_1}, \langle ., . \rangle_{\mathcal{H}_2}, \langle ., . \rangle_{\mathcal{G}_1}, \langle ., . \rangle_{\mathcal{G}_2}$: Scalar products on $\mathcal{H}_1, \mathcal{H}_2, \mathcal{G}_1, \mathcal{G}_2$.

◇ $\|.\|_{\mathcal{H}_1}, \|.\|_{\mathcal{H}_2}, \|.\|_{\mathcal{G}_1}, \|.\|_{\mathcal{G}_2}$: Norm of spaces $\mathcal{H}_1, \mathcal{H}_2, \mathcal{G}_1, \mathcal{G}_2$.

◇ Id : Identity of operator.

◇ $2^{\mathcal{H}}$: Set of parts of $\mathcal{H}$.

◇ $\mathcal{H}_1 \times \mathcal{H}_2$: Cartesian product.

◇ $\Gamma_0(\mathcal{H})$: Classe of lower semicontinuous convex functions $f : \mathcal{H} \to]-\infty, +\infty]$ such that $Dom f = \{x \in \mathcal{H} \mid f(x) < +\infty\} \neq \emptyset$.

◇ $\mathcal{B}(\mathcal{H}, \mathcal{G})$: Space of bounded linear operators from $\mathcal{H}$ to $\mathcal{G}$.

◇ L^* : Adjoint of the operator $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$.

◇ $\to$: Strong convergence.

◇ $\rightharpoonup$: Weak convergence.

◇ P_V : Projection onto the closed vector subspace V of $\mathcal{H}$.

◇ $P_{V^\perp}$: Orthogonal projection.

2.2. Notation relating to a function $f \in \Gamma_0(\mathcal{H})$

$\diamond$ Domain of f:

$$Domf = \{x \in \mathcal{H} \mid f(x) < +\infty\}$$

$\diamond$ Conjugate of f:

$$f^* : u \mapsto \sup_{x \in \mathcal{H}} (\langle x, u \rangle - f(x))$$

$\diamond$ The subdifferential of f at $x \in Domf$:

$$\partial f(x) = \{u \in \mathcal{H} \mid (\forall y \in \mathcal{H})\ \langle y - x, u \rangle + f(x) \leq f(y)\}$$

with inverse given by

$$(\partial f)^{-1} = \partial f^*$$

$\diamond$ The proximity operator of f is

$$prox_f x = \underset{y \in \mathcal{H}}{argmin}\ f(y) + \frac{1}{2}\|x - y\|^2$$

we have

$$J_{\partial f} = prox_f$$

2.3. Notations and definitions relating to a set-valued operator

• Let $A : \mathcal{H} \to 2^{\mathcal{H}}$ be a set-valued operator. The graph of A will be noted

$$GraA := \{(x, u) \in \mathcal{H}^2 \mid u \in Ax\}$$

and the sets

$$DomA := \{x \in \mathcal{H} \mid Ax \neq \emptyset\}$$

$$RanA := \{u \in \mathcal{H} \mid (\exists x \in H) \text{ such that } u \in Ax\}$$

and

$$ZerA := \{x \in \mathcal{H} \mid 0 \in Ax\}$$

denote the domain, the range, and the zeros of A. the inverse operator of A is defined by

$$A^{-1}u := \{x \in \mathcal{H} \mid u \in Ax\}$$

The resolvent of A is

$$J_A = (Id + A)^{-1}$$

Definition 2.1. The operator A is monotone if:

$$(\forall (x,u) \in GraA)\, (\forall (y,v) \in GraA)\, \langle x-y, u-v\rangle \geq 0.$$

Definition 2.2. The operator $A : \mathcal{H} \to 2^{\mathcal{H}}$ is maximally monotone if it is monotone and there exists no monotone operator $B : \mathcal{H} \to 2^{\mathcal{H}}$ such that $GraB$ properly contains $GraA$.

