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# Designs for Multiple Comparisons of Control versus Treatments

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SUMMARY

The emphasis in this study has been on the construction, characterization and properties of experimental designs suitable for the comparison of test treatments versus a control or standard treatment. Until the introduction of the BTIB designs by Bechhofer and Tamhane traditional experimental designs or modifications of them had been used for this purpose. The work by Bechhofer and Tamhane has brought the design aspect of multiple comparisons in to a sharper focus and has stimulated more research in this area including the work reported in this study.

Although the balanced designs introduced by Bechhofer and Tamhane represent a rather large class of designs it seemed quite natural to generalize these designs to partially balanced designs. Just as BIB designs were generalized to PBIB designs. We refer to these new designs as PBTIB designs either of Type I or Type I I. Both types are defined such that treatment-control comparisons are made with two different variances (rather than one as for the BTIB designs),but the covariance structure among the comparisons is different for the two types of designs, with that for Type I I designs being more general.

Both types of PBTIB designs together provide a rich class of designs for many design parameter combinations such as number of test treatments (p), number of blocks (b), block size (k), number of replicates for control (r0) and test treatments (r). these designs are related in a natural way to PBIB designs. We have made use in particular of group divisible PBIB designs and exploited the structure of their association schemes.

Although PTBIB designs provide useful supplements and alternatives to BTIB designs their analytical properties are not easily characterized. No design optimality criteria are appropriate. Hence to choose "best" designs introduce the concept of A-efficiency with respect to what we call near-balance control-treatment block designs.

## 1. INTRODUCTION:

Multiple com.parison or simultaneous statistical inference procedures are used extensively in analyzing date or, more specifically, in drawing inferences about a set of population means. In an experimental setting such population means or rather comparisons among them can be expressed as comparisons among treatment effects. These comparisons can be of various types, such as simple comparisons of two treatment effects or, more generally, linear combinations or contrasts involving several treatment effects.

Historically, it is very difficult to identify who started the work in this area of statistical inference. On the other hand, even though many people have contributed in varying degrees, the general forms of the current structure of the multiple comparison procedures can be attributed to three researchers : Duncan, Tukey, and Scheff.

In this study we shall confine ourselves to the investigation of simultaneous multiple comparisons of several test treatments with a control or standard. Recently , this problem has been given considerable attention in statistical research because of the important role that such multiple comparisons play in many applied fields, such as agriculture and biology , and in industry.

We discuss the problem from the point of view of different designs such as Completely Randomized (CR) designs , Randomized Completely Block (RCB) designs, and , in more detail, Incomplete Block (IB) designs. Although the CR and RCB designs are the easiest to use, quite often they are not practical. Therefore , the use of IB designs in which the number of units in each block is less than the number of treatments is often preferable.

In summary, this research is concerned with the construction and characterization of designs useful in the context of treatment-control comparisons and applicable under different experimental situations.

## 2. DEFINITION AND LINEAR MODEL:

Consider the usual additive model for a block design.

$$Y_{ijh} = \mu + \alpha_i + \beta_j + \epsilon_{ijh} \qquad i = 0,1,2,\dots\dots,p$$
$$j = 1,2,3,\dots\dots,b \qquad (1)$$
$$h = 1,2,3,\dots\dots,k$$

Where $Y_{ijh}$ is the observation with treatment $i$ on plot $h$ of block $\alpha_i$ represents the $i$ th treatment effect, $\beta_j$ represents the $j$ th block effect with $\sum_i \alpha_i = 0\,, \sum_j \beta_j$ and $\epsilon_{ijh} \sim ijdN(0, \sigma^2)$. Bechhofer and Tamhane (1981) consider a class of designs that are balanced in the sense that :

$$var(\widehat{\alpha_0} - \widehat{\alpha_i}) = \tau^2\sigma^2 \qquad i = 1,2,3,\dots\dots,p \qquad (2)$$
$$cov(\widehat{\alpha_0} - \widehat{\alpha_{i1}}\,, \widehat{\alpha_0} - \widehat{\alpha_{i2}}\,) = p\sigma^2 \qquad i1 \neq i2 = 1,2,3,\dots,p$$

Where $\tau$ and $p$ are parameters depending on the design employed , $\widehat{\alpha_0} - \widehat{\alpha_\iota}$ is the best linear unbiased estimate (BLUE).

Bechhofer and Tamhane (1981) give necessary and sufficient conditions for a design to be a Balanced Treatment Incomplete Block (BTIB) design.

## 3. PARTIALLY BALANCED TREATMENT INCOMPLETE BLOCK DESIGN:

In many practical situations balanced designs exist only for a limited number of parameter combinations. To provide further flexibility in constructing designs suitable for simultaneously comparing test treatments vs, control we shall introduce the notion of Partially Balanced Treatment Incomplete Block (PBTIB) Designs. Whereas the main property of BTIB designs is that the $p$ treatment-control differences are estimated with the same precision, for PBTIB designs, this property will be modified in that some of the $p$ differences will be estimated with higher precision then others.

Generally , the advantages of PBTIB are as follows:

1- They offer different combinations of parameters yielding a large number of designs from which an experimenter may select according to his needs.

2- They offer the possibility to choose smaller size plane then the give provided by BTIB designs.

Our model will be the usual additive model (1) and its reduce normal equations can be written as follows:

$$\left(R - \frac{1}{K}NN'\right)\hat{\alpha} = \left(T - \frac{1}{K}NB\right) \tag{3}$$

Where $R = diag(r_\circ, \dots \dots., r)(p+1)((p+1)$, where $r_\circ$ is the number of replications of the control and $r$ is the number of replications for each test treatment.

