Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 14, Number 3 (2018), pp. 377-392

http://www.ripublication.com

Blow-up and Global Solutions for Nonlinear Reaction-Diffusion Equations with Nonlinear Boundary Condition

Jie Wu[1] **and Zejian Cui**[2]

[1]*College of Mathematic and Information, China West Normal University, NanChong, 637009, China.*

[1]*Corresponding author*

[2]*College of Mathematic and Information, China West Normal University, NanChong, 637009, China.*

Abstract

In this paper, we study the blow-up of positive solutions for a nonlinear reaction-diffusion equation subject to nonlinear boundary conditions. We obtain the sufficient condition under the solutions which may exist globally or blow up in finite time. Moreover, an upper bound of the blow-up time, an upper estimate of the blow-up rate, and an upper estimate of the global solutions are also given. At last we give some examples of the theorem to obtain the possible application of this paper.

Keywords: Nonlinear reaction-diffusion equations, Blow-up solutions, Global solutions

1. INTRODUCTION

Blow-up solutions and global solutions for nonlinear reaction-diffusion equations reflect instability and stability of heat and mass transport processes respectively. There is a vast literature on global existence and the blow-up in finite time of solutions to nonlinear parabolic equations and systems. We refer the reader to [3,5,6,11,16] and the references therein. Papers [8,7,12-14] researched the following equations:

$$\begin{cases} u_t = \nabla\cdot(g(u)\nabla u), x\in D, t\in(0,T), \\ \dfrac{\partial u}{\partial n} = h(x,t,u), x\in\partial D, t\in(0,T), \\ u(x,0) = u_0(x) > 0, x\in D. \end{cases}$$

Where ∇ is the gradient operator and $\partial/\partial n$ is the outward normal derivative. Paper [15] dealt with the following equations:

$$\begin{cases} \beta_t(u) = Lu + f(u), x\in D, t\in(0,T) \\ \dfrac{\partial u}{\partial n} + \sigma(x,t)h(u) = 0, x\in\partial D, t\in(0,T) \\ u(x,0) = u_0(x) > 0, x\in D. \end{cases}$$

Where $Lu = a^{ij}(x)u_{ij} + b^i u_i$ is uniformly elliptic.Paper[17] resolved the following equations:

$$\begin{cases} \beta_t(u) = \nabla\cdot(g(u)\nabla u) + f(u), x\in D, t\in(0,T) \\ \dfrac{\partial u}{\partial n} + \sigma(x,t)h(u) = 0, x\in\partial D, t\in(0,T) \\ u(x,0) = u_0(x) > 0, x\in D. \end{cases}$$

The main purpose of the paper is to extend the results in [8,7,15,17] to more general parabolic equations. In this paper, we consider the following nonlinear reaction-diffusion equations：

$$\begin{cases} \alpha_t(\phi) = \nabla\cdot(\beta(\phi)\nabla\phi) + k(t)h(\phi), x\in D, t\in(0,T) \\ \dfrac{\partial\phi}{\partial n} + \sigma(x,t)\gamma(\phi) = 0, x\in\partial D, t\in(0,T) \\ \phi(x,0) = \phi_0(x), x\in D. \end{cases} \tag{1.1}$$

Where $D\subset R^n$ is bounded domain with smooth boundary $\partial\Omega$ and T is the maximum existence time of $u(x,t)$.

Set $D_T = D\times(0,T), R^+ = (0,\infty)$. We assume throughout this paper that the function α is a position $C^2(R^+)$ increasing function, the function β is a positive $C^1(R^+)$ function, the function h is a positive $C^2(R^+)$ function and satisfies $h(0)>0$,the function k is a positive $C^1(R^+)$ function the function γ is a positive $C^1(R^+)$ function, the function σ is a nonnegative $C^1(\overline{D}_T)$ function, the function ϕ_0 is a positive $C^3(R^+)$ function and satisfies the compatibility conditions.

The object of this paper is the blow-up solutions and global solutions of (1.1). We obtain some existence theorems for blow-up solutions, upper bounds of the blow-up time, upper estimates of the blow-up rate, existence theorems for global solutions, and upper estimates of global solutions.

