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Interval Valued Fuzzy Sequence Spaces Defined by a Modulus Function

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Abstract

In this paper, sequence spaces $c_0(E^2, M, p)$, $c(E^2, M, p)$ and $\ell_\infty(E^2, M, p)$ as the set of all null, convergent and bounded sequences of interval valued fuzzy numbers are defined with respect to modulus function M, respectively. In addition, it is shown that the spaces $c_0(E^2, M, p)$, $c(E^2, M, p)$ and $\ell_\infty(E^2, M, p)$ are complete metric spaces, cofinal and symmetric spaces.

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1. Introduction

Interval valued fuzzy sets (for brief: IVFS) were first suggested by Turksen [2] and Gorzalczany [3]. They are applied to the fields of approximate inference, signal transmission and controller, etc. Interval valued fuzzy numbers were defined by Guijun and Xiaoping [14]. At the same time, some elementary properties, series of decomposition and representation theorems were also discussed in [14]. Li [4] introduced distance between interval valued fuzzy sets. Additionally, Hong and Lee [5], Meenakshi and Kaliraja [1], Li [4] have been studied different properties of interval valued fuzzy numbers. The theoretical and practical applications of fuzzy sets have increased considerably since Zadeh's paper, (see also [12], [13]).

2. Preliminaries

By $\mathbb{R}, \mathbb{N}, \mathbb{Q}$ we denote the set of all real, natural and rational numbers, respectively. Let us suppose that I be the set of all closed and bounded intervals on $[0, 1]$, i.e. $[I] = \{x = [x^-, x^+] : 0 \leq x^- \leq x^+ \leq 1\}$ and X be an ordinary set. The mapping $u : X \to [I],\ x \to u(x)$ is called an interval valued fuzzy set on X, [14]. An interval valued fuzzy number (IVFN) is a function, u, from $\mathbb{R}$ to $[I]$, which satisfies the following properties:

1. u is normal, i.e., there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = [u^-(x_0), u^+(x_0)] = [1, 1]$.
2. u is fuzzy convex, i.e., for any $x, y \in \mathbb{R}$ and $\mu \in [0, 1]$, $u[\mu x + (1 - \mu)y] \geq \min\{u(x), u(y)\}$.
3. u^- and u^+ are upper semi-continuous.
4. The closure of $\{x \in \mathbb{R} : u^-(x) > 0,\ u^+(x) > 0\}$, denoted by u^0, is compact.

We denote the set of all interval valued triangular fuzzy numbers and fuzzy numbers by E^2 and E^1, on the set of real numbers, $\mathbb{R}$, respectively. In this paper, for brevity, we represent an interval valued fuzzy number(s) by IVFN. It is easy to see that, it can be written $u(x) = [u^-(x), u^+(x)]$ for each $u \in E^2$, where $u^-(x) \leq u^+(x)$ and $x \in \mathbb{R}$. Then $u^-(x) : \mathbb{R} \to I = [0, 1]$ and $u^+(x) : \mathbb{R} \to I$ are two ordinary fuzzy sets on $\mathbb{R}$. For brief, through the text, we shall write $u = [u^-, u^+]$ instead of $u(x) = [u^-(x), u^+(x)]$.

A quasi-vector space (over $\mathbb{R}$), denoted $(\mathbb{Q}, +, \mathbb{R}, *)$, is an abelian group $(\mathbb{Q}, +)$ with a mapping " $*$ " : $\mathbb{R} \times \mathbb{Q} \to \mathbb{Q}$, such that for $a, b, c \in \mathbb{Q}, \alpha, \beta, \gamma \in \mathbb{R}$:

- $\gamma * (a + b) = \gamma * a + \gamma * b$,
- $\alpha * (\beta * c) = (\alpha\beta) * c$,
- $1 * a = a$,
- $(\alpha + \beta) * c = \alpha * c + \beta * c$, if $\alpha\beta \geq 0$, [7].

An isomorphism f between linear spaces is a bijective linear map. Two linear spaces are called isomorphic if and only if there exists an isomorphism between them [6].