$N = (n_{ij})(p+1) \times b$ is the incidence matrix , where $n_{ij}$ is the number of replications of the $i^{th}$ treatment in the $j^{th}$ block :

$T = (T_0, \dots \dots.., T_P), (P+1) \times 1,$ where $T_i$ is the total of the $i^{th}$ treatment;

$B = (B_1, \dots \dots \dots, B_b), b \times 1,$ where $B_i$ is the total of the $j^{th}$ block;

$\hat{\alpha} = \left(\widehat{\alpha_0} - \hat{\alpha}_\iota, \dots \dots.., \hat{\alpha}_p\right), (p+1) \times 1,$ where $\hat{\alpha}_i$ is the estimator for the effect of the $i^{th}$ treatment;

$NN' = \sum_{j=1}^{b} \cap_{i1j} \cap_{i2j} = (\lambda_{i1i2})$, where $\lambda_{i1i2}$ denotes the number of times that treatment $i1$ $and$ $i2$ appear together in the same block over the whole design.

## 4. DEFINITION AND PROPERTIES OF PBTIB DESIGNS:

We shall define two types of PBTIB designs (Type I and II). To do this we partition the set of test treatments $T = (1,2, \ldots\ldots, p)$into two disjoint sets $T_1 = \left(1_1, \ldots, 1_q\right)$ and

$T_2 = (2_1, \ldots., 2_{p-q})$.

Definition 4.1:

A design for comparing $p$ test treatments with a control is said to be a PBTIB-Type I design if the variance-covariance structure for treatment-control comparisons has the following form:

$$\begin{aligned} & var(\widehat{\alpha_0} - \widehat{\alpha}_\iota) = \tau^2{}_1\sigma^2 && i\epsilon\, T_1 \\ & var\left(\widehat{\alpha_0} - \widehat{\alpha}_J\right) = \tau^2{}_2\sigma^2 && j\epsilon\, T_2 \\ & cov(\widehat{\alpha_0} - \widehat{\alpha}_\iota\,, \widehat{\alpha_0} - \widehat{\alpha}_\iota\,) = p_1\sigma^2 && 1, 1'\epsilon\, T_1 \\ & cov\left(\widehat{\alpha_0} - \widehat{\alpha}_J\,, \widehat{\alpha_0} - \widehat{\alpha}_J\,\right) = p_2\sigma^2 && j, j'\epsilon\, T_2 \\ & cov\left(\widehat{\alpha_0} - \widehat{\alpha}_\iota\,, \widehat{\alpha_0} - \widehat{\alpha}_J\,\right) = p_3\sigma^2 && 1\epsilon\, T_1 \quad , j\epsilon\, T_2 \end{aligned} \qquad (4)$$

Where the parameters $\tau_1, \tau_2, p_1, p_2, and\ \ p_3$ depend on the design employed.

Theorem 4.1:

An incomplete block design is a PBTIB-Type I design, if it satisfies the following conditions.

I $-$the experimental units are divided into $b$ block, where each block has $k$ units $k(p + 1)$.

II $-$there are $p$ test treatments, each of which occurs in $r$ block , and there is a control treatment which occurs $r_0$ times in the experiment. Each test treatment is applied to at most one unite in the same block, but the control may be applied to several units in a block.

III $-$the set of test treatments is denoted by $T = (1,2, \ldots\ldots, p)$, and $T$ can be divided into two disjoint subsets, $T_1\ and\ T_2$ as defined earlier. Each test treatment appears with the control treatment either $\lambda_1\ or\ \ \lambda_2$ times;

$$\lambda_{0i} = \begin{cases} \lambda_1: & 1\epsilon\, T_1 \\ \lambda_2: & 1\epsilon\, T_2 \end{cases}$$

Where $\lambda_{0i}$ is the number of times that test treatment 1 occurs with the control in the same block over the whole design.

IV $-$any two treatments in the $1^{th}$ group (i.e in $T_i$) occurs together in exactly block (i=1,2).

V $-$ any two treatments, one from $T_1$ and the other from $T_2$ occur together in exactly $\gamma_{12}$ block.

These five conditions imply that the information matrix of

$\left(\widehat{\alpha_0} - \widehat{\alpha_\iota}, \widehat{\alpha_0} - \widehat{\alpha_2}, \dots \dots \dots .., \widehat{\alpha_0} - \widehat{\alpha_p}\right)$ is of the following form:

$$C^* = \begin{pmatrix} aI_q + bJ_{q*q} & eJ_{q*\bar{q}} \\ eJ_{\bar{q}*q} & cI_{\bar{q}} + dJ_{\bar{q}*\bar{q}} \end{pmatrix} \tag{5}$$

$where\ \bar{q} = p - q,\ b = -\frac{\gamma^1}{k},\ c = r - \frac{r}{k} + \frac{\gamma^2}{k},\ a = r - \frac{r}{k} + \frac{\gamma^1}{k},\ d = -\frac{\gamma^2}{k},\ e = -\frac{\gamma_{12}}{k}$

Proof:

To look at the form of $C^{*-1}$

Let $c_{11} = aI_q + bJ_{q*q}$ , $c_{22} = cI_{\bar{q}} + dJ_{\bar{q}*\bar{q}}$ , $c_{12} = eJ_{q*\bar{q}} = c_{21}$ and then

$c_{11.2} = c_{11} - c_{12}c_{22}{}^{-1}c_{21}$ , $c_{22.1} = c_{22} - c_{21}c_{11}{}^{-1}c_{12}$