We proceed as follows. In Section 2, we establish the comparison principle, local existence and uniqueness of the solutions of (1.1). Section 3 studies the blow-up solutions and global solutions of (1.1). A few examples are presented in Section 4 to illustrate the application of the abstract results.

2. PRELIMINARIES

At first, we study local existence of the solution of (1.1). According to the classical parabolic equation theorem [4], a unique classical solution of (1.1) exists for $t<\delta_0$ if δ_0 is small enough, and $\phi>0$ in D_T by the maximum principle [1,9]. Denote by T the supremum of all $\delta(0<\delta<\infty)$ such that the solution exists for all $t<\delta$. We say that the solution $\phi(x,t)$ blows up if there exists a $0<T<\infty$ such that lim $\sup_{t\to T}\|\phi\|_{L^\infty(D)}=\infty$. If $T=\infty$, we call $\phi(x,t)$ a global solution. Furthermore, by the regularity theorem [10], $\phi(x,t)\in C^3(D\times(0,T))\bigcap C^2(\overline{D}\times(0,T))$.

Next we present a comparison lemma that will be used in the following sections.

Definition 2.1. Assume $\underline{\phi},\overline{\phi}\in C^3(D\times(0,T))\bigcap C^2(\overline{D}\times(0,T))$. $\underline{\phi}$ is a lower solution to problem (1.1) if

$$\begin{cases} \alpha_t(\underline{\phi}) - \nabla\cdot(\beta(\underline{\phi})\nabla\underline{\phi}) - k(t)h(\underline{\phi}) \le 0, x\in D, t\in(0,T), \\ \dfrac{\partial\underline{\phi}}{\partial n} + \sigma(x,t)\gamma(\underline{\phi}) \le 0, x\in\partial D, t\in(0,T), \\ \underline{\phi}(x,0) \le \phi_0(x), x\in D. \end{cases}$$

$\overline{\phi}$ is a upper solution to problem (1.1) if

$$\begin{cases} \alpha_t(\overline{\phi}) - \nabla\cdot(\beta(\overline{\phi})\nabla\overline{\phi}) - k(t)h(\overline{\phi}) \ge 0, x\in D, t\in(0,T), \\ \dfrac{\partial\overline{\phi}}{\partial n} + \sigma(x,t)\gamma(\overline{\phi}) \ge 0, x\in\partial D, t\in(0,T), \\ \overline{\phi}(x,0) \ge \phi_0(x), x\in D. \end{cases}$$

Lemma 2.2. If $\underline{\phi}$ is a lower solution and $\overline{\phi}$ is an upper solution to problem (1.1), then $\underline{\phi}\le\phi\le\overline{\phi}$, where ϕ is the solution to (1.1).

The proof of the comparison is similar to the one given in [1], so we omit it here. Our approach relies heavily on the maximum principles and upper-and lower-solution techniques. Therefore, the comparison lemma is our main tool.

3. BLOW-UP AND GLOBAL SOLUTIONS

We first consider the case which the initial value is a positive constant.

Theorem 3.1. Let $\phi(x,t)$ be a smooth solution of the following auxiliary problem

$$\begin{cases} \alpha_t(\phi) = \nabla\cdot(\beta(\phi)\nabla\phi) + k(t)h(\phi), x\in D, t\in(0,T) \\ \dfrac{\partial\phi}{\partial n} + \sigma(x,t)\gamma(\phi) = 0, x\in\partial D, t\in(0,T) \\ \phi(x,0) = \varepsilon, x\in D. \end{cases} \tag{3.1}$$

Where ε is a positive constant. Set $\delta = \beta(\varepsilon)/\alpha^{'}(\varepsilon)$. Assume that the following conditions are satisfied:

(1) Suppose the following:

(i) $k(t)\ge 1, k'(t)\le 0, \alpha'(s)>0, (\beta(s)\gamma(s))'\ge 0, (\beta(s)/\alpha'(s))'\ge 0, (h'(s)/\beta(s))'\ge 0,$

$((\beta(s)\gamma(s))/h(s))'\le 0 \ \ for \ \ s>0$; $\sigma_t(x,t)\le 0 \ in \ \overline{D}_T$.

