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Null-Controllability for Linear Control System using Fixed Point Theorem

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Abstract

Null controllbility of linear control system in this paper, was obtained by the help of Leray-Schauder fixed point theorem. To achieve this, sufficient conditions and assumptions which made the existence of at least a solution of the control system to be steered to zero in finite time were put in place.

AMS subject classification:
Keywords: Null-Controllability,Leray-Schauder theorem, admissible control.

1. Introduction

Let us assume that R^n is the n-dimentional Euclidean space. In this space, we shall discuss the linear control systems

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ \\ x(t_0) = x_0 \end{cases} \tag{1.1}$$

where $x \in R^n$, A and B are respectively $n \times n$ and $n \times m$ continuous linear matrix functions on the interval $I = [0, \infty)$. These functions have summable components.

Then, u is an m-vector valued measurable function with value $u(t)$ in a compact convex set Ω lying in R^m. We know that such a u is said to be admissible. For the above systems (1.1), we can consider subset C^m of R^m which is $m-$ dimensional unit cube. So, for each $t \in I, u : I \longrightarrow C^m$ where

$$C^m = \{u : u(t) \in \Omega; |u_j| \leq 1, j = 1, 2, 3, \ldots, m\}$$

We shall denote the solution of (1.1) by $x(t, u)$. Then, by the variation of parameter method, the solution of (1.1) above is given by

$$x(t, u) = X(t)x_0 + X(t) \int_0^t X^{-1}(s)B(s)u(s)ds \tag{1.2}$$

where $X(t)$ is the transitional matrix solution of

$$\begin{cases} \dot{x} = A(t)x(t) \\ x(t_0) = x_0 \end{cases} \tag{1.3}$$

with $X(0) = I$, the identity matrix. Since our aim is to achieve $null - controllability$, we impose on (1.1) the boundary condition

$$Tx = 0 \tag{1.4}$$

Here, we expect T to be bounded operator on $\left[[0, \infty), R^n\right]$, the space of all bounded and continuous operator from $[0, \infty)$ to R^n. According to K.Balachandram et al [2], if such system is uniformly assymptotically stable so that the solution $x(t_0, u)$ satisfies

$$||x(t_0, u)|| \leq Me^{-\alpha(t-t_0)}||u||, \text{ for some } \alpha > 0, M > 0.$$

and the control system is controllable, then the system system is $null - controllable.$

Also, by J.U. Onwuatu [7], if the constraint set Ω is an arbitrary compact subset of R^n and the system is uniformly asymptotically stable so that the solution $x(t_0, u)$ satisfies

$$||x(t_0, u)|| \leq Me^{-\alpha(t-t_0)}||u||, \text{ for some } \alpha > 0, M > 0.$$

and is proper in R^n, then the system is $Euclidean\ null - controllable.$

Here, we shall pursue this topic from the point of view of fixed point of the operator T. First of all, let us have the following preliminaries.

2. Preliminaries

Here, we put togather the necessary facts we need to establish the main result of this our paper. This will come in the form of definitions and Lemmas.

Definition 2.1. [1] The control system (1.1) is said to be *Euclidean controllable* if for each $x_0 \in R^n$ and each $x_1 \in R^n$, there exists a time $t_1 \geq 0$ and an admissible control u such that the solution $x(t,u) = x(t)$, say of (1.1) satisfies $x(0) = x_0$ and $x(t_1) = x_1$.

Definition 2.2. [1] The control system (1.1) is said to be $Euclidean\ null-controllable$ if in definition 2.1 above, $x_1 = 0$.

Definition 2.3. [8] Let X and Y be normed linear spaces. An operator $T : X \longrightarrow Y$ is said to be completely continuous if T maps bounded sets in X into relatively compact sets in Y.

Note: We have to note that completely continuous operator is said to be a compact Operator.