Lemma 2.1. [14] $u \in E^2$ if and only if u^- and u^+ are ordinary fuzzy numbers.

Let's suppose that $u_1, u_2 \in E^2$ and $\lambda \in \mathbb{R}$. Then the partial ordering relation and some algebraic operations on E^2 are defined as follows:

Ordering: $u_1 \leq u_2 \Leftrightarrow [u_1^-, u_1^+] \leq [u_2^-, u_2^+] \Leftrightarrow u_1^- \leq u_2^-$ and $u_1^+ \leq u_2^+$,

Addition: $u_1 + u_2 = \{v \in E^1 : u_1^- + u_2^- \leq v \leq u_1^+ + u_2^+\}$, Scalar multiplication:

$$\lambda u = \begin{cases} \{v \in E^1 : \lambda u^- \leq v \leq \lambda u^+\}, & \lambda \geq 0 \\ \{v \in E^1 : \lambda u^+ \leq v \leq \lambda u^-\}, & \lambda < 0 \end{cases}.$$

Multiplication:

$$u_1 u_2 = \{v \in E^1 : \min\{u_1^- u_2^-, u_1^- u_2^+, u_1^+ u_2^-, u_1^+ u_2^+\} \leq v \leq \max\{u_1^- u_2^-, u_1^- u_2^+, u_1^+ u_2^-, u_1^+ u_2^+\}\}.$$

Fuzzy numbers
u^- and u^+ are seen as lower and upper fuzzy number of u, respectively. If u^- and u^+ are both triangular fuzzy numbers, then u is called as an interval valued triangular fuzzy number.

Define a map $\overline{d} : E^1 \times E^1 \longrightarrow \mathbb{R}$ by $\overline{d}(u, v) = \sup_{0 \leq \alpha \leq 1} d([u]^\alpha, [v]^\alpha)$. It is known that E^1 is a complete metric space with the metric $\overline{d}$ [8]. A sequence $v = (v_k)$ of fuzzy numbers is a function v from the set $\mathbb{N}$ the set of all positive integers, into E^1, and fuzzy number v_k denotes the value of the function at k and is called the k^{th} term of the sequence. Let $c_0(E^1)$, $c(E^1)$ and $\ell_\infty(E^1)$ denote of all null, convergent and bounded sequences of fuzzy numbers, respectively. In [9], it is shown that $c_0(E^1)$, $c(E^1)$ and $\ell_\infty(E^1)$ are complete metric spaces with the metric $D_\infty(u, v) = \sup_{k \in \mathbb{N}} \overline{d}(u_k, v_k)$.

Definition 2.2. Let's suppose that $u = [u^-, u^+]$ be one of the IVFN. If $u^- = u^+$, then u is called degenerate interval valued fuzzy number.

It can be easily seen that a degenerate interval valued fuzzy number is an ordinary fuzzy number i.e., if $u^- = u^+$ then $u \in E^1$.

Definition 2.3. [11] Let $\tau^2 \subset E^2$ and let us consider function $||.|| : \tau^2 \to \mathbb{R}$. The function $\| \,.\, \|$ is called module on the set τ^2 if it has the following properties:

M1. $||u|| = \theta \Leftrightarrow u = \theta$, where θ is zero element of E^2,
M2. $||\lambda u|| = |\lambda| ||u||$,
M3. $||u + v|| \leq ||u|| + ||v||$.

If the function $||.|| : \tau^2 \to \mathbb{R}$ satisfy the following conditions M1, M2 and M3 then τ^2 is called module space of the IVFN. And if τ^2 is complete with respect to the module $||.||$ then τ^2 is called complete module space of the IVFN.

Let $u, v \in E^2$ and we give

$$D(u, v) = \max\left\{\overline{d}(u^-, v^-), \overline{d}(u^+, v^+)\right\} \quad (1)$$

The module of the $u \in$ IVFN is defined as the non-negative real number. Additionally, $D(u, \theta)$ corresponds to the distance from u to θ, [10].

In [4], it is shown that E^2 is metric space with the metric defined by (1).