(ii) $\int_{\varepsilon}^{+\infty}\beta(s)/h(s)ds<\infty$

Then, $\phi(x,t)$ must blow up in finite T and $T\le\delta^{-1}\int_{\varepsilon}^{+\infty}\beta(s)/h(s)ds.$ Moreover,

$$\sup_{x\in\overline{D}}\phi(x,t)\le G^{-1}(\delta(T-t)),\ for\ \ 0<t<T.$$

Where $G(\tau)=\int_{\tau}^{+\infty}\beta(s)/h(s)ds$ and G^{-1} is the inverse function of G.

(2) Suppose the following:

(i) $0<k(t)\le 1, k'(t)\ge 0, \alpha'(s)>0, (\beta(s)\gamma(s))'\ge 0, (\beta(s)/\alpha'(s))'\le 0, (h'(s)/\beta(s))'\le 0,$

$((\beta(s)\gamma(s))/h(s))'\ge 0 \ \ for \ \ s>0$; $\sigma_t(x,t)\ge 0 \ in \ \overline{D}_T$.

(ii) $\int_{\varepsilon}^{+\infty}\beta(s)/h(s)ds=\infty$

Then, $\phi(x,t)$ exists globally and

$$\sup_{x\in\overline{D}}\phi(x,t)\le G_1^{-1}(\delta t+G_1(\varepsilon))\ \ for\ \ 0<t,$$

Where $G_1(\tau)=\int_0^{\tau}\beta(s)/h(s)ds$ and G_1^{-1} is the inverse function of G_1.

Proof. Similarly as [2,5,10], consider the auxiliary function

$$J=\delta k(t)h(\phi)-\beta(\phi)\phi_t \tag{3.2}$$

From which we find that

$$J_t=\delta k'(t)h(\phi)+\delta k(t)h'(\phi)\phi_t-\beta'(\phi)\phi_t^2-\beta(\phi)\phi_{tt} \tag{3.3}$$

$$\nabla J = \delta k(t)h'(\phi)\nabla\phi - \beta'(\phi)\nabla\phi\phi_t - \beta(\phi)\nabla\phi_t \tag{3.4}$$

And

$$\Delta J = \nabla\cdot\nabla J = \delta k(t)h''(\phi)|\nabla\phi|^2 + \delta k(t)h'(\phi)\Delta\phi - \beta''(\phi)|\nabla\phi|^2\phi_t$$

$$-\beta'(\phi)\Delta\phi\phi_t - 2\beta'(\phi)\nabla\phi\nabla\phi_t - \beta(\phi)\Delta\phi_t \tag{3.5}$$

By (3.1), we have

$$\phi_t = \frac{\beta(\phi)|\nabla\phi|^2}{\alpha'(\phi)} + \frac{\beta(\phi)\Delta\phi}{\alpha'(\phi)} + \frac{h(\phi)k(t)}{\alpha'(\phi)} \tag{3.6}$$

Substituting (3.6) into (3.3), we get

$$J_t = \delta k'(t)h(\phi) + \delta k(t)h'(\phi)\phi_t - \beta'(\phi)\phi_t^2 - \beta(\phi)[\frac{\beta'(\phi)|\nabla\phi|^2 + \beta(\phi)\Delta\phi + k(t)h(\phi)}{\alpha'(\phi)}]_t$$

$$= \delta k'(t)h(\phi) - \beta'(\phi)\phi_t^2 + [\delta k(t)h'(\phi) + \frac{k(t)\beta(\phi)h(\phi)\alpha''(\phi)}{(\alpha''(\phi))^2} - \frac{k(t)h'(\phi)\beta(\phi)}{\alpha'(\phi)}]\phi_t$$

$$+[\frac{\beta^2(\phi)\alpha''(\phi)}{(\alpha'(\phi))^2} - \frac{\beta(\phi)\beta'(\phi)}{\alpha'(\phi)}]\phi_t\Delta\phi - \frac{\beta^2(\phi)}{\alpha'(\phi)}\Delta\phi_t$$

$$+[\frac{\beta(\phi)\beta'(\phi)\alpha''(\phi)}{(\alpha'(\phi))^2} - \frac{\beta(\phi)\beta''(\phi)}{\alpha'(\phi)}]\phi_t|\nabla\phi|^2 - \frac{2\beta(\phi)\beta'(\phi)}{\alpha'(\phi)}\nabla\phi\nabla\phi_t + \frac{k(t)h(\phi)\beta'(\phi)}{\alpha'(\phi)} \tag{3.7}$$