Definition 2.4. [8] A set of functions $\{f_n\}$ defined on a real interval I is said to be equi-continuous on I if given $\epsilon > 0$, there exist a $\delta > 0$ independent of the particular function in the set and also for any $t_1, t_2 \in I$, we have $|f_n(t_2) - f_n(t_1)| < \epsilon$ whenever $|t_2 - t_1| < \delta$.

Lemma 2.5. (Leray Schauder Theorem) [5] Let S be a Bannach Space and let $T(\lambda, x)$ be an operator depending continuously on the parameter $\lambda, (0 < \lambda \leq 1)$ and such that for each λ in this range, T is continuous and is completely continuous map of S into S. Suppose further that there exists a constant $\beta_0, 0 < \beta_0 < \infty$, independent of λ, such that for every $x \in S$ satisfying

$$x - \lambda T(\lambda, x) = 0, \ (0 < \lambda \leq 1) \tag{2.1}$$

we have

$$||x|| \leq \beta_0 \tag{2.2}$$

Then for every λ in this range, there exists at least one $x \in S$ satisfying (2.1).

Lemma 2.6. (Ascoli - Axzela Theorem) [6] If a sequence $\{f_n\}_{n=1}^{\infty}$ in $C(X)$ is bounded and equi-continuous, then it has a uniformly subsequence.

Note that in this statement

(a) "$F \subset C(X)$ is bounded" means that there exists a positive constant $M < \infty$ such that $|f(x)| \leq M$ for each $x \in X$ and each $f \in F$, and

(b) "$F \in C(X)$ is equi-continuous" means that for every $\epsilon > 0$ there exists $\delta > 0$ (which depends only on ϵ such that for $x, y, \in X, d(x,y) < \delta \Rightarrow |f(x) - f(y)| < \epsilon \ \forall f \in F$. where d is the metric in X.

Lemma 2.7. (Lebesque Theorem) [8] Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of a measurable functions on E that converges pointwise $a.e$ on E to g, such that $|f_n| \leq g$ for all $n \in N$.

if $\lim_{n\to\infty}\int_E g_n = \int_E g$, then

$$\lim_{n\to\infty}\int_E f_n = \int_E f.$$

Note: The Leray–Schauder Theorem given above (Lemma 2.1) is an important tool used in the proof of the existence of periodic solutions of differential equations under a prescribed condition. We have to observe that, in general, the theorem is used together with Schaefer's Lemma in establishing the existence of a desired operator T, say, and the choice of a suitable Bannach space X for $\lambda = 1$.

This then means that $x = T(1, x)$. This choice is made with out loss of generality. By this we may imply that a fixed point $x \in X$ exists, thereby implying the existence of a periodic solution of the equation concerned.

In our present study, all that is required is the performance of two jobs; namely: the establishment of a suitable operator from a well defined Bannach space into itself,and the provision of a bound satisfying equation (2.2) as stated above.

Now, suppose C is the Bannach space of all bounded continuous functions from $I = [0, \infty)$ to R^n with the sup norm. By this, we mean $g \in C$ to imply

$$||g|| = \sup_{t\in I}|g(t)| \tag{2.3}$$

Also, we want the norm of x to be

$$||x|| = \sum_{i=1}^{n}|x_i|, x \in \mathbb{R}^n \tag{2.4}$$

Suppose C_1 is the set of those functions $f \in C$ such that the limit $\lim_{t\to\infty} f(t)$ exists and is finite. Obviously, C_1 is a closed subspace of C. If μ is a fixed positive integer, we write

$$\begin{cases} S_\mu = \{x \in \mathbb{R}^n : ||x|| \le \mu\} \\ S^\mu = \{f \in C_1 : ||f|| \le \mu\} \end{cases} \tag{2.5}$$

The operator $T : C_1 \longrightarrow \mathbb{R}^n$ is assumed to be a bounded linear operator. If $X(t)$ is the transitional matrix solution of the system (1.3), then let us define the operator T_0 as follows

$$T_0 = TX(t),\ t \in T$$

that is

$$T_0 = \{Tx_1(t), Tx_2(t), \dots\}.$$

So, T_0 is the constant matrix whose columns are the values of T at the corresponding columns of X.