In an equivalent manner: A is maximally monotone if it is monotone and, in addition

$$(\langle x-y, u-v\rangle \geq 0 \text{ for every } (y,v) \in GraA) \;\Rightarrow\; (x,u) \in GraA.$$

• Let $A : \mathcal{H}_1 \times \mathcal{H}_2 \longrightarrow 2^{\mathcal{H}_1\times\mathcal{H}_2} : (x_1,x_2) \longmapsto A_1x_1 \times A_2x_2$ be a set-valued

◇ $DomA = \{x = (x_1,x_2) \in \mathcal{H}_1 \times \mathcal{H}_2 \mid A_1x_1 \neq \emptyset \text{ and } A_2x_2 \neq \emptyset\}$

◇ $GraA = \{(x,u) \in (\mathcal{H}_1 \times \mathcal{H}_2) \times (\mathcal{H}_1 \times \mathcal{H}_2) \mid u_1 \in A_1x_1 \text{ and } u_2 \in A_2x_2\}$

◇ the inverse operator of A is $A^{-1} : \mathcal{H}_1 \times \mathcal{H}_2 \to 2^{\mathcal{H}_1\times\mathcal{H}_2}$ defined by

$$A^{-1}u := \{x = (x_1,x_2) \in \mathcal{H}_1 \times \mathcal{H}_2 \mid u_1 \in A_1x_1 \text{ and } u_2 \in A_2x_2\}$$

◇ $ZerA = \{(x_1,x_2) \in \mathcal{H}_1 \times \mathcal{H}_2 \mid 0 \in A_1x_1 \text{ and } 0 \in A_2x_2\}$.

◇ $RanA = \{(u_1,u_2) \in \mathcal{H}_1 \times \mathcal{H}_2 \mid (\exists (x_1,x_2) \in \mathcal{H}_1 \times \mathcal{H}_2)\, u_1 \in A_1x_1 \text{ and } u_2 \in A_2x_2\}$.

◇ The resolvant of A is

$$\begin{aligned} P = J_Ax &\Leftrightarrow x \in (Id + A)P \\ &\Leftrightarrow (x_1 - p_1, x_2 - p_2) \in A_1p_1 \times A_2p_2 \\ &\Leftrightarrow x_1 - p_1 \in A_1p_1 \textit{ and } x_2 - p_2 \in A_2p_2 \\ &\Leftrightarrow p_1 = J_{A_1}x_1 \textit{ and } p_2 = J_{A_2}x_2 \\ &\Leftrightarrow P = (J_{A_1}x_1, J_{A_2}x_2) \\ &\Leftrightarrow J_A = (J_{A_1}, J_{A_2}) \end{aligned}$$

Proposition 2.3. [4, Proposition 20.22 and 20.23] Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. Let $A_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ and $A_2 : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ be maximally monotone operators.
Then the operator

$$A : \mathcal{H}_1 \times \mathcal{H}_2 \to 2^{\mathcal{H}_1\times\mathcal{H}_2} : (x_1,x_2) \mapsto A_1x_1 \times A_2x_2$$

is also maximally monotone.

• Let $L_1 : \mathcal{H}_1 \to \mathcal{G}_1$ and $L_2 : \mathcal{H}_2 \to \mathcal{G}_2$ be linear bounded operators. The operator L is defined by:

$$\begin{array}{rrcl} L: & \mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2 & \longrightarrow & \mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2 \\ & (x_1, x_2) & \longmapsto & (L_1 x_1, L_2 x_2) \end{array}$$

Then L verifies the following properties:

⋄ $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$.

⋄ The adjoint of L is

Let $y \in \mathcal{G}$ $\forall x \in \mathcal{H}$ $\langle L^* y, x\rangle_{\mathcal{H}} = \langle y, Lx\rangle_{\mathcal{G}}$

$$\begin{aligned} \langle L^* y, x\rangle_{\mathcal{H}} &= \langle y, Lx\rangle_{\mathcal{G}} \\ &= \langle (y_1, y_2), (L_1 x_1, L_2 x_2)\rangle_{\mathcal{G}} \\ &= \langle y_1, L_1 x_1\rangle_{\mathcal{G}_1} + \langle y_2, L_2 x_2\rangle_{\mathcal{G}_2} \\ &= \langle L_1^* y_1, x_1\rangle_{\mathcal{G}_1} + \langle L_2^* y_2, x_2\rangle_{\mathcal{G}_2} \\ &= \langle (L_1^* y_1, L_2^* y_2), (x_1, x_2)\rangle_{\mathcal{H}} \\ &= \langle (L_1^* y_1, L_2^* y_2), x\rangle_{\mathcal{H}} \end{aligned}$$

therefore $L^* y = (L_1^* y_1, L_2^* y_2)$.

3. Algorithm and convergence

In this section, we propose an algorithm for solve the problems of monotone inclusions **(1.5)** and **(1.6)** and show its convergence in real Hilbertians spaces, using the primal-dual composite method of partial inverses.

In order to do this, we must first recall the primal-dual composite method of partial inverses. This method is presented in the following theorem.