It follows from (3.5) and (3.7) that

$$\frac{\beta(\phi)}{\alpha'(\phi)}\Delta J - J_t = \beta'(\phi)\phi_t^2 - [\delta k(t)h'(\phi) + \frac{k(t)\beta(\phi)h(\phi)\alpha''(\phi)}{(\alpha''(\phi))^2} - \frac{k(t)\beta(\phi)h'(\phi)}{\alpha'(\phi)}]\phi_t$$

$$-\frac{\beta(\phi)\beta'(\phi)\alpha''(\phi)}{(\alpha'(\phi))^2}\phi_t|\nabla\phi|^2 - \frac{\beta^2(\phi)\alpha''(\phi)}{(\alpha'(\phi))^2}\phi_t\Delta\phi$$

$$+\frac{\delta k(t)\beta(\phi)h''(\phi)}{\alpha'(\phi)}|\nabla\phi|^2 + \frac{\delta k(t)\beta(\phi)h'(\phi)}{\alpha'(\phi)}\Delta\phi \tag{3.8}$$

By (3.6) we have

$$\Delta\phi = \frac{\alpha'(\phi)\phi_t}{\beta(\phi)} - \frac{\beta'(\phi)|\nabla\phi|^2}{\beta(\phi)} - \frac{k(t)h(\phi)}{\beta(\phi)} \tag{3.9}$$

Substituting (3.9) into (3.8), we get

$$\begin{aligned}
\frac{\beta(\phi)}{\alpha'(\phi)}\Delta J - J_t &= \beta'(\phi)\phi_t^2 - [\delta k(t)h'(\phi) + \frac{k(t)\beta(\phi)h(\phi)\alpha''(\phi)}{(\alpha''(\phi))^2} - \frac{k(t)\beta(\phi)h'(\phi)}{\alpha'(\phi)}]\phi_t \\
&- \frac{\beta(\phi)\beta'(\phi)\alpha''(\phi)}{(\alpha'(\phi))^2}\phi_t|\nabla\phi|^2 - \frac{\beta^2(\phi)\alpha''(\phi)}{(\alpha'(\phi))^2}\phi_t(\frac{\alpha'(\phi)\phi_t}{\beta(\phi)} - \frac{\beta'(\phi)|\nabla\phi|^2}{\beta(\phi)} - \frac{k(t)h(\phi)}{\beta(\phi)}) \\
&+ \frac{\delta k(t)\beta(\phi)h''(\phi)}{\alpha'(\phi)}|\nabla\phi|^2 + \frac{\delta k(t)\beta(\phi)h'(\phi)}{\alpha'(\phi)}(\frac{\alpha'(\phi)\phi_t}{\beta(\phi)} - \frac{\beta'(\phi)|\nabla\phi|^2}{\beta(\phi)} - \frac{k(t)h(\phi)}{\beta(\phi)}) \\
&= [\beta'(\phi) - \frac{\beta(\phi)\alpha''(\phi)}{\alpha'(\phi)}]\phi_t^2 + \frac{k(t)\beta(\phi)h'(\phi)}{\alpha'(\phi)}\phi_t \\
&+ [\frac{\delta k(t)\beta(\phi)h''(\phi)}{\alpha'(\phi)} - \frac{\delta k(t)\beta'(\phi)h'(\phi)}{\alpha'(\phi)}]|\nabla\phi|^2 - \frac{\delta k(t)h(\phi)h'(\phi)}{\alpha'(\phi)}
\end{aligned} \tag{3.10}$$