Then for any $y \in \mathbb{R}^n$, we can write

$$[T_0X]_y = T(X_y). \tag{2.6}$$

Under the above conditions, we can see, provided that the matrix $[T_0X]$ is a non-singular, that, the solution of the control system (1.1) satisfying (1.4) if it exists, is

$$x(t,u) = p(t) - x(t)[T_0X]^{-1}T_p \tag{2.7}$$

where

$$p(t) = X(t)\int_0^t X^{-1}(s)B(s)u(s)ds \tag{2.8}$$

This is because, the solution of (1.1) as give in (1.2) is

$$x(t,u) = X(t)x_0 + X(t)\int_0^t X^{-1}(s)B(s)u(s)ds \tag{$\bigstar$}$$

i.e $x(t,u) = X(t)x_0 + p(t)$, where $p(t)$ is defined in (2.8). So, applying T to both sides of ($\bigstar$), we get
$Tx(t,u) = TX(t)x_0 + Tp(t)$, (since T is linear)
We know that $Tx = 0$ (from (1.4)). This means

$$Tx(t,u) = 0.$$

So we have

$$TX(t)x_0 + Tp(t) = 0.$$

Thus, $TXx_0 + Tp = 0$. Consequently, we get from (2.6) that

$$[T_0X]x_0 = -Tp.$$

Then, since $[T_0X]$ is assumed to be non - singular, we have

$$x_0 = -[T_0X]^{-1}Tp.$$

If we substitute for x_0 in ($\bigstar$) above, we have the desired result (2.7).

3. Main Result

The main result of this our paper is stated as theorem 3.1 below.

Theorem 3.1. Consider the Linear control systems (1.1). Let us assume the followings:

(a) $\lim_{t\to\infty} X(t) = X(\infty)$,say, exists and finite;

(b) $N = \int_0^\infty q(s)ds < \infty$ where $q(s) = \sup_{\substack{u\,(t)\in C^m \\ t\in I}} ||X^-1(t)B(t)||$, and

(c) If $T_f = [T_0X]^{-1}Tp$, $\underset{f\in S^\mu}{max}||T_f|| = M$ and $\underset{t\geq 0}{max}||X(t)|| = 1$, we have $L(N-M) \leq \mu$;

Under these conditions, there exists at least one solution of (1.1) satisfying (1.4). That is, the system (1.1) is Euclidean null - controllable.

Proof. Now, let us consider the operator $p : S^{\mu} \longrightarrow C^1$ which maps the function $pf \in C^1$ defined as

$$(pf)(t) = X(t)[\int_0^t X^{-1}(s)B(s)u(s)ds - T_f] \tag{3.1}$$

From the hypothesis (c) of this theorem,and also, from the definition of our C^1, we can see that $PS^{\mu} \subset S^{\mu}$, that is P maps S^{μ} into itself.
Now, if we fix $f \in S^{\mu}$ and set $x_f(t) = (pf)(t)$, which has finite limit $\underset{t\longrightarrow\infty}{lim} x_f(t) = x_f(\infty)$, say. (is finite).
Then from the theorem, we have

$$\begin{aligned}
&||x_f(t) - x_f(\infty)|| \\
&= \left\| X(t)\left\{\int_0^t X^{-1}(s)B(s)u(s)ds - T_f\right\} - X(\infty)\left\{\int_0^\infty X^{-1}(s)B(s)u(s)ds - T_f\right\}\right\| \\
&= - \left\| T_f[X(t) - X(\infty)] + X(t)\int_0^t X^{-1}(s)B(s)u(s)ds - X(\infty)\int_0^\infty X^{-1}(s)B(s)u(s)ds\right\| \\
&\leq -M\|X(t) - X(\infty)\| + \left\| X(t)\int_0^t X^{-1}(s)B(s)u(s)ds - X(\infty)\int_0^\infty X^{-1}(s)B(s)u(s)ds\right\|
\end{aligned}$$