Theorem 3.1. [2, Théorème 3.2] Let $\mathcal{H}$ and $\mathcal{G}$ be real Hilbert spaces, let $A : \mathcal{H} \to 2^{\mathcal{H}}$ and $B : \mathcal{G} \to 2^{\mathcal{G}}$ be maximally monotone operator, and let $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$. Set $Q = (Id + L^* L)^{-1}$ and assume that

$$Zer(A + L^* BL) \neq \emptyset \tag{3.1}$$

Let $(\lambda_n)_{n\in\mathbb{N}}$ be sequence in $]0\,,2[$, let $(a_n)_{n\in\mathbb{N}}$ be sequence in $\mathcal{H}$, and let $(b_n)_{n\in\mathbb{N}}$ be sequence in $\mathcal{G}$ such that:

$$\sum_n \lambda_n (2 - \lambda_n) = +\infty \;and\; \sum_n \lambda_n \sqrt{\|a_n\|^2 + \|b_n\|^2} < +\infty \tag{3.2}$$

Let $x_0 \in \mathcal{H}$, $v_0 \in \mathcal{G}$, and set $y_0 = Lx_0$, $u_0 = -L^* v_0$ and

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} p_n = J_A(x_n + u_n) + a_n \\ q_n = J_B(y_n + v_n) + b_n \\ r_n = x_n + u_n - p_n \\ s_n = y_n + v_n - q_n \\ t_n = Q(r_n + L^* s_n) \\ w_n = Q(p_n + L^* q_n) \\ x_{n+1} = x_n - \lambda_n t_n \\ y_{n+1} = y_n - \lambda_n L t_n \\ u_{n+1} = u_n + \lambda_n (w_n - p_n) \\ v_{n+1} = v_n + \lambda_n (L w_n - q_n) \end{array} \right. \tag{3.3}$$

Then the following hold:

1. $x_n - w_n + Q(a_n + L^* b_n) \to 0$ and $y_n - L w_n + L Q(a_n + L^* b_n) \to 0$.

2. $u_n - r_n + t_n - a_n + Q(a_n + L^* b_n) \to 0$ and $v_n - s_n + L t_n - b_n + L Q(a_n + L^* b_n) \to 0$.

   Moreover, there exist a solution $\bar{x}$ to **(1.3)** and a solution $\bar{v}$ to **(1.4)** such that the following hold:

3. $-L^* \bar{v} \in A\bar{x}$ and $\bar{v} \in BL\bar{x}$.

4. $x_n \rightharpoonup \bar{x}$ and $v_n \rightharpoonup \bar{v}$.

The main algorithm that we used for solve the problem **1.1** is the following:

Algorithm 3.2. (main algorithm) Set $Q_1 = (Id + L_1^* L_1)^{-1}$ and $Q_2 = (Id + L_2^* L_2)^{-1}$, let $(\lambda_n)_{n\in\mathbb{N}}$ be sequence in $]0,2[$. Let $(a_{1,n}, a_{2,n})_{n\in\mathbb{N}}$ be sequence in $\mathcal{H}_1 \times \mathcal{H}_2$ and let $(b_{1,n}, b_{2,n})_{n\in\mathbb{N}}$ be sequence in $\mathcal{G}_1 \times \mathcal{G}_2$ such that:

$$\sum_n \lambda_n (2 - \lambda_n) = +\infty \tag{3.4}$$

and

$$\sum_n \lambda_n \sqrt{\|a_{1,n}\|_{\mathcal{H}_1} + \|a_{2,n}\|_{\mathcal{H}_2} + \|b_{1,n}\|_{\mathcal{G}_1} + \|b_{2,n}\|_{\mathcal{G}_2}} < +\infty \tag{3.5}$$

Let $(x_{1,n}, x_{2,n})_{n\in\mathbb{N}}$ and $(v_{1,n}, v_{2,n})_{n\in\mathbb{N}}$ be sequences generated by the following:

$$
(\forall n \in \mathbb{N}) \quad
\left\lfloor
\begin{array}{l}
Initialization \\
\left\lfloor
\begin{array}{l}
x_{1,0} \in \mathcal{H}_1 \\
x_{2,0} \in \mathcal{H}_2 \\
v_{1,0} \in \mathcal{G}_1 \\
v_{2,0} \in \mathcal{G}_2 \\
y_{1,0} = L_1 x_{1,0} \\
y_{2,0} = L_2 x_{2,0} \\
u_{1,0} = -L_1^* v_{1,0} \\
u_{2,0} = -L_2^* v_{2,0}
\end{array}
\right. \\
p_{1,n} = J_{A_1}(x_{1,n} + u_{1,n}) + a_{1,n} \\
p_{2,n} = J_{A_2}(x_{2,n} + u_{2,n}) + a_{2,n} \\
q_{1,n} = J_{B_1}(y_{1,n} + v_{1,n}) + b_{1,n} \\
q_{2,n} = J_{B_2}(y_{2,n} + v_{2,n}) + b_{2,n} \\
r_{1,n} = x_{1,n} + u_{1,n} - p_{1,n} \\
r_{2,n} = x_{2,n} + u_{2,n} - p_{2,n} \\
s_{1,n} = y_{1,n} + v_{1,n} - q_{1,n} \\
s_{2,n} = y_{2,n} + v_{2,n} - q_{2,n} \\
t_{1,n} = Q_1(r_{1,n} + L_1^* s_{1,n}) \\
t_{2,n} = Q_2(r_{2,n} + L_2^* s_{2,n}) \\
w_{1,n} = Q_1(p_{1,n} + L_1^* q_{1,n}) \\
w_{2,n} = Q_2(p_{2,n} + L_2^* q_{2,n}) \\
x_{1,n+1} = x_{1,n} - \lambda_n t_{1,n} \\
x_{2,n+1} = x_{2,n} - \lambda_n t_{2,n} \\
y_{1,n+1} = y_{1,n} - \lambda_n L_1 t_{1,n} \\
y_{2,n+1} = y_{2,n} - \lambda_n L_2 t_{2,n} \\
u_{1,n+1} = u_{1,n} + \lambda_n (w_{1,n} - p_{1,n}) \\
u_{2,n+1} = u_{2,n} + \lambda_n (w_{2,n} - p_{2,n}) \\
v_{1,n+1} = v_{1,n} + \lambda_n (L_1 w_{1,n} - q_{1,n}) \\
v_{2,n+1} = v_{2,n} + \lambda_n (L_2 w_{2,n} - q_{2,n})
\end{array}
\right. \tag{3.6}
$$

Using this algorithm we obtain the results of convergence indicated in the following theorem:

Theorem 3.3. In problem **1.1**, assume that

$$Zer(A + L^* BL) \neq \emptyset \tag{3.7}$$

and let $(x_{1,n}, x_{2,n})_{n\in\mathbb{N}}$ and $(v_{1,n}, v_{2,n})_{n\in\mathbb{N}}$ be sequences generated by the algorithm **3.1**. Then the following hold:

1. $x_{1,n} - w_{1,n} + Q_1(a_{1,n} + L_1^* b_{1,n}) \to 0$ and $x_{2,n} - w_{2,n} + Q_2(a_{2,n} + L_2^* b_{2,n}) \to 0$.
2. $y_{1,n} - L_1 w_{1,n} + L_1 Q_1(a_{1,n} + L_1^* b_{1,n}) \to 0$ and $y_{2,n} - L_2 w_{2,n} + L_2 Q_2(a_{2,n} + L_2^* b_{2,n}) \to 0$.

3. $u_{1,n} - r_{1,n} + t_{1,n} - a_{1,n} + Q_1(a_{1,n} + L_1^* b_{1,n}) \to 0$ and $u_{2,n} - r_{2,n} + t_{2,n} - a_{2,n} + Q_2(a_{2,n} + L_2^* b_{2,n}) \to 0$.

4. $v_{1,n} - s_{1,n} + L_1 t_{1,n} - b_{1,n} + L_1 Q_1(a_{1,n} + L_1^* b_{1,n}) \to 0$ and $v_{2,n} - s_{2,n} + L_2 t_{2,n} - b_{2,n} + L_2 Q_2(a_{2,n} + L_2^* b_{2,n}) \to 0$.
   Moreover, there exist a solution $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2)$ to **(1.5)** and a solution $\bar{\mathbf{v}} = (\bar{v}_1, \bar{v}_2)$ to **(1.6)** such that the following hold:

5. $-L^*\bar{\mathbf{v}} \in A\bar{\mathbf{x}}$ and $\bar{\mathbf{v}} \in BL\bar{\mathbf{x}}$.

6. $\mathbf{x}_n \rightharpoonup \bar{\mathbf{x}}$ and $\mathbf{v}_n \rightharpoonup \bar{\mathbf{v}}$.