According to (3.2), we have

$$\begin{aligned}
\frac{\beta(\phi)}{\alpha'(\phi)}\Delta J + \frac{h'(\phi)}{\alpha'(\phi)}J - J_t &= [\beta'(\phi) - \frac{\beta(\phi)\alpha''(\phi)}{\alpha'(\phi)}]\phi_t^2 + \frac{k(t)\beta(\phi)h'(\phi)}{\alpha'(\phi)}\phi_t \\
&+ [\frac{\delta k(t)\beta(\phi)h''(\phi)}{\alpha'(\phi)} - \frac{\delta k(t)\beta'(\phi)h'(\phi)}{\alpha'(\phi)}]|\nabla\phi|^2 - \frac{\delta k(t)h(\phi)h'(\phi)}{\alpha'(\phi)} \\
&+ \frac{h'(\phi)}{\alpha'(\phi)}(\delta k(t)h(\phi) - \beta(\phi)\phi_t) \\
&= \alpha'(\phi)(\frac{\beta(\phi)}{\alpha'(\phi)})'\phi_t^2 + \frac{\delta k(t)\beta^2(\phi)}{\alpha'(\phi)}(\frac{h'(\phi)}{\beta(\phi)})'|\nabla\phi|^2 \\
&+ (k(t)-1)\frac{\delta\beta(\phi)h'(\phi)}{\alpha'(\phi)}\phi_t
\end{aligned} \tag{3.11}$$

For (1): According to supposition (1)(i), the right side of equation (3.11) is nonnegative, i.e.,

$$\frac{\beta(\phi)}{\alpha'(\phi)}\Delta J+\frac{h'(\phi)}{\alpha'(\phi)}J-J_t\geq 0 \tag{3.12}$$

By the maximum principle for parabolic problems we know that J can attain its maximum only for $t=0$ or on ∂D. For $t=0$,

$$J(x,0)=\left\{\delta k(t)h(\phi)-\frac{\beta(\phi)}{\alpha'(\phi)}[\nabla\cdot(\beta(\phi)\nabla\phi)+k(t)h(\phi)]\right\}|_{t=0}=\delta k(0)h(\varepsilon)-k(0)\frac{\beta(\varepsilon)}{\alpha'(\varepsilon)}h(\varepsilon)=0$$

We claim that J cannot take a positive maximum at any point $(x_0,t_0)\in\partial D\times(0,T)$.

Indeed, if J takes a positive maximum at point $(x_0,t_0)\in\partial D\times(0,T)$. then

$$J(x_0,t_0)>0 \quad \text{and} \quad \frac{\partial J}{\partial n}>0 \quad \text{at} \quad (x_0,t_0). \tag{3.13}$$

From $J(x_0,t_0)>0$ we have $\phi_t(x,t)<\delta k(t)h(\phi)/\beta(\phi)$ at (x_0,t_0). Note that

$$\frac{\partial\phi}{\partial n}+\sigma(x,t)\gamma(\phi)=0 \quad \text{on} \quad \partial D\times(0,T).$$

Thus

$$\begin{aligned}\frac{\partial J}{\partial n}\Big|_{\substack{x=x_0\\ t=t_0}} &=\delta k'(t)h(\phi)+\delta k(t)h'(\phi)\frac{\partial\phi}{\partial n}-\beta'(\phi)\frac{\partial\phi}{\partial n}\phi_t-\beta(\phi)\frac{\partial\phi_t}{\partial n}\\
&=\delta k'(t)h(\phi)+\beta'(\phi)\sigma(x,t)\gamma(\phi)\phi_t+\beta(\phi)\sigma(x,t)\gamma'(\phi)\phi_t+\sigma_t(x,t)\gamma(\phi)\beta(\phi)\\
&\quad-\delta k(t)\sigma(x,t)h'(\phi)\gamma(\phi)\\
&=\delta k'(t)h(\phi)+\sigma(x,t)(\beta(\phi)\gamma(\phi))'\phi_t+\sigma_t(x,t)\beta(\phi)\gamma(\phi)-\delta k(t)\sigma(x,t)h'(\phi)\gamma(\phi)\\
&\leq\delta k'(t)h(\phi)+\sigma(x,t)(\beta(\phi)\gamma(\phi))'\frac{\delta k(t)h(\phi)}{\beta(\phi)}+\sigma_t(x,t)\beta(\phi)\gamma(\phi)-\delta k(t)\sigma(x,t)h'(\phi)\gamma(\phi)\\
&=\delta k'(t)h(\phi)+\frac{\delta\sigma(x,t)k(t)}{\beta(\phi)}\left(\frac{\beta(\phi)\gamma(\phi)}{h(\phi)}\right)'+\sigma_t(x,t)\beta(\phi)\gamma(\phi)\end{aligned}$$