We know that $\int_0^t (.)ds = \int_0^\infty (.)ds - \int_t^\infty (.)ds$
So, the above estimate becomes

$$\begin{aligned}
||x_f(t) - x_f(\infty)|| &\leq -M||X(t) - X(\infty)|| + N||X(t) - X(\infty)|| \\
&\quad + \left\| X(\infty)\int_0^\infty X^{-1}(s)B(s)u(s)ds\right\|. \\
&\leq (N - M)||(X(t) - X(\infty)|| + X(\infty)||\int_t^\infty g(s)ds||.
\end{aligned}$$

So, for a given $\epsilon > 0$, there exists a $t_0(\epsilon) > 0$ such that $||x_f(t) - x_f(\infty)|| < \epsilon$ for each $t > t_0(\epsilon)$, and each $f \in S^{\mu}$. So, $\{x_f\}$ is uniformly convergent. From Functional analysis, it can be shown that $\{x_f\}$ is equicontinuous and also uniformly bounded. Then, by Ascoli - Axzela lemma, (Lemma 2.2), $\{x_f\}$ is relatively compact in C^1. Hence, the operator P is completely continuous and maps the bounded set $\{f \in C^1 : ||f|| \leq \mu\}$ into the relatively compact set $\{x_f(s)\} \in C$.

We now show that the operator P is continuous. For this, we consider a sequence $\{f_n\} \in S^{\mu}$ and $\{f \in S^{\mu}\}$ such that

$$\lim_{n\to\infty} ||f_n - f|| = 0$$

We assume that $\lim_{n\to\infty} ||U_n - u|| = 0$ and $\{U_n\}$ is a sequence of admissible controls. Let $Pf_n = x_n$ and $Pf = x$. Then as each is generated by a corresponding admissible control U, we have

$$\begin{aligned} ||x_n - x|| &\leq L\left\{||Tf_n - Tf|| + \int_0^{\infty} ||X^{-1}(s)B(s)[u_n - u](s)||ds\right\} \\ &\leq L\left\{[T_0X]^{-1}TX(t)\int_0^t X^{-1}(s)B(s)[u_n - u](s)ds \right. \\ &\qquad \left. + \int_0^{\infty} ||X^{-1}(s)B(s)[u_n - u](s)||ds\right\}. \end{aligned}$$

Each of the integrands above tends to zero as $n \longrightarrow \infty$ and at the same time, each is uniformly bounded. Hence, by the Lebesque theorem, (Lemma 2.3 above) we have

$$\lim_{n\to\infty} ||x_n - x|| = 0$$

Hence, P is continuous on S. Then from the hypothesis (c), where $L(N - M) \leq \mu$ together with the hypothesis (b) we get

$$L(N - M) \leq ||X(t)\int_0^t X^{-1}(s)B(s)u(s)ds - X(t)[T_0X]^{-1}Tp||$$

which, from (2.7), is equivalent to $||x(t, u)|| = ||x||$ say.
So, we have $L(N - M) \leq \mu$.
This measns $||x|| \leq \mu$ and μ being a positive integer, plays the role of B_0 in (2.2) above. Since the operator P is continuous and completely continuous on the space S^{μ}, and there is a positve constant μ such that $||x|| \leq \mu$, we get from Lerray - Schauder fixed point Theorem, (Lemma 2.1) that P has a fixed point which corresponds to at least one solution of the control system (1.1) satisfying (1.4). So, the control (1.1) is null-controllable. ■

4. Conclusion

We now conclude that null-controllability of linear system which was achieved using Leray Schauder fixed point techniques, can also be achieved using other fixed point theorems.

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