To prove that $c_{11.2}$ *and* $c_{22.1}$ are symmetric and of the form $vI + uJ$, we follow the steps below:

$$\text{From } c_{12}c_{22}{}^{-1}c_{21} = \left(eJ_{q*\bar{q}}\right)\left[cI_{\bar{q}} + dJ_{\bar{q}*\bar{q}}\right]^{-1}\left(eJ_{\bar{q}*q}\right)$$
$$= \left(eJ_{q*\bar{q}}\right)\left[fI_{\bar{q}} + gJ_{\bar{q}*\bar{q}}\right]^{-1}\left(eJ_{\bar{q}*q}\right) \tag{6}$$
$$= fe^2\bar{q}J_{q*q} + ge^2\bar{q}^2J_{q*q}$$
$$= \Upsilon J_{q*q}$$

It follows that

$$c_{11.2} = \left(aI_q + bJ_{q*q}\right) - \Upsilon J_{q*q} \tag{7}$$
$$= aI_q + (b - \Upsilon)J_{q*q}$$

Similarly , $c_{22.1}$ will be of the form (7) and hence $c_{11.2}{}^{-1}$ *and* $c_{22.1}{}^{-1}$ are of the form (7), i.e.

$$c_{11.2}{}^{-1} = \begin{pmatrix} \tau_1^2 & \cdots & (p_1) \\ & \ddots & \vdots \\ & & \tau_1^2 \end{pmatrix} \quad , \quad c_{22.1}{}^{-1} = \begin{pmatrix} \tau_2^2 & \cdots & (p_2) \\ & \ddots & \vdots \\ & & \tau_2^2 \end{pmatrix} \tag{8}$$

Looking at the other sub matrices of $C^{*-1}$ , we find

$$c_{11.2}{}^{-1}c_{12}c_{22}{}^{-1} = \left[vI_q + uJ_{q*q}\right]\left(eJ_{q*\bar{q}}\right)\left[cI_{\bar{q}} + dJ_{\bar{q}*\bar{q}}\right]^{-1}$$
$$= \left[vI_q + uJ_{q*q}\right]\left(eJ_{q*\bar{q}}\right)\left[hI_{\bar{q}} + sJ_{\bar{q}*\bar{q}}\right] \tag{9}$$
$$= mJ_{q*\bar{q}}$$

Where , say , $m = p_3$.

Hence these conditions lead us to designs that have the properties of definition 4.1 . in order to express the parameter $\tau_1, \tau_2, p_1, p_2, and\ \ p_3$ in terms of the entries of the $C^*$

matrix $(s')$ $a, b, c, d, and\ e.$ see Rasheed ( ) by substitution we obtain.

$$c_{11.2}{}^{-1} = \frac{1}{a} I_q - \left( \frac{b - \frac{\bar{q}e^2}{c} + \frac{d\bar{q}^2 e^2}{c(c+\bar{q}d)}}{a\left(a + \bar{q}\left(b - \frac{\bar{q}e^2}{c} + \frac{d\bar{q}^2e^2}{c(c+\bar{q}d)}\right)\right)} \right) J_{q*q} \qquad (10)$$

Hence we have

$$\tau_1{}^2 = \frac{1}{a} - \frac{b - \frac{\bar{q}e^2}{c} + \frac{d\bar{q}^2 e^2}{c(c+\bar{q}d)}}{a\left(a + \bar{q}\left(b - \frac{\bar{q}e^2}{c} + \frac{d\bar{q}^2e^2}{c(c+\bar{q}d)}\right)\right)} \qquad (11)$$

And $$p_1 = -\frac{b - \frac{\bar{q}e^2}{c} + \frac{d\bar{q}^2 e^2}{c(c+\bar{q}d)}}{a\left(a + \bar{q}\left(b - \frac{\bar{q}e^2}{c} + \frac{d\bar{q}^2e^2}{c(c+\bar{q}d)}\right)\right)} \qquad (12)$$

The same way can find $\tau_2{}^2$ and $p_2$ .

Finally , the expression for $p_3$ can be obtained by substituting (10) in-$c_{11.2}{}^{-1}c_{12}c_{22}{}^{-1}$ ,which yields

$$p_3 = \frac{e(q-\bar{q})\left(b - \frac{\bar{q}e^2}{c} + \frac{d\bar{q}^2 e^2}{c(c+\bar{q}d)}\right) - ae}{a(c+\bar{q}d)\left(a + \bar{q}\left(b - \frac{\bar{q}e^2}{c} + \frac{d\bar{q}^2e^2}{c(c+\bar{q}d)}\right)\right)} \qquad (13)$$

Example 4.1:

The following design is a PBTIB-Type I design with the parameters

$p = 5$ , $k = 3$ , $r = 3$ , $r_0 = 9$ , $b = 8$ , and the two groups $T_1 = (2,3,4)$ , and $T_2 = (1,5)$

With $\lambda_1 = 2$ , $\lambda_2 = 3$ , $\gamma^1 = 1$ , $\gamma^2 = 0$ , and $\gamma^1{}_{12} = 1$.

$$D_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 3 & 4 & 2 & 3 \\ 1 & 2 & 5 & 3 & 4 & 5 & 4 & 5 \end{pmatrix} ,$$

$$C^* = \begin{pmatrix} 2 & -,33 & -,33 & -,33 & -,33 \\ -,33 & 2 & -,33 & -,33 & -,33 \\ -,33 & -,33 & 2 & -,33 & -,33 \\ -,33 & -,33 & -,33 & 2 & 0 \\ -,33 & -,33 & -,33 & 0 & 2 \end{pmatrix}$$

Where $C^*$ is the information matrix of $(\widehat{\alpha_0} - \widehat{\alpha_1}, \widehat{\alpha_0} - \widehat{\alpha_2}, \dots \dots \dots .., \widehat{\alpha_0} - \widehat{\alpha_5})$ and

$$C^{*-1} = \begin{pmatrix} .619 & .19 & .19 & .166 & .166 \\ .19 & .619 & .19 & .166 & .166 \\ .19 & .19 & .619 & .166 & .166 \\ .166 & .166 & .166 & .583 & .083 \\ .166 & .166 & .166 & .083 & .583 \end{pmatrix}$$

Hence $\tau_1{}^2 = .619$ , $\tau_2{}^2 = .583$ , $p_1 = .19$ , $p_2 = .083$ , $p_3 = .166$

These designs have desirables but there are relatively few PBTIB-Type I designs. In order to generate more useful PBTIB designs, we consider designs of Type I I which are defined as follows.