According to supposition (1)(i), left hand side of the above inequality is not positive, i.e.,

$$\frac{\partial J}{\partial n} \le 0.$$

This contradicts the inequality (3.13). Thus $J \le 0$ in $\overline{D} \times [0,T)$. From the assumption

$f \in C^2(R^+)$ and $f(0) > 0$, we have

$$\frac{\beta(\phi)}{\delta k(t) h(\phi)} \phi_t \ge 1 \text{ in } \overline{D} \times [0,T). \tag{3.14}$$

Integrate (3.14) over $[0,t]$ to get, for each fixed x and $k(t) \ge 1$, we have

$$\int_0^t \frac{\beta(\phi)}{h(\phi)} \phi_t dt \ge \int_0^t \delta dt$$

$$\int_{\phi(x,0)}^{\phi(x,t)} \frac{\beta(s)}{h(s)} ds \ge \delta t$$

$$\int_{\varepsilon}^{\phi(x,t)} \frac{\beta(s)}{h(s)} ds \ge \delta t$$

By using assumption (1)(ii), it follows that $\phi(x,t)$ must blow up in finite time T. Furthermore, the following inequality must hold

$$T \le \frac{1}{\delta} \int_{\varepsilon}^{+\infty} \frac{\beta(s)}{h(s)} ds.$$

Integrating inequality (3.14) over $[t,s] (0 < t < s < T)$, that

$$G(\phi(x,t)) \ge G(\phi(x,t)) - G(\phi(x,s)) = \int_{\phi(x,t)}^{\phi(x,s)} \frac{\beta(s)}{h(s)} ds \ge \delta(s-t) \quad ,$$

Thus

$$\phi(x,t) \le G^{-1}(\delta(T-t)) \text{ for } 0<t<T.$$

The proof of theorem 3.1(1) is complete.

For (2): According to supposition (2)(i) and the equation (3.11), we have

$$\frac{\beta(\phi)}{\alpha'(\phi)}\Delta J + \frac{h'(\phi)}{\alpha'(\phi)} J - J_t \le 0$$

Repeating the above proof process, we get

$$J \ge 0 \text{ in } \overline{D}\times[0,T).$$

Which implies from the assumption $f \in C^2(R^+)$ and $f(0) > 0$,

$$\phi_t(x,t) \le \frac{\delta k(t) h(\phi)}{\beta(\phi)} \quad \text{in} \quad \overline{D}\times[0,T). \tag{3.15}$$

For each fixed $x \in \overline{D}$ and $0 < k(t) \le 1$, we obtain by integration (3.15)

$$\int_{\varepsilon}^{\phi(x,t)} \frac{\beta(s)}{h(s)} ds \le \delta t \tag{3.16}$$

It follows from assumption (2)(ii) that $\phi(x,t)$ must be a global solution. From the inequality (3.16), we get

$$G_1(\phi(x,t)) - G_1(\varepsilon) = \int_{\varepsilon}^{\phi(x,s)} \frac{\beta(s)}{h(s)} ds \le \delta t$$

And

$$\phi(x,t) \le G_1^{-1}(\delta t + G(\varepsilon))$$

The proof of theorem 3.1 is complete.

The following theorem considers that the initial value is a function.

Theorem 3.2. Assuming $\phi(x,t)$ is a smooth solution of (1.1). set

$\delta^* = \beta(M)/\alpha'(M), \delta_* = \beta(m)/\alpha'(m)$, where $M = \max_{x\in\overline{D}} \phi_0(x), m = \min_{x\in\overline{D}} \phi_0(x)$.

(1)Suppose the following:

(i) $k(t) \geq 1, k'(t) \leq 0, \alpha'(s) > 0, (\beta(s)\gamma(s))' \geq 0, (\beta(s)/\alpha'(s))' \geq 0, (h'(s)/\beta(s))' \geq 0,$

$(\beta(s)\gamma(s)/h(s))' \leq 0$ *for* $s > 0$; $\sigma_t(x,t) \leq 0$ *in* $\overline{D}_T$.