Lemma 2.4. [10] Define the module

$$D(u,\theta) = ||u||_{E^2} = \max\left\{\overline{d}(u^-,\overline{0}), \overline{d}(u^+,\overline{0})\right\}. \tag{2}$$

Then E^2 is complete module space of the IVFN with the module defined by (2).

Definition 2.5. [10] A sequence space of IVFN is subspace of $w(E^2)$, where $w(E^2) = \{(u_k) = ([u_k^-, u_k^+])_{k\in\mathbb{N}} : u : \mathbb{N} \to E^2, k \to u(k) = [u_k^-, u_k^+]$ and $u_k^-, u_k^+ \in E^1\}$. If $(u_k) \in w(E^2)$ then (u_k) is called a sequence of IVFN.

Definition 2.6. [10] A sequence $(u_k) \in w(E^2)$ is said to be bounded if and only if there exists two IVFN t and T such that $t \leq u_k \leq T$ for all $k \in \mathbb{N}$.

Definition 2.7. [10] Let $\lambda(E^2)$ be a sequence space of the IVFN. If the function $||.|| : \lambda(E^2) \to \mathbb{R}$ satisfies M1, M2 and M3 then $\lambda(E^2)$ is called module sequence space of the IVFN. In addition this, if $\lambda(E^2)$ is complete with respect to a module then $\lambda(E^2)$ is called complete module sequence space of the IVFNs.

3. The Spaces $c_0(E^2, M, p)$, $c(E^2, M, p)$ and $\ell_\infty(E^2, M, p)$

Let us suppose that M is a modulus function and $u = (u_k) = ([u_k^-, u_k^+])$ be a sequence space of IVFN. In this section, we define the sequence spaces $c_0(E^2, M, p)$, $c(E^2, M, p)$ and $\ell_\infty(E^2, M, p)$ as the set of all null, convergent and bounded sequences of IVFN with respect to modulus function M, respectively, that is

$$c_0(E^2, M, p) = \{u = ([u_k^-, u_k^+]) \in w(E^2) : \lim_k M(D(u_k, \theta))^{p_k} = 0\}, \tag{3}$$

$$c(E^2, M, p) = \{u = ([u_k^-, u_k^+]) \in w(E^2) : \lim_k M(D(u_k, u_0))^{p_k} = 0\} \tag{4}$$

and

$$\ell_\infty(E^2, M, p) = \{u = ([u_k^-, u_k^+]) \in w(E^2) : \sup_k M(D(u_k, \theta))^{p_k} < \infty\}, \tag{5}$$

where $p = (p_k)$ is bounded sequence of real numbers, $\mathbb{R}$. If $M(x) = x$, i.e., M is an unit function, and $p = (p_k) = (1, 1, ..., 1, ...)$ then the sets $c_0(E^2, M, p), c(E^2, M, p)$ and $\ell_\infty(E^2, M, p)$ are induced to the sets $c_0(E^2), c(E^2)$ and $\ell_\infty(E^2)$ as showed in the following, respectively:

$$c_0(E^2) = \{u = ([u_k^-, u_k^+]) \in w(E^2) : \lim_k D(u_k, \theta) = 0\},$$

$$c(E^2) = \{u = ([u_k^-, u_k^+]) \in w(E^2) : \lim_k (D(u_k, u_0)) = 0\},$$

$$\ell_\infty(E^2) = \{u = ([u_k^-, u_k^+]) \in w(E^2) : \sup_k D(u_k, \theta) < \infty\},$$

which were defined by Zararsız in [11].

We may begin with the following results which are essential in the text.

Theorem 3.1. The sequence spaces $c_0(E^2, M, p)$, $c(E^2, M, p)$ and $\ell_\infty(E^2, M, p)$ are complete metric spaces with the metric given by

$$\mathfrak{D}(u, v) = \mathfrak{D}(u_k, v_k) = \sup_{k\in\mathbb{N}} M(D(u_k, v_k))^{p_k} \tag{6}$$

where $u = [u_k^-, u_k^+], v = [v_k^-, v_k^+]$ are in $\{c_0(E^2, M, p), c(E^2, M, p), \ell_\infty(E^2, M, p)\}$.