Definition 4.2:

An incomplete block design is said to be a PBTIB-Type I design if the variance-covariance structure for treatment-control comparisons has the following form:

$$\begin{aligned} &var(\widehat{\alpha_0} - \hat{\alpha}_i) = \tau^2{}_1\sigma^2 && i\epsilon\, T_1 \\ &var(\widehat{\alpha_0} - \hat{\alpha}_j) = \tau^2{}_2\sigma^2 && j\epsilon\, T_2 \\ &cov(\widehat{\alpha_0} - \hat{\alpha}_i , \widehat{\alpha_0} - \hat{\alpha}_i ) = p_1\sigma^2 && 1, 1'\epsilon\, T_1 \\ &cov(\widehat{\alpha_0} - \hat{\alpha}_j , \widehat{\alpha_0} - \hat{\alpha}_j ) = p_2\sigma^2 && j, j'\epsilon\, T_2 \\ &cov(\widehat{\alpha_0} - \hat{\alpha}_i , \widehat{\alpha_0} - \hat{\alpha}_j ) = p_3\sigma^2 && 1\epsilon\, T_1 \quad , j\epsilon\, T_2 \end{aligned} \qquad (14)$$

Theorem 4.2:

An incomplete block design is a PBTIB-Type II design, if it satisfies the following conditions:

(I) , (II), (III), are the same as in Type I .(Theorem 4.1)

IV −each group T; considered by itself has either a BIB or PBIB association structure , with at least one of these groups having PBIB design properties. (if both have a BIB structure, then Type II designs become Type I designs)

i.e. any two treatments in the $i^{th}$ group appear together in either $\gamma^2{}_1$ blocks or $\gamma^2{}_2$ blocks with $\gamma^2{}_1 \neq \gamma^2{}_2$ in at least one of the two groups.

V—any two treatments, one from $T_1$ and the other from $T_2$ occur together in $\gamma_{12}$ blocks with $\gamma_{12} = \gamma^2{}_1 or\ \gamma^2{}_2$

These five conditions lead to the information matrix $C^*$ have the following form:

$$C_{ij} = a_0 B_0^i + a_1 B_1^i + a_2 B_2^i \quad , \qquad i = 1,2. \qquad (15)$$

Where :

$$a_0 = r - \frac{r}{k} \quad , \quad a_1 = -\frac{\gamma^2{}_1}{k} \quad , \quad a_2 = -\frac{\gamma^2{}_2}{k}$$

And the $B_j^i$ are appropriate association matrices with

$$\sum_{j=0}^{1\,or\,2} \mathrm{B}_{\mathrm{j}}^{1} = J_{q*q} \;\; and \;\; \sum_{j=0}^{1\,or\,2} \mathrm{B}_{\mathrm{j}}^{2} = J_{\bar{q}*\bar{q}} \tag{16}$$

$c_{12} = eJ_{q*\bar{q}} = c_{21}$

where $e = -\frac{\gamma_{12}}{k}$

proof: to prove that $c_{11.2}$ *and* $c_{22.1}$ are symmetric and of the form:

$\mathrm{v}_0\mathrm{B}_0^{\mathrm{i}} + \mathrm{v}_1\mathrm{B}_1^{\mathrm{i}} + \mathrm{v}_2\mathrm{B}_2^{\mathrm{i}}$ , $\mathrm{i} = 1,2$ we follow the steps below:

$$\begin{aligned} c_{12}{c_{22}}^{-1}c_{21} &= \left(eJ_{q*\bar{q}}\right)\left[\mathrm{v}_0\mathrm{B}_0^{\mathrm{i}} + \mathrm{v}_1\mathrm{B}_1^{\mathrm{i}} + \mathrm{v}_2\mathrm{B}_2^{\mathrm{i}}\right]^{-1}\left(eJ_{\bar{q}*q}\right) \\ &= \left(eJ_{q*\bar{q}}\right)\left[\mathrm{d}_0\mathrm{B}_0^{\mathrm{i}} + \mathrm{d}_1\mathrm{B}_1^{\mathrm{i}} + \mathrm{d}_2\mathrm{B}_2^{\mathrm{i}}\right]\left(eJ_{\bar{q}*q}\right) \qquad (17) \\ &= \mathrm{d}_0e^2\bar{q}J_{q*q} + \mathrm{d}_1e^2\bar{q}\,wJ_{q*q} + \mathrm{d}_2e^2\bar{q}\,(q-w-1)J_{q*q} \\ &= \mathrm{Q}J_{q*q} \end{aligned}$$

Then $c_{11.2} = c_{11} - c_{12}{c_{22}}^{-1}c_{21}$

$$\begin{aligned} &= \mathrm{a}_0\mathrm{B}_0^1 + \mathrm{a}_1\mathrm{B}_1^1 + \mathrm{a}_2\mathrm{B}_2^1 - \mathrm{Q}J_{q*q} \\ &= \mathrm{a}_0\mathrm{B}_0^1 + \mathrm{a}_1\mathrm{B}_1^1 + \mathrm{a}_2\mathrm{B}_2^1 - \mathrm{Q}(\mathrm{B}_0^1 + \mathrm{B}_1^1 + \mathrm{B}_2^1) \qquad (18) \\ &= \mathrm{v}_0\mathrm{B}_0^1 + \mathrm{v}_1\mathrm{B}_1^1 + \mathrm{v}_2\mathrm{B}_2^1 \end{aligned}$$