Proof. We define $\mathcal{H}$ and $\mathcal{G}$ as the real Hilbert spaces obtained by endowing the cartesian products:

$$\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2 \textit{ and } \mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2$$

with the scalar products respectively defined by

$$\langle ., . \rangle_{\mathcal{H}} : (\mathbf{x}, \mathbf{y}) \mapsto \langle x_1, y_1 \rangle_{\mathcal{H}_1} + \langle x_2, y_2 \rangle_{\mathcal{H}_2}$$

and

$$\langle ., . \rangle_{\mathcal{G}} : (\mathbf{u}, \mathbf{v}) \mapsto \langle u_1, v_1 \rangle_{\mathcal{G}_1} + \langle u_2, v_2 \rangle_{\mathcal{G}_2}$$

where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ denote generic elements in $\mathcal{H}$, and $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ the element in $\mathcal{G}$.
Let us set

$$A : \mathcal{H} \to 2^{\mathcal{H}} : (x_1, x_2) \mapsto A_1 x_1 \times A_2 x_2 \tag{3.8}$$

$$B : \mathcal{G} \to 2^{\mathcal{G}} : (v_1, v_2) \mapsto B_1 v_1 \times B_2 v_2 \tag{3.9}$$

Since the operators A_1, A_2, B_1, and B_2 are maximally monotone, then A and B are also maximally monotone [4, prop 20.22 et 20.23].
We also take

$$L : \mathcal{H} \to \mathcal{G} : (x_1, x_2) \mapsto (L_1 x_1, L_2 x_2) \tag{3.10}$$

note that L is linear bounded (see section **2**), and its adjoint $L^* \in \mathcal{B}(\mathcal{G}, \mathcal{H})$ is:

$$L^* : \mathcal{G} \to \mathcal{H} : (v_1, v_2) \mapsto (L_1^* v_1, L_2^* v_2) \tag{3.11}$$

Moreover, set

$$Z = \{(\bar{\mathbf{x}}, \bar{\mathbf{v}}) = ((\bar{x}_1, \bar{x}_2), (\bar{v}_1, \bar{v}_2)) \in \mathcal{H} \times \mathcal{G} \mid -L^*\bar{\mathbf{v}} \in A\bar{\mathbf{x}} \text{ and } \bar{\mathbf{v}} \in BL\bar{\mathbf{x}}\} \tag{3.12}$$

It is shown in [9, 19] that under the condition **(3.7)**,

$$Zer(A + L^*BL) \neq \emptyset \;\Leftrightarrow\; Zer(-LA^{-1}(-L^*) + B^{-1}) \neq \emptyset \;\Leftrightarrow\; Z \neq \emptyset.$$

Now set,

$$(\forall n \in \mathbb{N}) \begin{cases} \mathbf{a}_n = (a_{1,n}, a_{2,n}) \\ \mathbf{b}_n = (b_{1,n}, b_{2,n}) \\ \mathbf{p}_n = (p_{1,n}, p_{2,n}) \\ \mathbf{q}_n = (q_{1,n}, q_{2,n}) \\ \mathbf{r}_n = (r_{1,n}, r_{2,n}) \\ \mathbf{s}_n = (s_{1,n}, s_{2,n}) \\ \mathbf{t}_n = (t_{1,n}, t_{2,n}) \\ \mathbf{w}_n = (w_{1,n}, w_{2,n}) \\ \mathbf{x}_n = (x_{1,n}, x_{2,n}) \\ \mathbf{y}_n = (y_{1,n}, y_{2,n}) \\ \mathbf{u}_n = (u_{1,n}, u_{2,n}) \\ \mathbf{v}_n = (v_{1,n}, v_{2,n}) \end{cases} \tag{3.13}$$

Thus

$$(\forall n \in \mathbb{N})\; J_A(\mathbf{x}_n + \mathbf{u}_n) = J_{A_1 \times A_2}(x_{1,n} + u_{1,n}, x_{2,n} + u_{2,n}) \tag{3.14}$$

$$= (J_{A_1}(x_{1,n} + u_{1,n}), J_{A_2}(x_{2,n} + u_{2,n})) \tag{3.15}$$

and

$$(\forall n \in \mathbb{N})\; J_B(\mathbf{y}_n + \mathbf{v}_n) = J_{B_1 \times B_2}(y_{1,n} + v_{1,n}, y_{2,n} + v_{2,n}) \tag{3.16}$$