Proof. We give the proof for the case $c(E^2, M, p)$. It is very easy to see that $\mathfrak{D}$ is a metric defined by (6). Let us suppose that $(u_k^n) = (u_0^n, u_1^n, u_2^n, \dots)$ be a Cauchy sequence in $c(E^2, M, p)$ for each n. Then, for every $\epsilon > 0$, there exist a $n_0 > 0$ such that $\mathfrak{D}(u_k^n, u_k^m) = \sup_{k\in\mathbb{N}} M\left(D(u_k^n, u_k^m)\right)^{p_k} < \epsilon$, for all $n, m \geq n_0$. Since M is a modulus function, we have $\overline{d}(u_k^{-n}, u_k^{-m}) < \epsilon$ and $\overline{d}(u_k^{+n}, u_k^{+m}) < \epsilon$, for all $n, m \geq n_0$. From here, we can conclude that (u_k^{-n}) and (u_k^{+m}) are Cauchy sequence of fuzzy numbers in $c(E^1)$. Since $c(E^1)$ is complete (u_k^n) is convergent in $c(E^1)$ for all $n \in \mathbb{N}$.

Let us suppose that $\lim_n u_k^n = u_k$ for each $k \in \mathbb{N}$. Because of the fact that $\mathfrak{D}(u_k^n, u_k^m) < \epsilon$ for all $n, m \geq k$, we can write the followings below:

$$\lim_{m\to\infty} \mathfrak{D}(u_k^n, u_k^m) = \mathfrak{D}(u_k^n, \lim_{m\to\infty} u_k^m) = \mathfrak{D}(u_k^n, u_k) < \varepsilon.$$

This implies that $u_k^n \to u_k$, $(n \to \infty)$ for all $n \geq k_0$ in $c(E^2, M, p)$. On the other hand, since

$$\begin{aligned}\mathfrak{D}(u_k, u_k^n - u_k^n) &= \lim_k M\left(D(u_k, u_k^n - u_k^n)\right)^{p_k} \\ &\leq \lim_k M(D(u_k, u_k^n))^{p_k} + \lim_k M(D(\theta, u_k^n))^{p_k} < \infty\end{aligned}$$

this shows that $u = (u_k) \in c(E^2, M, p)$. ■

Theorem 3.2. Let us suppose that p_k is equal to 1, for all $k \in \mathbb{N}$. Then inclusions $c_0(E^2, M, p) \subseteq c(E^2, M, p) \subseteq \ell_\infty(E^2, M, p)$ hold.

Proof. The inclusion $c_0(E^2, M, p) \subseteq c(E^2, M, p)$ is clear. Let us show that the inclusion between $c(E^2, M, p)$ and $\ell_\infty(E^2, M, p)$ is valid. Let us suppose that $u \in c(E^2, M, p)$. This says to us that, the following equations exist:

$$\lim_k D([u_k^-, u_k^+], [u_0^-, u_0^+])^{p_k} = 0 \Rightarrow \lim_k M(\max\{\overline{d}(u_k^-, u_0^-), \overline{d}(u_k^+, u_0^+)\}) = 0.$$

Thus, we obtain $\overline{d}(u_k^-, u_0^-) < \epsilon$ and $\overline{d}(u_k^+, u_0^+) < \epsilon$. This means that $(u_k^-), (u_k^+) \in c(E^1)$ and $(u_k^+) \in c(E^1)$. Since $c(E^1) \subset \ell_\infty(E^1)$ we see that $u = (u_k) = ([u_k^-, u_k^+]) \in \ell_\infty(E^2, M, p)$. This step completes the proof. ■

Theorem 3.3. The spaces $c_0(E^1)$, $c(E^1)$ and $\ell_\infty(E^1)$ consisting of the null, convergent and bounded sequences of fuzzy numbers are subsets of the spaces $c_0(E^2, M, p)$, $c(E^2, M, p)$ and $\ell_\infty(E^2, M, p)$, respectively.