Because of the PBIB association structure the inverse of $c_{11.2}$ will be of the same form, the same holds for $c_{22.1}$ and its inverse. Now we need to find the form of ${c_{11.2}}^{-1}c_{12}{c_{22}}^{-1} =$

$$\begin{aligned} &= [\mathrm{v}_0\mathrm{B}_0^1 + \mathrm{v}_1\mathrm{B}_1^1 + \mathrm{v}_2\mathrm{B}_2^1]\left(eJ_{q*\bar{q}}\right)[\mathrm{a}_0\mathrm{B}_0^2 + \mathrm{a}_1\mathrm{B}_1^2 + \mathrm{a}_2\mathrm{B}_2^2]^{-1} \\ &= \left[\mathrm{v}_0e\,J_{q*\bar{q}} + \mathrm{v}_1e\,wJ_{q*\bar{q}} + \mathrm{v}_2e\,(q-w-1)J_{q*\bar{q}}\right][\,\mathrm{d}_0\mathrm{B}_0^2 + \mathrm{d}_1\mathrm{B}_1^2 + \mathrm{d}_2\mathrm{B}_2^2] \\ &= \left(DJ_{q*\bar{q}}\right)[\,\mathrm{d}_0\mathrm{B}_0^2 + \mathrm{d}_1\mathrm{B}_1^2 + \mathrm{d}_2\mathrm{B}_2^2] \\ &= \mathrm{d}_0DJ_{q*\bar{q}} + \mathrm{d}_1\mathrm{D}\mu J_{q*\bar{q}} + \mathrm{d}_2\mathrm{F}(\bar{q} - \mu - 1)J_{q*\bar{q}} \\ &= nJ_{q*\bar{q}} \end{aligned}$$

Hence these conditions lead us to designs that have the properties of definition 4.2. unfortunately , there are no explicit expressions for the parameters as we have seen them for Type I.

Example 4.2:

The following design is a PBTIB-Type II design with the parameters

$p = 5\,, k = 2\,, r = 4\,, r_0 = 4\,, b = 12\,,$ and the two groups $T_1 = (1,2,4,5)$ , and $T_2 = (3)$

With $\lambda_1 = 1\,, \lambda_2 = 0\,,$ the treatments of the first group: have a PBIB association structure with $\gamma_1^1 = 1$ *and* $\gamma_2^1 = 0\,,$ *and* $\gamma_{12} = 1.$

$$D = \begin{pmatrix} 1 & 3 & 4 & 1 & 2 & 5 & 1 & 2 & 3 & 1 & 2 & 4 \\ 2 & 5 & 0 & 3 & 4 & 0 & 5 & 0 & 4 & 0 & 3 & 5 \end{pmatrix} ,$$

$$C^{*} = \begin{pmatrix} 2 & -,5 & 0 & -,5 & -,5 \\ -,5 & 2 & -,5 & 0 & -,5 \\ 0 & -,5 & 2 & -,5 & -,5 \\ -,5 & 0 & -,5 & 2 & -.5 \\ -,5 & -,5 & -,5 & -.5 & 2 \end{pmatrix}$$

Where $C^{*}$ is the information matrix of $(\widehat{\alpha_0} - \widehat{\alpha_1}, \widehat{\alpha_0} - \widehat{\alpha_2}, \ldots\ldots\ldots.., \widehat{\alpha_0} - \widehat{\alpha_5})$ and

$$C^{*-1} = \begin{pmatrix} .83 & .42 & .33 & .42 & .5 \\ .42 & .83 & .42 & .33 & .5 \\ .33 & .42 & .83 & .43 & .5 \\ .42 & .33 & .42 & .83 & .5 \\ .5 & .5 & .5 & .5 & 1 \end{pmatrix}$$

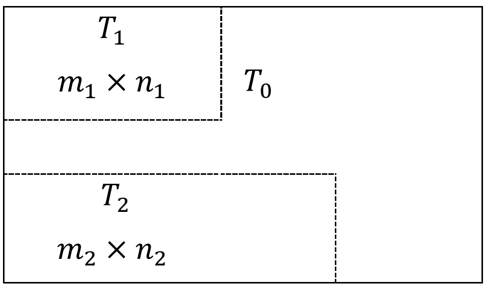
Hence ${\tau_1}^2 = .83$ , ${\tau_2}^2 = 1$ , $p_{11} = .42$ , $p_{12} = .33$ , $p_1 = .5$

## 5. CONSTRUCTION OF PBTIB DESIGN

We start with a GD-PBIB(2) design with $t$ treatments in $b$ blocks of size $k$ with $t > p$. Then we replace the "excess" treatments $p + 1, P + 2, \ldots\ldots, t$ not necessarily in order by zero to obtain a PBTIB design.

There are different methods for selecting the treatments to be replaced by zeros, depending on the PBIB design employed . in general , the procedure will be as follows :

1) Start with the association scheme arrays of the GD-PBIB(2) design which is an $m \times n$ array of the treatments $(1,2, \ldots\ldots\ldots, mn = t > p)$.
2) Choose a set of $(t - p)$ treatments that can be replaced with zeros and leave the remaining treatments in the association scheme to be arranged into two groups (arrays). Each group will be an array having the property of group divisible association scheme. Thus $T_1$ will consist of the treatments of the first group and $T_2$ will consist of the treatments of the second group ,and $T_0$ will consist of the $(t - p)$ treatments that we replaced with zeros. Therefore, schematically the GD association scheme $(m, n)$ will be partitioned as follows :




(19)

otherwise we will end up with either a PBTIB design with more than two groups or with only one group.