(ii) $\int_m^{+\infty} \beta(s)/h(s)ds < \infty$

Then, $\phi(x,t)$ must blow up in finite T and $T \leq \delta_*^{-1} \int_\varepsilon^{+\infty} \beta(s)/h(s)ds.$ Moreover,

$$\sup_{x\in\overline{D}} \phi(x,t) \leq G^{-1}(\delta^*(T-t)) \, for \, 0 < t < T$$

Where $G(\tau) = \int_\tau^{+\infty} \beta(s)/h(s)ds$ and G^{-1} is the inverse function of G.

(2)Suppose the following:

(i) $0 < k(t) \leq 1, k'(t) \geq 0, \alpha'(s) > 0, (\beta(s)\gamma(s))' \geq 0, (\beta(s)/\alpha'(s))' \leq 0, (h'(s)/\beta(s))' \leq 0,$

$(\beta(s)\gamma(s)/h(s))' \geq 0$ *for* $s > 0$; $\sigma_t(x,t) \geq 0$ *in* $\overline{D}_T$.

(ii) $\int_M^{+\infty} \beta(s)/h(s)ds = \infty.$

Then, $\phi(x,t)$ exists globally and

$$\sup_{x\in\overline{D}} \phi(x,t) \leq G_1^{-1}(\delta^* t + G_1(M)) \, for \, 0 < t,$$

Where $G_1(\tau) = \int_\tau^{+\infty} \beta(s)/h(s)ds$ and G_1^{-1} is the inverse function of G_1.

Proof. Let $\overline{\phi}(x,t)$ and $\underline{\phi}(x,t)$ be smooth positive solutions of the problems

$$\begin{cases} \alpha_t(\overline{\phi}) = \nabla \cdot (\beta(\overline{\phi})\nabla\overline{\phi}) + k(t)h(\overline{\phi}), x \in D, t \in (0,T) \\ \dfrac{\partial \overline{u}}{\partial n} + \sigma(x,t)\gamma(\overline{\phi}) = 0, x \in \partial D, t \in (0,T) \\ \overline{\phi}(x,0) = M, x \in D. \end{cases} \tag{3.17}$$

And

$$\begin{cases} \alpha_t(\underline{\phi}) = \nabla \cdot (\beta(\underline{\phi})\nabla\underline{\phi}) + k(t)h(\underline{\phi}), x \in D, t \in (0,T) \\ \dfrac{\partial \underline{\phi}}{\partial n} + \sigma(x,t)\gamma(\underline{\phi}) = 0, x \in \partial D, t \in (0,T) \\ \underline{\phi}(x,0) = m, x \in D. \end{cases} \tag{3.18}$$

Respectively.

(1) By Theorem 3.1(1), $\underline{\phi}(x,t)$ blows up globally in $\overline{D}$ and

$$\sup_{x\in\overline{D}} \underline{\phi}(x,t) \le G^{-1}(\delta^*(T-t)) \quad \text{for} \quad 0 < t < T.$$

It follows from theorem 3.1(1) that $\underline{\phi}(x,t)$ blows up globally in $\overline{D}$ at the blow-up time

$$T \le \frac{1}{\delta^*}\int_m^{+\infty} \frac{\beta(s)}{h(s)}\, ds.$$

By lemma 2.2, we have

$$\underline{\phi}(x,t) \le \phi(x,t) \le \overline{\phi}(x,t).$$

Hence, $\phi(x,t)$ blows up globally in $\overline{D}$ and

$$T \le \frac{1}{\delta^*}\int_m^{+\infty} \frac{\beta(s)}{h(s)}\, ds.$$

As well as

$$\sup_{x\in\overline{D}} \phi(x,t) \le G^{-1}(\delta^*(T-t)) \quad \text{for} \quad 0 < t < T.$$

The proof of theorem 3.2(1) is complete.