Proof. We consider only $c_0(E^1) \subset c_0(E^2, M, p)$. If we take $M(x) = x$ and $p = (p_k) = (1, 1, ..., 1, ...)$ then the proof is clear since every elements of $c_0(E^1)$ is a degenerate sequences of IVFN (see, Definition 2.2). ■

Theorem 3.4. For any $(u_k), (v_k) \in c(E^2, M, p)$, if $\lim_k u_k = u_0$ and $\lim_k v_k = v_0$ then we have

1. $\lim_k (u_k + v_k) = u_0 + v_0$,
2. $\lim_k (u_k - v_k) = u_0 - v_0$,
3. $\lim_k (u_k v_k) = u_0 v_0$.

Proof. Since the proof can also be obtained in the similar way for (2) and (3) we will only deal with (1). Let us suppose that $\lim_k u_k = u_0$ and $\lim_k v_k = v_0$. Then, for all $k \geq m$, we have

$$\mathfrak{D}(u_k, u_0) = M\left(D(u_k, u_0)\right)^{p_k} < \frac{1}{2}\varepsilon \text{ and } \mathfrak{D}(v_k, v_0) = M\left(D(v_k, v_0)\right)^{p_k} < \frac{1}{2}\varepsilon,$$

where $m \in \mathbb{N}$. Then we can write

$$\mathfrak{D}(u_k + v_k, u_0 + v_0) = M\left(D(u_k, u_0)\right)^{p_k} + M\left(D(v_k, v_0)\right)^{p_k} < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.$$

This shows to us that $\lim_k (u_k + v_k) = u_0 + v_0$. This step completes the proof of the theorem. ■

Definition 3.5. Let us suppose that $\lambda(E^2, M, p)$, $\mu(E^2, M, p)$ are sets of the sequences of IVFN and $\lambda(E^2, M, p) \subset \mu(E^2, M, p)$. Then $\lambda(E^2, M, p)$ is called cofinal in $\mu(E^2, M, p)$ if for $(u_k) \in \lambda(E^2, M, p)$ there is $(v_k) \in \mu(E^2, M, p)$ such that $\mathfrak{D}(u_k, \theta) \leq \mathfrak{D}(v_k, \theta)$ for all $k \in \mathbb{N}$.

If $\lambda(E^2, M, p)$ is cofinal in $\mu(E^2, M, p)$ then $\lambda(E^2, M, p)^\alpha = \mu(E^2, M, p)^\alpha$; the converse of this assertion is not true, where the α- duals of the spaces $\lambda(E^2, M, p)$ and $\mu(E^2, M, p)$ are denoted by $\lambda(E^2, M, p)^\alpha$, $\mu(E^2, M, p)^\alpha$, respectively.

Theorem 3.6. The interval valued fuzzy sequence spaces $c_0(E^2, M, p)$, $c(E^2, M, p)$ are cofinal in $\ell_\infty(E^2, M, p)$.

Proof. Let us suppose that $\lambda(E^2, M, p)$ denote any of the spaces $c_0(E^2, M, p)$, $c(E^2, M, p)$ and suppose that $\mathfrak{D}(u_k, \theta) \leq \mathfrak{D}(v_k, \theta)$ holds for some $(v_k) \in \ell_\infty(E^2, M, p)$.

Then we can easily see that

$$\sup_k M(D(u_k,\theta))^{p_k} \le \sup_k M(D(v_k,\theta))^{p_k}$$

and

$$\lim_k M(D(u_k,\theta))^{p_k} \le \lim_k M(D(v_k,\theta))^{p_k}.$$

This step completes the proof of the theorem. ■

From definition 3.6, we see that the α-duals of the spaces $c_0(E^2, M, p)$ and $c(E^2, M, p)$ are equal of the α -dual of the space $\ell_\infty(E^2, M, p)$, i.e., $c_0(E^2, M, p)^\alpha = \ell_\infty(E^2, M, p)^\alpha$ and $c(E^2, M, p)^\alpha = \ell_\infty(E^2, M, p)^\alpha$

If we take $M(x) = x$ and $p_k = 1$ for all $k \in \mathbb{N}$ then we see that $c_0(E^2, M, p)^\alpha = c(E^2, M, p)^\alpha = \ell_\infty(E^2, M, p)^\alpha = \ell_1(E^2)$, where $\ell_1(E^2) = \{u = (u_k) \in w(E^2) :$

$$\sum_k \max\{\overline{d}(u_k^-, \overline{0}), \overline{d}(u_k^+, \overline{0})\} < \infty, \}$$

(see, [11]).