3) Finally, the resulting design will be a PBTIB design of either type I $or$ II , depending on the structure of the treatments within each group. More specifically, if the association scheme for a group has the form of an array, then this association scheme has a PBIB characterization , but if the association scheme for a group has the form of a vector, then it has a BIB characterization.

<u>Example 5.1:</u>

The association scheme for the GD-PBIB(2) design with parameters

$t = 9\,, k = 3\,, r = 3\,,\ b = 9\,, \bar{\lambda}_1 = 0\,,\ \bar{\lambda}_2 = 1$ (design SR23 ,Clatwortry 1973).

$$\begin{pmatrix} 1 & 4 & 7 & 1 & 2 & 3 & 1 & 2 & 3 \\ 2 & 5 & 8 & 5 & 6 & 4 & 6 & 4 & 5 \\ 3 & 6 & 9 & 9 & 7 & 8 & 8 & 9 & 7 \end{pmatrix}$$

Is derived from the array $(m, n) = (3{,}3)$

$$\begin{matrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{matrix}$$

Suppose we want to construct a PBTIB design for $p = 8$ Suppose treatments 7 is chosen to be replaced by zero and the group divisible association scheme will have the following partitioning:

| | | |
|---|---|---|
| 1 | 4 | 7 |
| 2<br>3 | 5<br>6 | 8<br>9 |

Where $T_1 = (1{,}4)$, and $T_2 = (2{,}3{,}5{,}6{,}8{,}9)$ , $T_0 = (7)$, then

$\alpha_1 = 1\,, \alpha_2 = 0\,, \beta_1 = 0\,, \beta_2 = 1$. the parameters of the resulting PBTIB design can be also be written in terms of the parameters of the PBIB design as follows :

$b = 9\,, k = 3\,, r = 3\,, r_0 = 3\,,\ \lambda_1 = \alpha_1\overline{\lambda_1} + \beta_1\overline{\lambda_2} = 0\,,\ \lambda_2 = \alpha_2\overline{\lambda_1} + \beta_2\overline{\lambda_2} = 1$

$\gamma_1^1 = \gamma_2^1 = \overline{\lambda_1} = \gamma_1^2 = 0\,, \gamma_2^2 = \overline{\lambda_2} = \gamma_{12} = 1\,,\ n_{11} = 1\,, n_{12} = 6\,, n_{21} = 2\,, n_{22} = 5\,,$

With the following association scheme for the two groups $T_1$ *and* $T_2$ :

| | Oth | 1st | 2nd |
|---|---|---|---|
| $T_1$ | 1 | 4 | 2,3,5,6,8,9 |
| | 4 | 1 | 2,3,5,6,8,9 |
| $T_2$ | 2 | 5 8 | 1,3,4,6,9 |
| | 3 | 6 9 | 1,2,4,5,8 |
| | 5 | 2 3 | 1,3,4,6,9 |
| | 6 | 3 9 | 1,2,4,5,8 |
| | 8 | 2 5 | 1,3,4,6,9 |
| | 9 | 3 9 | 1,2,4,5,8 |

From the types of the association scheme for $T_1$ *and* $T_2$ , we know that this is PBTIB-type II design.

We can still choose any treatments to be replaced by zero. Suppose we choose treatment 6 to be zero; then the association scheme after rearranging it will be as follows:

| | | |
|---|---|---|
| 1 4<br>3 9 | 7<br>6 |
| 2 5 | | 8 |

Where now $T_1 = (1,3,4,9)$ , and $T_2 = (2,5,8,)$ , $T_0 = (6,7)$, $\alpha_1 = 1, \alpha_2 = 0$, $\beta_1 = 1, \beta_2 = 2$. the resulting PBTIB design with $p = 7$ will be given as :

$$\begin{pmatrix} 1 & 4 & 0 & 1 & 2 & 3 & 1 & 2 & 3 \\ 2 & 5 & 8 & 5 & 0 & 4 & 0 & 4 & 5 \\ 3 & 0 & 9 & 9 & 0 & 8 & 8 & 9 & 0 \end{pmatrix}$$

With the parameters in terms of the parameters of the PBIB design as follows :

$b = 9, k = 3, r = 3, r_0 = 6,\ \lambda_1 = \alpha_1\overline{\lambda_1} + \beta_1\overline{\lambda_2} = 1,\ \lambda_2 = \alpha_2\overline{\lambda_1} + \beta_2\overline{\lambda_2} = 2$

$\overline{\lambda_2} = \gamma_2^1 = 1, \gamma_2^2 = \gamma_1^2 = \overline{\lambda_1} = 0, \overline{\lambda_2} = \gamma_{12} = 1,\ n_{11} = 1, n_{12} = 5, n_{21} = 2, n_{22} = 4,$

With the following association scheme for the two groups $T_1$ $and$ $T_2$.

| | Oth | 1st | 2nd |
|---|---|---|---|
| $T_1$ | 1 | 4 | 2,3,5,8,9 |
| | 3 | 9 | 1,2,4,5,8 |
| | 4 | 1 | 2,3,5,8,9 |
| | 9 | 3 | 1,2,4,5,8 |
| $T_2$ | 2 | 5,8 | 1,3,4, 9 |
| | 5 | 2,8 | 1,3,4, 9 |
| | 8 | 2,5 | 1,3,4, 9 |

If we decide to continue constructing more PBTIB designs from this design for $p$ other than( 6 or 7) , we need to be careful at this stage in choosing treatments to be replaced by zero because we need to retain the partitioning of the association scheme into two groups of treatments.