$$= (J_{B_1}(y_{1,n} + v_{1,n}), J_{B_2}(y_{2,n} + v_{2,n})) \tag{3.17}$$

thus, we derive from **(3.13)**, **(3.15)**, and **(3.17)** that the algorithm **(3.6)** reduces to **(3.3)**. Furthermore, since

$$\sum_{n\in\mathbb{N}} \sqrt{\|\mathbf{a}_n\|^2 + \|\mathbf{b}_n\|^2} = \sum_{n\in\mathbb{N}} \sqrt{\langle \mathbf{a}_n, \mathbf{a}_n \rangle_H + \langle \mathbf{b}_n, \mathbf{b}_n \rangle_G} \tag{3.18}$$

$$= \sum_{n\in\mathbb{N}} \sqrt{\langle a_{1,n}, a_{1,n} \rangle_{\mathcal{H}_1} + \langle a_{2,n}, a_{2,n} \rangle_{\mathcal{H}_2} + \langle b_{1,n}, b_{1,n} \rangle_{\mathcal{G}_1} + \langle b_{2,n}, a_{2,n} \rangle_{\mathcal{G}_1}} \tag{3.19}$$

$$= \sum_{n\in\mathbb{N}} \sqrt{\|a_{1,n}\|^2_{\mathcal{H}_1} + \|a_{2,n}\|^2_{\mathcal{H}_2} + \|b_{1,n}\|^2_{\mathcal{G}_1} + \|b_{2,n}\|^2_{\mathcal{G}_2}} < +\infty \tag{3.20}$$

then, le theorem **3.1(1)** and **(2)** imply that **(1)**, **(2)**, **(3)**, and **(4)** are satisfied, and theorem **3.1(3)** and **(4)** ensure that there is a solution

$$(\bar{\mathbf{x}}, \bar{\mathbf{v}}) = ((\bar{x}_1, \bar{x}_2), (\bar{v}_1, \bar{v}_2)) \in Z$$

such that

$$(\mathbf{x}_n, \mathbf{v}_n) = ((x_{1,n}, x_{2,n}), (v_{1,n}, v_{2,n})) \rightharpoonup (\bar{\mathbf{x}}, \bar{\mathbf{v}}) = ((\bar{x}_1, \bar{x}_2), (\bar{v}_1, \bar{v}_2)). \;\square$$

■

4. Application

We illustrate in this section an application of the problem **1.1**.

Problem 4.1. Let m be a strictly positive integer, and let $\mathcal{H}_1$, $\mathcal{H}_2$, $(\mathcal{G}_{1,i})_{1\leq i\leq m}$, $\mathcal{G}_2$ be real Hilbert spaces. Let $\mathbf{z} = (z_1, z_2) \in \mathcal{H}_1 \times \mathcal{H}_2$, and let $f_1 \in \Gamma_0(\mathcal{H}_1)$ and $f_2 \in \Gamma_0(\mathcal{H}_2)$. For every $i \in \{1, ..., m\}$, $g_{1,i} \in \Gamma_0(\mathcal{G}_{1,i})$, $\mathbf{o} = (o_{1,i}, o_2) \in \mathcal{G}_{1,i} \times \mathcal{G}_2$, $l_{1,i} \in \mathcal{B}(\mathcal{H}_1, \mathcal{G}_{1,i})$, $g_2 \in \Gamma_0(\mathcal{G}_2)$, and $l_2 \in \mathcal{B}(\mathcal{H}_2, \mathcal{G}_2)$.
Solve the primal problem:

$$\underset{\bar{\mathbf{x}}\in\mathcal{H}}{\text{minimize}}\ f(\bar{\mathbf{x}}) + L^* g L\bar{\mathbf{x}} - K(\bar{\mathbf{x}}, \mathbf{z}) \tag{4.1}$$

and the dual problem

$$\underset{\bar{\mathbf{v}}\in\mathcal{H}}{\text{minimize}}\ f^*(\mathbf{z} - L^*\bar{\mathbf{v}}) + L^* g^* \bar{\mathbf{v}} + H(\bar{\mathbf{v}}, \mathbf{o}) \tag{4.2}$$

with

$$\mathcal{G}_1 = \bigoplus_1^m \mathcal{G}_{1,i} \tag{4.3}$$

$$f : \mathbf{x} = (x_1, x_2) \mapsto (f_1(x_1), f_2(x_2)) \tag{4.4}$$

$$g : \mathbf{v} = ((v_{1,i})_i, v_2) \mapsto ((g_{1,i}(v_{1,i}))_i, g_2(v_2)) \tag{4.5}$$

$$L : \mathbf{x} = (x_1, x_2) \mapsto ((l_{1,i}x_1)_i, l_2 x_2) \tag{4.6}$$

$$K(\mathbf{x}, \mathbf{z}) = (\langle x_1, z_1\rangle_{\mathcal{H}_1}, \langle x_2, z_2\rangle_{\mathcal{H}_2}) \tag{4.7}$$

$$H(\mathbf{v}, \mathbf{o}) = ((\langle v_{1,i}, o_{1,i}\rangle_{\mathcal{G}_{1,i}})_i, \langle v_2, o_2\rangle_{\mathcal{G}_2}) \tag{4.8}$$