Definition 3.7. A sequence space $\lambda(E^2, M, p)$ is said to be symmetric if, when u in $\lambda(E^2, M, p)$, then v is in $\lambda(E^2, M, p)$ when the coordinates of v are those of u, but in a different order.

Theorem 3.8. If $M(x) = x$ and $p_k = 1$, $(\forall k \in \mathbb{N})$ then the spaces $c(E^2, M, p)$ and $c_0(E^2, M, p)$ are symmetric spaces.

Proof. We consider only $c(E^2, M, p)$ since the proof can also be obtained in the similar way for $c_0(E^2, M, p)$. Let us consider the sequence (u_k) in $c(E^2, M, p)$ defined by means of [10] as $(u_k) =$

$$\left(\left[\begin{cases} \frac{k}{2k-1}x, & x \in \left[0, \frac{2k-1}{k}\right] \\ 1, & x \in \left[\frac{2k-1}{k}, \frac{2k+1}{k}\right] \\ -\frac{k}{2k+1}(x-4), & x \in \left[\frac{2k+1}{k}, 4\right] \\ 0, & \text{otherwise} \end{cases}, \begin{cases} \frac{k}{2k-1}x - 1, & x \in \left[1, \frac{3k-1}{k}\right] \\ 1, & x \in \left[\frac{3k-1}{k}, \frac{3k+1}{k}\right] \\ -\frac{k}{2k+1}(x-5), & x \in \left[\frac{3k+1}{k}, 5\right] \\ 0, & \text{otherwise} \end{cases}\right]\right),$$

Clearly we see that $\lim_k u_k =$

$$\lim_k \left[\begin{cases} \frac{k}{2k-1}x, & x \in \left[0, \frac{2k-1}{k}\right] \\ 1, & x \in \left[\frac{2k-1}{k}, \frac{2k+1}{k}\right] \\ -\frac{k}{2k+1}(x-4), & x \in \left[\frac{2k+1}{k}, 4\right] \\ 0, & \text{otherwise} \end{cases}, \begin{cases} \frac{k}{2k-1}x-1, & x \in \left[1, \frac{3k-1}{k}\right] \\ 1, & x \in \left[\frac{3k-1}{k}, \frac{3k+1}{k}\right] \\ -\frac{k}{2k+1}(x-5), & x \in \left[\frac{3k+1}{k}, 5\right] \\ 0, & \text{otherwise} \end{cases} \right]$$

$$= \left[\begin{cases} \frac{1}{2}x, & x \in [0,2] \\ -\frac{1}{2}(x-4), & x \in [2,4] \\ 0, & \text{otherwise} \end{cases}, \begin{cases} \frac{1}{2}x-1, & x \in [0,2] \\ -\frac{1}{2}(x-5), & x \in [2,4] \\ 0, & \text{otherwise} \end{cases} \right].$$

Let (v_k) be a rearrangement of (u_k) which is defined by

$$(v_k) = (u_1, u_3, u_2, u_4, u_5, u_7, u_6, u_8, ...).$$

From here, we can obtain the following:

$$\lim_k v_k = \left(\left[\begin{cases} \frac{1}{2}x, & x \in [0,2] \\ -\frac{1}{2}(x-4), & x \in [2,4] \\ 0, & otherwise \end{cases}, \begin{cases} \frac{1}{2}x-1, & x \in [1,3] \\ -\frac{1}{2}(x-5), & x \in [3,5] \\ 0, & otherwise \end{cases} \right] \right).$$

It means that the sequences (u_k) and (v_k) has the same limit points. Therefore, from Definition 3.7, we see that $c(E^2, M, p)$ is symmetric space. ■

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