Finally, we can go one step further by replacing more treatment by zero to construct a PBTIB-type I design. One of the possibilities to do that is to choose treatment 4,8 and 9 in addition to 6 and 7 to be zero. Then the resulting PBTIB-type I design with $p = 4$ will be given as:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & 0 & 3 \\ 3 & 5 & 5 & 2 & 3 & 1 & 2 & 3 \end{pmatrix}$$

With the following parameters:

$b = 8$ , $k = 3$ , $r = 3$ , $r_0 = 12$ , $\lambda_1 = 3$ , $\lambda_2 = 4$

$\gamma_1^1 = \gamma_2^1 = 1$ , $\gamma_2^2 = \gamma_1^2 = 0$, $\gamma_{12} = 1$ , $n_{11} = 0$ , $n_{12} = 3$ , $n_{21} = 1$ , $n_{22} = 2$ ,

Where the partition of the association scheme will be as follows :

| | | |
|---|---|---|
| 3 | 9 | 6 |
| 1 | 4 | 7 |
| 2 | 5 | 8 |

With $T_1 = (1,3)$ , and $T_2 = (2,5)$ , $T_0 = (4,6,7,8,9)$, then $\alpha_1 = 2$ , $\alpha_2 = 1$ , $\beta_1 = 3$ , $\beta_2 = 4$. as illustrated below:

| | Oth | 1st | 2nd |
|---|---|---|---|
| $T_1$ | 1 | - | 3,2,5 |
| | 3 | - | 1,2,5 |
| $T_2$ | 2 | 5 | 1,3 |
| | 5 | 2 | 1,3 |

## 6. EFFICIENCY OF PBTIB DESIGNS

Different optimality criteria have been proposed to choose the “best” design capable of realizing the experimental goals , some of the criteria that are usually used are $E-, A-, D-$ optimality.

In search of the most efficient control vs, treatment design for a given $(p, k)$, we propose first choosing only the designs that are $(m-s)$ –optimal. An $(m-s)$ – optimal design is obtained by first maximizing $Tr(c)$ and then choosing the design which minimizes $Tr(c^2)$.

Then we measure the A-efficiency, for those PBTIB designs in which each test treatment appears $r$ times and the control appears $r_0$ times relative to a design that can be as optimal as possible and with the same number of replicates for the test treatment, i.e. $r$ and for the control , i.e $.ro$. we will call such a design a Near-Balance Control Treatment Block (NBCTB) design.

To find the efficiency of the PBTIB-type I design we then proceed as follows :

i. Choose the PBTIB designs that have the property of (m-s)-optimality.

ii. The A-efficiency for the chosen PBTIB-type I design will be defined as follows:

$$A-efficiency = \frac{Tr\left(C^{*-1}\right) for\ NBCTB\ Design}{Tr\left(C^{*-1}\right) for\ PBTIB\ Design}$$

Where $Tr\left(C^{*-1}\right) for\ PBTIB - Type$ I $Design$ is :

$$Tr\left(C^{*-1}\right) = \frac{q}{a} - \left( \frac{q\left(b - \frac{\bar{q}e^2}{c} + \frac{d\bar{q}^2e^2}{c(c+\bar{q}d)}\right)}{a\left(a + \bar{q}\left(b - \frac{\bar{q}e^2}{c} + \frac{d\bar{q}^2e^2}{c(c+\bar{q}d)}\right)\right)} \right) + \frac{\bar{q}}{a} - \left( \frac{\bar{q}\left(b - \frac{qe^2}{c} + \frac{dq^2e^2}{c(c+qd)}\right)}{a\left(a + q\left(b - \frac{qe^2}{c} + \frac{dq^2e^2}{c(c+qd)}\right)\right)} \right)$$

And $Tr\left(C^{*-1}\right) for\ NBCTB\ Designs$:

$$Tr(C^{*}) = \frac{p}{r} + \frac{2p}{r(1-2p)}$$

The same procedure can be used for PBTIB – Type II designs, but no explicit formula for $Tr\left(C^{*-1}\right)$ exists for PBTIB design.

$(i)$ example of PBTIB – Type II designs :

Let us consider designs $D_1, D_2, D_3$ , which are PBTIB – Type I designs with $p = 4$, $k = 4$, $r = 8$, $r_0 = 16$, $b = 12$,

Generated from PBIB designs S , R95, R96, respectively, of Clatworthy (1973).

Example 6.1:

$$D_1 = \begin{pmatrix} 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 \\ 4 & 0 & 0 & 4 & 0 & 0 & 4 & 0 & 0 & 4 & 0 & 0 \\ 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 & 1 \\ 0 & 0 & 4 & 0 & 0 & 4 & 0 & 0 & 4 & 0 & 0 & 4 \end{pmatrix}$$

Where $T_1 = (1,4)$, and $T_2 = (2,3)$, $\lambda_1 = 0$, $\lambda_2 = 12$,

$\gamma_1^2 = \gamma_2^2 = 4$, $\gamma_2^1 = \gamma_1^1 = 8$,

$\gamma_{12} = 4$, $n_{11} = 2, n_{12} = 1, n_{21} = 3, n_{22} = 0$,

$$D_2 = \begin{pmatrix} 1 & 3 & 4 & 2 & 0 & 0 & 1 & 3 & 4 & 2 & 0 & 0 \\ 2 & 0 & 0 & 1 & 3 & 4 & 2 & 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 3 & 4 & 2 & 0 & 0 & 1 & 3 & 4 & 2 \\ 0 & 1 & 0 & 4 & 2 & 3 & 0 & 1 & 0 & 4 & 2 & 3 \end{pmatrix}$$