Algorithm 4.2. (resulting algorithm) Set $Q_1 = \left(Id + \sum_{i=1}^{i=m} l_{1,i}^* l_{1,i}\right)^{-1}$ and $Q_2 = (Id + l_2^* l_2)^{-1}$, let $(\lambda_n)_{n\in\mathbb{N}}$ be sequence in $]0, 2[$. Let $(a_{1,n}, a_{2,n})_{n\in\mathbb{N}}$ be sequence in $\mathcal{H}_1 \times \mathcal{H}_2$ and let for every $i \in \{1, \ldots, m\}$ $(b_{1,i,n}, b_{2,n})_{n\in\mathbb{N}}$ be sequence in $\mathcal{G}_{1,i} \times \mathcal{G}_2$ such that:

$$\sum_n \lambda_n(2 - \lambda_n) = +\infty \tag{4.9}$$

and

$$\sum_n \lambda_n \sqrt{\|a_{1,n}\|_{\mathcal{H}_1} + \|a_{2,n}\|_{\mathcal{H}_2} + \sum_{i=1}^{i=m} \|b_{1,i,n}\|_{\mathcal{G}_{1,i}} + \|b_{2,n}\|_{\mathcal{G}_2}} < +\infty \tag{4.10}$$

Let $(x_{1,n}, x_{2,n})_{n\in\mathbb{N}}$ and $((v_{1,i,n})_{1\le i\le m}, v_{2,n})_{n\in\mathbb{N}}$ be sequences generated by the following:

$$
(\forall n \in \mathbb{N}) \quad
\left\lfloor
\begin{array}{l}
Initialization \\
\left\lfloor
\begin{array}{l}
x_{1,0} \in \mathcal{H}_1 \\
x_{2,0} \in \mathcal{H}_2 \\
pour\ i = 1, \ldots, m \\
\left\lfloor
\begin{array}{l}
v_{1,i,0} \in \mathcal{G}_{1,i} \\
y_{1,i,0} = l_{1,i} x_{1,0}
\end{array}
\right. \\
v_{2,0} \in \mathcal{G}_2 \\
y_{2,0} = l_2 x_{2,0} \\
u_{1,0} = -\sum_{i=1}^{i=m} l_{1,i}^* v_{1,i,0} \\
u_{2,0} = -l_2^* v_{2,0}
\end{array}
\right. \\
p_{1,n} = prox_{f_1}(x_{1,n} + u_{1,n} + z_1) + a_{1,n} \\
p_{2,n} = prox_{f_2}(x_{2,n} + u_{2,n} + z_2) + a_{2,n} \\
r_{1,n} = x_{1,n} + u_{1,n} - p_{1,n} \\
r_{2,n} = x_{2,n} + u_{2,n} - p_{2,n} \\
pour\ i = 1, \ldots, m \\
\left\lfloor
\begin{array}{l}
q_{1,i,n} = prox_{g_{1,i}}(y_{1,i,n} + v_{1,i,n} - o_{1,i}) + b_{1,i,n} + o_{1,i} \\
q_{2,n} = prox_{g_2}(y_{2,n} + v_{2,n} - o_2) + b_{2,n} + o_2 \\
s_{1,i,n} = y_{1,i,n} + v_{1,i,n} - q_{1,i,n} \\
s_{2,n} = y_{2,n} + v_{2,n} - q_{2,n}
\end{array}
\right. \\
t_{1,n} = Q_1(r_{1,n} + \sum_{i=1}^{i=m} l_{1,i}^* s_{1,i,n}) \\
t_{2,n} = Q_2(r_{2,n} + l_2^* s_{2,n}) \\
w_{1,n} = Q_1(p_{1,n} + \sum_{i=1}^{i=m} l_{1,i}^* q_{1,i,n}) \\
w_{2,n} = Q_2(p_{2,n} + l_2^* q_{2,n}) \\
x_{1,n+1} = x_{1,n} - \lambda_n t_{1,n} \\
x_{2,n+1} = x_{2,n} - \lambda_n t_{2,n} \\
u_{1,n+1} = u_{1,n} + \lambda_n (w_{1,n} - p_{1,n}) \\
u_{2,n+1} = u_{2,n} + \lambda_n (w_{2,n} - p_{2,n}) \\
pour\ i = 1, \ldots, m \\
\left\lfloor
\begin{array}{l}
y_{1,i,n+1} = y_{1,i,n} - \lambda_n l_{1,i} t_{1,n} \\
y_{2,n+1} = y_{2,n} - \lambda_n l_2 t_{2,n} \\
v_{1,i,n+1} = v_{1,i,n} + \lambda_n (l_{1,i} w_{1,i,n} - q_{1,n}) \\
v_{2,n+1} = v_{2,n} + \lambda_n (l_2 q_{2,n} - q_{2,n})
\end{array}
\right.
\end{array}
\right.
\tag{4.11}
$$