(2) It follows from theorem 3.1(2) that $\bar{\phi}(x,t)$ must be a global solution and

$$\sup_{x\in\bar{D}}\bar{\phi}(x,t)\le G_1^{-1}(\delta^* t+G_1(M)) \quad \text{for} \quad t>0.$$

Using comparison principle, we say that $\bar{\phi}(x,t)$ is an upper solution of (1.1). Thus $\phi(x,t)$ exists globally and

$$\sup_{x\in\bar{D}}\phi(x,t)\le G_1^{-1}(\delta^* t+G_1(M)) \quad \text{for} \quad t>0.$$

The proof of Theorem 3.2 is complete.

4. APPLICATIONS OF THE RESULT

In this section, we consider two special equations. We apply the results of Section 3 to obtain the behavior of the solution of (1.1).

Example 1. Suppose w is a smooth positive solution of the problem (see[5,10])

$$\begin{cases} w_t=\Delta w+h(w), x\in D, t\in(0,T) \\ \dfrac{\partial w}{\partial n}+\sigma(x,t)w=0, x\in\partial D, t\in(0,T) \\ w(x,0)=w_0(x), x\in D. \end{cases}$$

Where $0<m\le w_0(x)\le M$. Now

$$\alpha(s)=s, \beta(s)=1, k(t)=1, \gamma(s)=s$$

Assuming $h(0)>0,$ by Theorem 3.2, it is easy to obtain:

(I) If $h''(s)\ge 0, sh'(s)\ge h(s)$ for $s>0, \sigma_t(x,t)\le 0$ in $\bar{D}_T$ and $\int^{+\infty} 1/h(s)ds<+\infty,$ then

$w(x,t)$ must blow up in finite time T and $T\le\int_m^{+\infty}1/h(s)ds.$ Moreover,

$$\sup_{x\in\in\bar{D}} w(x,t) \le G^{-1}(T-t) \quad \text{for} \quad 0<t<T,$$

Where $G(\tau)=\int_\tau^{+\infty} 1/h(s)ds.$

(II) If $h''(s)\le 0, sh'(s)\le h(s)$ for $s>0, \sigma_t(x,t)\ge 0$ in $\bar{D}_T$ and $\int^{+\infty} 1/h(s)ds=+\infty,$ then

$w(x,t)$ exists globally and

$$\sup_{x\in\in\bar{D}} w(x,t) \le G_1^{-1}(t+G_1(M)) \quad \text{for} \quad t>0,$$

Where $G_1(\tau)=\int_0^\tau 1/h(s)ds.$

Example 2. Similarly as [3], we discuss the following equations

$$\begin{cases} ((\lambda+w)\ln^p(\lambda+e^w))_t=\nabla\cdot(\ln^\sigma(\lambda+e^w)\nabla w)+(r^t+1)(\lambda+w)\ln^q(\lambda+e^w), x\in D, t\in(0,T),\\ \dfrac{\partial w}{\partial n}=0, x\in\partial D, t\in(0,T),\\ w(x,0)=w_0(x)>0, x\in D, \end{cases}$$

Where p,q,σ,r is nonnegative and $\lambda>1, 0<r<1$ are constants. Now

$$\alpha(s)=(\lambda+w)\ln^p(\lambda+e^w), \beta(s)=\ln^\sigma(\lambda+e^w), k(t)=r^t+1, h(s)=(\lambda+w)\ln^q(\lambda+e^w), \gamma(s)=0$$

In this case $q-1>\sigma\ge p\ge 0$, From Theorem3.2(1), the solution $w(x,t)$ of the problem must blow up in finite time T and

$$T\le\frac{\ln^{\sigma-q+1}(\lambda+e^m)}{\delta_*(q-\sigma-1)},$$

where $m=\min_{x\in\bar{D}} w_0(x)$ and $\delta_*=\ln^{\sigma-p+1}(\lambda+e^m)(\ln(\lambda+e^m)+p)^{-1}.$

where $\delta^*=\ln^{\sigma-p+1}(\lambda+e^M)(\ln(\lambda+e^M)+p)^{-1}$ and $M=\max_{x\in\bar{D}} w_0(x).$

Moreover, $w(x,t)\le e^{[\delta^*(q-\sigma-1)(T-t)]^{\frac{1}{\sigma-q+1}}}-1.$

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