Where $T_1 = (1)$, and $T_2 = (2,3,4)$, $\lambda_1 = 12$, $\lambda_2 = 8$,

$\gamma_1^2 = \gamma_2^2 = 6$, $\gamma_{12} = 4$, $n_{11} = 3, n_{12} = 0, n_{21} = 1, n_{22} = 2$, and

$$D_3 = \begin{pmatrix} 3 & 4 & 1 & 0 & 0 & 4 & 2 & 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & 0 & 0 & 1 & 2 & 3 & 4 & 2 & 3 & 4 & 0 \\ 0 & 0 & 2 & 2 & 0 & 3 & 4 & 0 & 3 & 4 & 1 & 1 \\ 4 & 2 & 3 & 4 & 3 & 1 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

Where $T_1 = (1,4)$, and $T_2 = (2,3)$, $\lambda_1 = 10$, $\lambda_2 = 9$,

$\gamma_1^1 = \gamma_2^1 = 4$, $\gamma_2^2 = \gamma_1^2 = 5$,

$\gamma_{12} = 5$, $n_{11} = 1, n_{12} = 2, n_{21} = 0, n_{22} = 3$, Applying the $(m-s)$-optimality criteria to choose the best design among these three designs, we find that

$$Tr({C^*}_1) = Tr({C^*}_2) = Tr({C^*}_3) = 24,$$

And $Tr\left({{C^*}_1}^2\right) = 162$ , $Tr\left({{C^*}_2}^2\right) = 163.5$ , $Tr\left({{C^*}_3}^2\right) = 161.625$

Hence $D_3$ is $(m-s)$-optimality relative to $D_1$ and $D_2$ .

Now in order to compute the $A-efficiency$ of $D_3$ , we need to construct first the appropriate NBCTB design. Obviously , it is given by

$$\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \end{matrix}$$

Hence , the $A-efficiency$ for $D_3$ will be given as

$$A-efficiency = \frac{Tr\left({C^*}^{-1}\right) for\ NBCTB\ Design}{Tr\left({C^*}^{-1}\right) for\ PBTIB\ Design}$$

$$= \frac{.75}{.838}$$

$$= 89.5\%$$

<u>$(ii)$ example of PBTIB-Type II designs.</u>

Let us consider designs $D_4 , D_5$ , which are PBTIB – Type II designs with $p = 7$ , $k = 3$ , $r = 6$ , $r_0 = 12$ , $b = 18$ ,

Obtained from the PBIB designs SR24 and R60,respectively, of Clatworthy (1973).

$$D_4 = \begin{pmatrix} 1 & 4 & 7 & 1 & 4 & 2 & 1 & 4 & 3 & 1 & 4 & 3 & 1 & 3 & 2 & 1 & 2 & 5 \\ 2 & 5 & 0 & 6 & 7 & 6 & 6 & 2 & 5 & 2 & 5 & 7 & 5 & 4 & 6 & 0 & 3 & 6 \\ 3 & 6 & 0 & 5 & 0 & 0 & 0 & 0 & 7 & 6 & 0 & 0 & 0 & 0 & 7 & 0 & 4 & 7 \end{pmatrix}$$

Where $T_1 = (1,4,7)$, and $T_2 = (2,3,5,6)$, $\lambda_1 = 4$ , $\lambda_2 = 2$,
$\gamma_1^1 = \gamma_2^1 = 0$ , $\gamma_2^2 = 2$ , $\gamma_1^2 = 0$,
$\gamma_{12} = 2$ , $n_{11} = 2$ , $n_{12} = 4$ , $n_{21} = 1$ , $n_{22} = 5$ ,

$$D_5 = \begin{pmatrix} 7 & 0 & 3 & 1 & 4 & 7 & 1 & 2 & 3 & 6 & 0 & 5 & 4 & 5 & 6 & 0 & 0 & 2 \\ 4 & 5 & 0 & 2 & 6 & 0 & 7 & 0 & 6 & 1 & 4 & 3 & 1 & 2 & 0 & 5 & 3 & 7 \\ 1 & 2 & 6 & 3 & 5 & 0 & 4 & 5 & 0 & 0 & 2 & 7 & 7 & 0 & 3 & 1 & 4 & 6 \end{pmatrix}$$

Where $T_1 = (1,4,7)$ , and $T_2 = (2,3,5,6)$, $\lambda_1 = 2$ , $\lambda_2 = 4$ , $\gamma_1^1 = \gamma_2^1 = 3$ , $\gamma_2^2 = 1$ , $\gamma_1^2 = 3$, $\gamma_{12} = 1$ , $n_{11} = 4$ , $n_{12} = 2$ , $n_{21} = 5$ , $n_{22} = 1$ , Following the same procedure that we used for the PBTIB $-$ Type I , by using first the $(m-s)$-optimality criteria , we find that :

$$Tr({C^*}_4) = Tr({C^*}_5) = 28$$

And $Tr\left({{C^*}_5}^2\right) = 115.55$ , $Tr\left({{C^*}_4}^2\right) = 126.22$ , therefore , $D_5$ is $(m-s)$ -optimality relative to $D_4$ .

Now , the appropriate NBCTB design is:

| | | | | | |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 2 | 2 | 2 | 2 | 2 |
| 3 | 3 | 3 | 3 | 3 | 3 |
| 4 | 4 | 4 | 4 | 4 | 4 |
| 5 | 5 | 5 | 5 | 5 | 5 |
| 6 | 6 | 6 | 6 | 6 | 6 |
| 7 | 7 | 7 | 7 | 7 | 7 |

Hence , the $A-efficiency$ for $D_5$ will be $= \frac{1.75}{2.373} = 73.7\%$

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