Corollary 4.3. In problem **4.1**, assume that

$$\mathbf{z} \in Ran(\partial f + L^* \partial g(L - \mathbf{o})) \tag{4.12}$$

and let $(x_{1,n}, x_{2,n})_{n\in\mathbb{N}}$ and $((v_{1,i,n})_{1\leq i\leq m}, v_{2,n})_{n\in\mathbb{N}}$ be sequences generated by the algorithm **4.1**.
Then the following hold:

1. $\mathbf{z} - L^*\bar{\mathbf{v}} \in \partial f$ and $\bar{\mathbf{v}} \in \partial g L(\bar{\mathbf{x}} - \mathbf{o})$
2. $\mathbf{x}_n \rightharpoonup \bar{\mathbf{x}}$ and $\mathbf{x}_n \rightharpoonup \bar{\mathbf{v}}$

Proof. Set

$$\begin{cases} A_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1} : x_1 \mapsto -z_1 + \partial f_1(x_1) \\ A_2 : \mathcal{H}_2 \to 2^{\mathcal{H}_2} : x_2 \mapsto -z_2 + \partial f_2(x_2) \\ B_2 : \mathcal{G}_2 \to 2^{\mathcal{G}_2} : v_2 \mapsto +\partial g_2(v_2 - o_2) \\ l_2 : \mathcal{H}_2 \to \mathcal{G}_2 : x_2 \mapsto l_2 x_2 \\ \mathcal{G}_1 = \bigoplus_{1}^{m} \mathcal{G}_{1,i} \\ B_1 : \mathcal{G}_1 \to 2^{\mathcal{G}_1} : (v_{1,i})_i \mapsto \prod_{i=1}^{i=m} \partial g_{1,i}(v_{1,i} - o_{1,i}) \\ l_1 : \mathcal{H}_1 \to \mathcal{G}_1 : x_1 \mapsto (l_{1,i} x_1)_i \end{cases} \tag{4.13}$$

Then

$$L^* : \mathcal{G} \to \mathcal{H} : ((v_{1,i})_{1\leq i\leq m}, v_2) \mapsto (\sum_{i=1}^{i=m} l_{1,i}^* v_{1,i}, l_2^* v_2) \tag{4.14}$$

and so, the problem **4.1** reduces to the problem **1.1**. Moreover,

$$J_A : \mathbf{x} = (x_1, x_2) \mapsto (prox_{f_1}(x_1 + z_1), prox_{f_2}(x_2 + z_2)) \tag{4.15}$$

and

$$J_B : \mathbf{y} = ((y_{1,i})_{1\leq i\leq m}, y_2) \mapsto ((prox_{g_{1,i}}(y_{1,i} - o_{1,i}) + o_{1,i})_{1\leq i\leq m}, prox_{g_2}(y_2 - o_2) + o_2) \tag{4.16}$$

Now set,

$$\begin{cases} b_{1,n} = (b_{1,i,n})_{1\leq i\leq m} \\ q_{1,n} = (q_{1,i,n})_{1\leq i\leq m} \\ s_{1,n} = (s_{1,i,n})_{1\leq i\leq m} \\ v_{1,n} = (v_{1,i,n})_{1\leq i\leq m} \\ y_{1,n} = (y_{1,i,n})_{1\leq i\leq m} \end{cases} \tag{4.17}$$

and conequently, the algorithm **4.11** becomes the algorithm **3.6**.
Thus, the assertions follow from **3.2(5)** and **(6)**. ■

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